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Lesson 22: Congruence Criteria for Triangles—SAS

Student Outcomes

- Students learn why any two triangles that satisfy the SAS congruence criterion must be congruent.

Lesson Notes

In Lesson 22, we begin to investigate criteria, or the indicators, of triangle congruence. Students are introduced to the concept in 8th grade, but have justified the criteria of triangle congruence (i.e., ASA, SAS, and SSS) in a more hands-on manner, manipulating physical forms of triangles through rigid motions to justify whether a pair of triangles is congruent or not. In this course, students formally prove the triangle congruency criteria.

Classwork

Opening Exercise (7 minutes)

> **Answer the following question. Then discuss your answer with a partner.**
>
> **Is it possible to know that there is a rigid motion that takes one triangle to another without actually showing the particular motion?**
>
> ***Yes, it is possible. Our indicators—ASA, SAS, etc— tell us that a rigid motion will map one triangle completely onto another, without actually having to display the rigid motion.***

Remember, prior to the Common Core standards, we used a pattern such as ASA to forecast that all three sides and all three angles between two triangles are congruent. We are no longer using this idea. Even if we know this information, the real question is whether there is exists rigid motions mapping one triangle to the other. Thus, these patterns are indicators of whether rigid motions exist to take one triangle to the other.

You might anticipate your students saying, "Well, if ALL the corresponding parts are equal in measure, don't we OBVIOULSY have congruent triangles?" The answer is, "NO! We have agreed to use the word 'congruent' to mean 'there exists a composition of basic rigid motion of the plane that maps one figure to the other." We will see that SAS, ASA, and SSS implies the existence of the rigid motion needed, but precision demands that we explain how and why."

Discussion (20 minutes)

> **It is true that we will not need to show the rigid motion to be able to know that there is one. We are going to show that there are criteria that refer to a few parts of the two triangles and a correspondence between them that guarantee congruency (i.e. existence of rigid motion). We start with the Side-Angle-Side criteria.**
>
> **Side-Angle-Side triangle congruence criteria (SAS):** **Given two triangles $\triangle ABC$ and $\triangle A'B'C'$ so that $AB = A'B'$ (Side), $\angle A = \angle A'$ (Angle), $AC = A'C'$ (Side). Then the triangles are congruent.**
>
> **The steps below show the most general case for determining a congruence between two triangles that satisfy the SAS criteria. Note that not all steps are needed for every pair of triangles. For example, sometimes the triangles will already share a vertex. Sometimes a reflection will be needed, sometimes not. What is important is that we can always use the steps below—some or all of them—to determine a congruence between the two triangles that satisfy the SAS criteria.**

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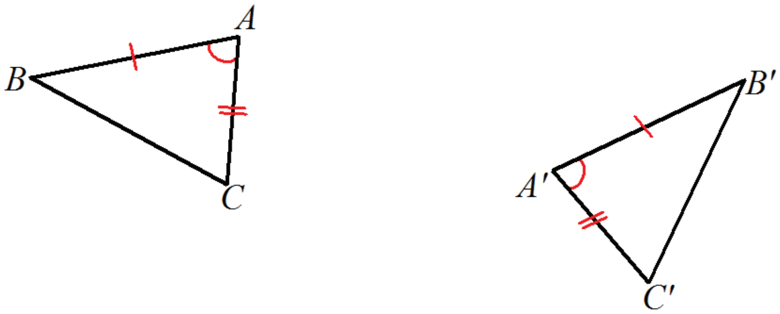
Proof: Provided the two distinct triangles below, assume $AB = A'B'$ (Side), $\angle A = \angle A'$ (Angle), $AC = A'C'$ (Side).



By our definition of congruence, we will have to find a composition of rigid motions will map $\triangle A'B'C'$ to $\triangle ABC$. So we must find a congruence F so that $F(\triangle A'B'C') = \triangle ABC$. First, use a translation T to map a common vertex.

Which two points determine the appropriate vector?

A', A.

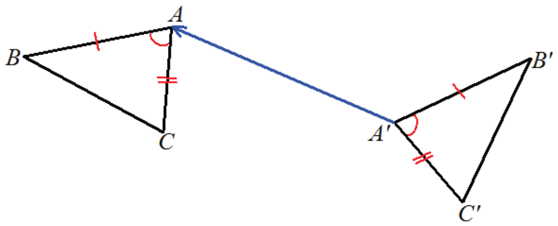
Can any other pair of points be used? Why or why not?

