# Lesson 18: Probability

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# What is Probability?

- The odds of a certain event E occurring, denoted $P(E) \in [0, 1]$
- When discrete, it can be computed as

$$\frac{\#\text{ of times E occurs}}{\#\text{ total events}}$$

# Complementary Counting

- It's often easier to try to count the opposite of $E$
- Formally, since $P(E^c) = 1 - P(E)$, we only need to find $P(E^c)$ to get $P(E)$
- Note that complementary counting doesn't give you a specific way to solve questions, rather a general heuristic to make problem solving a lot easier

# Complementary Counting

## 2011 AIME II #12

Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Let the probability that each delegate sits next to at least one delegate from another country be $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

- What is the complement of this scenario?
- We have a delegate which doesn't sit next to another country if all 3 delegates from a single country are together
- WLOG that the seats are numbered but the delegates are indistinguishable. What is the total # of ways to be seated?
- $\frac{9!}{(3!)^3} = 1680$
- Now, how many ways can we arrange the delegates if one country's members are all together?

# 2011 AIME II #12

- $9 * \frac{6!}{(3!)^2} = 180$. This can happen to any of the countries, so we get $180 * 3 = 540$. Are we done?
- No! We overcounted when two countries are both in blocks
- If two countries are in blocks, we have $9 \cdot 4 = 36$ ways to arrange
- If all three countries are in blocks, we have $9 \cdot 2 = 18$ ways to arrange
- In total, our complement is

$$3 \cdot 180 - 3 \cdot 36 + 18 = 450$$

- Hence, we get

$$1 - \frac{450}{1680} = 1 - \frac{15}{56} = \frac{41}{56}$$

and our answer is $41 + 56 = \boxed{97}$.

# Conditional Probability

- Sometimes questions will be in the format of: Compute the probability of $A$ given $B$. This is known as conditional probability
- Denoted $P(A|B) \in [0, 1]$
- We have that
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$
- Though the concept of conditional probability isn't difficult, it is worth keeping in mind that some traditional logic will not follow under a condition
- Conditional Probability questions can often be solved by considering the nice properties of the imposed condition

# Conditional Probability

## 2015 HMMT C9

Calvin has a bag containing 50 red balls, 50 blue balls, and 30 yellow balls. Given that after pulling out 65 balls at random (without replacement), he has pulled out 5 more red balls than blue balls, what is the probability that the next ball he pulls out is red?

- Why does it suffice to find the probability that a yellow ball is pulled?
- If the probabilities are $r, b, y$, we have that $r + b + y = 1$ and $b - r = \frac{1}{13}$, so having one of the variables yields the other two
- How do we find the expected number of yellow balls remaining?
- Can anyone see any symmetry in the imposed? condition
- Call the number of red, blue, yellow balls $R, B, Y$
- $P(R, B, Y) = P(B, R, Y)$

# 2015 HMMT C9

- $P(R, B, Y) = P(50 - R, 50 - B, 30 - Y)$
- Combined, $P(R, B, Y) = P(50 - B, 50 - R, 30 - Y)$, AND we have

$$R - B = 5 \iff (50 - B) - (50 - R) = 5$$

- So, all cases where $R - B = 5$ can be paired in this way
- The probability of getting $Y$ and $30 - Y$ is the same, so $\mathbb{E}(Y) = 15$, and $y = \frac{15}{65} = \frac{3}{13}$
- Hence, $r = \frac{1-y-\frac{1}{13}}{2} = \boxed{\frac{9}{26}}$

# Conditional Probability

## 2016 AIME II #2

There is a 40% chance of rain on Saturday and a 30% of rain on Sunday. However, it is twice as likely to rain on Sunday if it rains on Saturday than if it does not rain on Saturday. The probability that it rains at least one day this weekend is $\frac{a}{b}$, where $a$ and $b$ are relatively prime positive integers. Find $a + b$.

- Let $A$ denote the event of rain on Saturday and $B$ denote the event of rain on Sunday.
- We know $P(A) = \frac{2}{5}$, $P(B) = \frac{3}{10}$, and $P(B|A) = 2P(B|(\text{not } A))$.
- $\frac{P(A \cap B)}{P(A)} = 2\frac{P((\text{not } A) \cap B)}{P(\text{not } A)}$
- Note that $P((\text{not } A) \cap B) = P(B) - P(A \cap B) = \frac{3}{10} - P(A \cap B)$ and $P(\text{not } A) = \frac{3}{5}$.

$$\frac{P(A \cap B)}{\frac{2}{5}} = 2\frac{\frac{3}{10} - P(A \cap B)}{\frac{3}{5}}$$

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# 2016 AIME II #2

- Solving $P(A \cap B) = \frac{6}{35}$.
- Recall we want $P(A) + P(B) - P(A \cap B) = \frac{2}{5} + \frac{3}{10} - \frac{6}{35} = \frac{37}{70} \implies \boxed{107}$.

# Conditional Probability

## 2011 AIME I #12

Six men and some number of women stand in a line in random order. Let $p$ be the probability that a group of at least four men stand together in the line, given that every man stands next to at least one other man. Find the least number of women in the line such that $p$ does not exceed 1 percent.

- Let $A$ denote the event of four men standing together in line and let $B$ denote the event of every man standing next to at least one other man.
- We want to find $P(A|B) = \frac{P(A \cap B)}{P(B)}$.
- We begin by calculating $P(B)$.
- Suppose that there are $n$ women in line. We note that the total number of ways to arrange them in line is $\binom{n+6}{6}$ (assuming all men and all women are indistinguishable), so this is the denominator of $P(B)$ and $P(A \cap B)$

- Now, note that in order for event $B$ to happen, the men must be split into blocks of size at least 2.
- We now have the cases are 2/2/2, 2/4, 3/3, and 6.
- How many ways are there for the case 2/2/2 to happen?
- If we were to arrange the $n$ women in a row, then there are $n+1$ places to put a block of 2 men, so there are $\binom{n+1}{3}$ possibilities here.
- Similarly, the 3/3 and 6 cases give $\binom{n+1}{2}$ and $\binom{n+1}{1}$ possibilities, respectively.
- However, the 2/4 case has 2 differently sized blocks, so we not only choose where to insert the blocks but also which one is which, so there are $2\binom{n+1}{2}$ possibilities here.
- Thus, we find that there are $\binom{n+1}{3}+3\binom{n+1}{2}+\binom{n+1}{1}$ possibilities in $P(B)$
- We can also find that there are $2\binom{n+1}{2}+\binom{n+1}{1}$ possibilities in $P(A \cap B)$.

# 2011 AIME I #12

- Thus, we find that

$$P(A|B) = \frac{2\binom{n+1}{2} + \binom{n+1}{1}}{\binom{n+1}{3} + 3\binom{n+1}{2} + \binom{n+1}{1}} = \frac{6(n+1)}{n^2 + 8n + 6} \leq \frac{1}{100}$$

- Thus, expanding gives us $n(n - 592) \geq 594$, so we find that $n = 594$ is the least $n$ that works.