No. We use A' and A because only these angles are congruent by assumption.

State the vector in the picture below that can be used to translate $\triangle A'B'C'$:

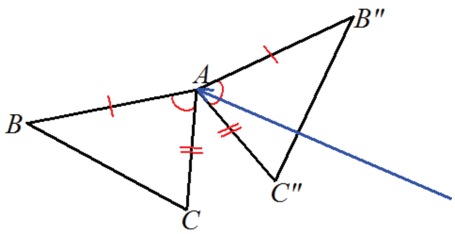
$\overrightarrow{A'A}$

Using a dotted line, draw an intermediate position of $\triangle A'B'C'$ as it moves along the vector:



After the translation (below), $T_{\overrightarrow{A'A}}(\triangle A'B'C')$ shares one vertex with $\triangle ABC$, A. In fact, we can say

$T_{\overrightarrow{A'A}}(\triangle A'B'C') = \triangle AB''C''$.



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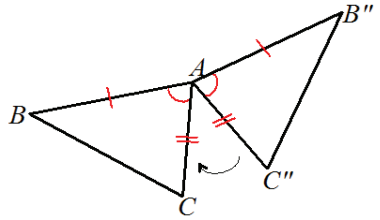
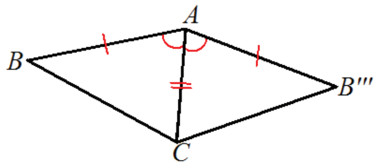
Next, use a clockwise rotation $R_{\angle CAC''}$ to bring the sides AC" to AC (or counterclockwise rotation to bring AB" to AB).



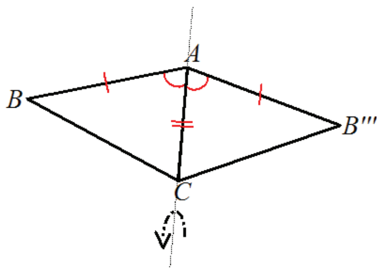
A rotation of appropriate measure will map $\overrightarrow{AC}$" to $\overrightarrow{AC}$, but how can we be sure that vertex C" maps to C? Recall that part of our assumption is that the lengths of sides in question are equal, ensuring that the rotation maps C'' to C ($AC = AC''$, the translation performed is a rigid motion, and thereby did not alter the length when AC' became AC'').



After the rotation, $R_{\angle CAC''}(\triangle AB''C'')$, a total of two vertices are shared with $\triangle ABC$, A and C. Therefore,

$R_{\angle CAC''}(\triangle AB"C") = \triangle AB'''C.$

Finally, if B''' and B are on opposite sides of the line that joins AC, a reflection Λ_{AC} brings B''' to the same side as B.



Since a reflection is a rigid motion and it preserves angle measures, we know that $\angle B'''AC = \angle BAC$ and so $\overrightarrow{AB'''}$ maps to $\overrightarrow{AB}$. However, just because $\overrightarrow{AB'''}$ coincides with $\overrightarrow{AB}$, can we be certain that B''' actually maps to B? We can, because not only are we certain that the rays coincide, but by our assumption, $AB = AB'''$ (our assumption began as $AB = A'B'$, but the translation and rotation have preserved this length now as AB'''). Together, these two pieces of information ensure that the reflection over AC brings B''' to B.

Another way to visually confirm this would be to draw the marks of the perpendicular bisector construction for AC.

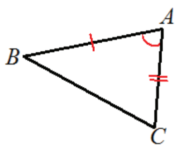
Write the transformations used to correctly notate the congruence (the composition of transformations) that takes $\triangle A'B'C' \cong \triangle ABC$:

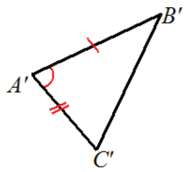
F Translation

G Rotation

H Reflection

$H(G(F(\triangle A'B'C') = \triangle ABC$





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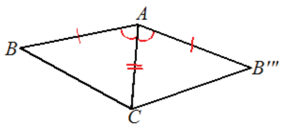
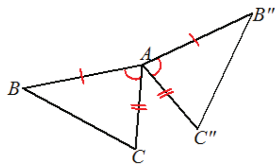
We have now shown a sequence of rigid motions that takes $\triangle A'B'C'$ to $\triangle ABC$ with the use of just three criteria from each triangle: two sides and an included angle. Given any two distinct triangles, we could perform a similar proof. There is another situation, when the triangles are not distinct, where a modified proof will be needed to show that the triangles map onto each other. Examine these below.

Example 1 (5 minutes)

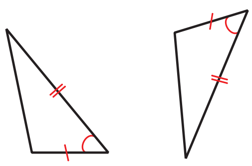
Students try an example based on Class Notes.

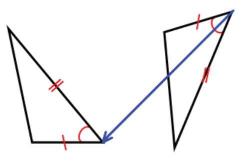
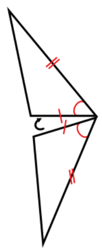
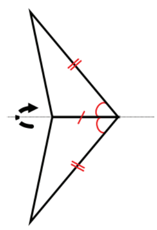
What if we had the SAS criteria for two triangles that were not distinct? Consider the following two cases. How would the transformations needed to demonstrate congruence change?

Case	Diagram	Transformations Needed
Shared Side		***Reflection***
Shared Vertex		***Rotation, Reflection***

Exercises 1-3 (7 minutes)

1. **Given: Triangles with a pair of corresponding sides of equal length and a pair of included angles of equal measure. Sketch and label 3 phases of the sequence of rigid motions that prove the two triangles to be congruent.**



Translation	***Rotation***	***Reflection***
		

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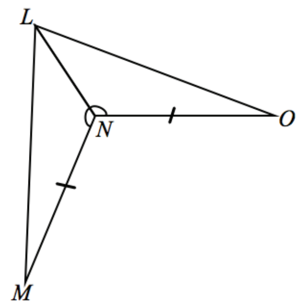
Directions: **Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.**

2. Given: $\angle LMN = \angle LNO, MN = OM$

   Do $\triangle LMN$ and $\triangle LOM$ meet the SAS criteria?



   $\angle LMN = \angle LNO$ — Given

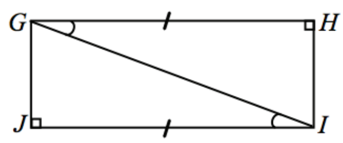
   $MN = OM$ — Given

   $LN = LN$ — Common Side

   The triangles map onto one another with a reflection over LN.

3. Given: $\angle HGI = \angle IHG, HG = JI$

   Do $\triangle HGI$ and $\triangle JIG$ meet the SAS criteria?



   $\angle HGI = \angle JIG$ — Given

   $HG = JI$ — Given

   $GI = GI$ — Common Side

   The triangles map onto one another with a 180˚ rotation about the midpoint of the diagonal.

Exit Ticket (7 minutes)

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Exit Ticket

If two triangles satisfy the SAS criteria, describe the rigid motion(s) that would map one onto the other in the following cases:

1. The two triangles shared a single common vertex?

2. The two triangles were distinct from each other?

3. The two triangles shared a common side?

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Exit Ticket Sample Solutions

1. **The two triangles shared a single common vertex?**
   Rotation, reflection

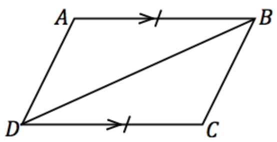
2. **The two triangles were distinct from each other?**
   Translation, rotation, reflection

3. **The two triangles shared a common side?**
   Reflection

Problem Set Sample Solutions

Directions: **Justify whether the triangles meet the SAS congruence criteria; explicitly state which pairs of sides or angles are congruent and why. If the triangles do meet the SAS congruence criteria, describe the rigid motion(s) that would map one triangle onto the other.**

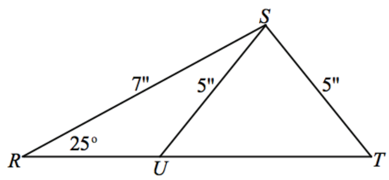
1. **Given:** $AB \parallel CD$, $AB = CD$
   Do $\triangle ABD$ **and** $\triangle CDB$ **meet the SAS criteria?**



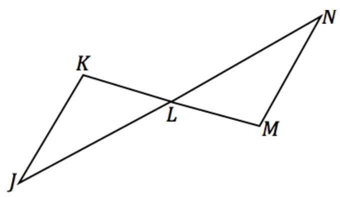
$AB = CD$	*Given*
$BD = BD$	*Shared Side*
$\angle ABC = \angle CDB$	*Alternate Interior Angles*

The triangles map onto one another with a 180˚ rotation about the midpoint of the diagonal.

2. **Given:** $\angle R = 25°$, $RT = 7"$, $SU = 5"$, $ST = 5"$
   Do $\triangle RSU$ **and** $\triangle RST$ **meet the SAS criteria?**
   Not enough information given.



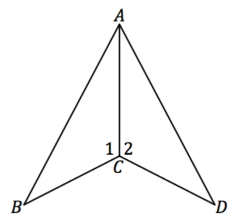
3. **Given:** KM **and** JN **bisect each other.**
   Do $\triangle JKL$ **and** $\triangle NML$ **meet the SAS criteria?**



$\angle KLJ = \angle MLN$	*Vert.* $\angle s$
$KM = LM$	*Bisected Line*
$JL = JN$	*Bisected Line*

The triangles map onto one another with a 180˚ rotation about L.

4. **Given:** $\angle 1 = \angle 2$, $BC = DC$
   Do $\triangle ABC$ **and** $\triangle ADC$ **meet the SAS criteria?**



$\angle 1 = \angle 2$	*Given*
$BC = DC$	*Given*
$AC = AC$	*Common Side*

The triangles map onto one another with a reflection over AC.

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5. Given: AE bisects angle $\angle BCD$, $BC = DC$

   Do $\triangle CAB$ and $\triangle CAD$ meet the SAS criteria?

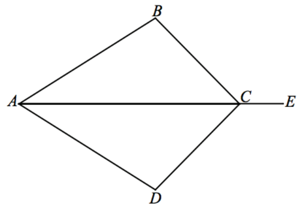
   $\angle BCA = \angle DCA$ *Bisected Angle*

   $BC = DC$ *Given*

   $AC = AC$ *Common Side*

   $\triangle CAD \cong \triangle CAB$ *SAS*

   The triangles map onto one another with a reflection over AC.



6. Given: SU and RT bisect each other

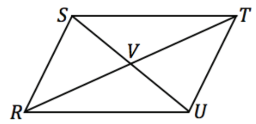
   Do $\triangle SVR$ and $\triangle UVT$ meet the SAS criteria?

   $SV = UV$ *Bisected Line*

   $RV = VT$ *Bisected Line*

   $\angle SVR = \angle UVT$ *Vert. $\angle s$*

   The triangles map onto one another with a 180° rotation about V.



7. Given: $JM = KL$, $JM \perp ML$, $KL \perp ML$

   Do $\triangle JML$ and $\triangle KLM$ meet the SAS criteria?

   $JM = KL$ *Given*

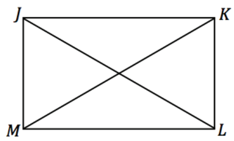
   $JM \perp ML$, $KM \perp ML$ *Given*

   $\angle JML = 90°$, $\angle KLM = 90°$ *Def. of $\perp$*

   $\angle JML = \angle KLM$

   $ML = ML$ *Common Side*

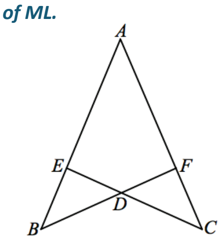
   The triangles map onto one another with a reflection over the perpendicular bisector of ML.



8. Given: $\overline{BF} \perp \overline{AC}$, $\overline{CE} \perp \overline{AB}$

   Do $\triangle BED$ and $\triangle CFD$ meet the SAS criteria?

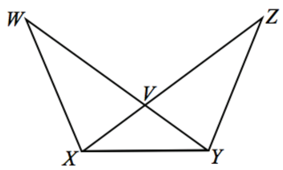
   Not enough information given.



9. Given: $\angle VXY = \angle VYX$

   Do $\triangle VXW$ and $\triangle VYZ$ meet the SAS criteria?

   Not enough information given.



10. Given: $\triangle RST$ is isosceles, $SY = TZ$

    Do $\triangle RSY$ and $\triangle RTZ$ meet the SAS criteria?

    $\triangle RST$ is isosceles *Given*

    $RS = RT$ *Isosceles $\triangle$*

    $\angle RSY = \angle RTZ$ *Isosceles $\triangle$*

    $SY = TZ$ *Given*

    The triangles map onto one another with a reflection over the bisector of $\angle SRT$.