Lesson 3: Arithmetic and Geometric Sequences

Student Outcomes

- Students learn the structure of arithmetic and geometric sequences.

Lesson Notes

In this lesson, students will use their knowledge of sequences developed in Lessons 1 and 2 to differentiate between arithmetic and geometric sequences.

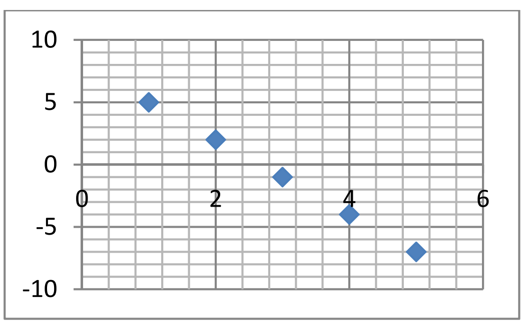
Classwork

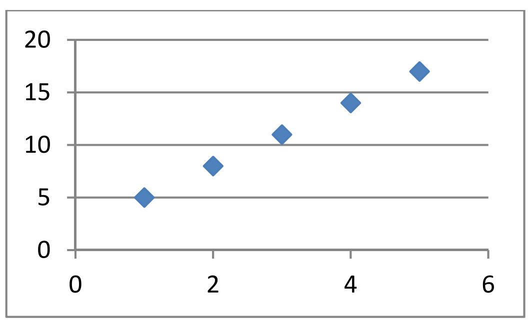
Opening (12 minutes)

In Lessons 1 and 2, students wrote formulas for arithmetic sequences and graphed the terms of the sequence as ordered pairs in the Cartesian plane. Continue to emphasize the point that $A(n)$ does not mean multiply A and n. It means the n^{th} term of the sequence defined by the formula (function) A.

Write the following sequences that were discussed last lesson on the board, and allow students to discuss ideas as a class. Scaffold their discussion as needed with the following:

- Yesterday we talked about the sequences created by Akelia.
  What sequence does $A(n+1) = A(n) - 3$ and $A(1) = 5$ for $n \geq 1$ yield?
  - $5, 2, -1, -4, -7, \ldots$
- What sequence does $A(n+1) = A(n) + 3$ and $A(1) = 5$ for $n \geq 1$ yield?
  - $5, 8, 11, 14, 17, \ldots$
- Johnny suggests that they plot the sequences to help them understand the patterns. What do they discover?





Take time to allow students to discuss the patterns and explain these patterns in words. We want students to notice that the data forms a linear pattern. Ask students what the slope of the line connecting the points would be. Is there a relationship between the slope of that line and the sequence itself? We are trying to get students to see that for both sequences, any two consecutive terms differ by the same number (in the first, it is -3, in the second sequence it is 3).

Likewise, the line containing a sequence has a constant average rate of change over all equal-length intervals. Students might not use the words "average rate of change"; in this case, introduce the term into the discussion yourself to put that concept in their minds for a later lesson.

Ask students to write a definition of an arithmetic sequence in their own words.

> *Scaffolding:*
>
> Is every arithmetic sequence given by a formula of the form $f(n) = mn + b$? Is every formula of the form $f(n) = mn + b$ going to yield an arithmetic sequence?

Arithmetic Sequence: A sequence is called *arithmetic* if there is a real number d such that each term in the sequence is the sum of the previous term and d.

Arithmetic sequences are often called "linear sequences." Have students explain why.

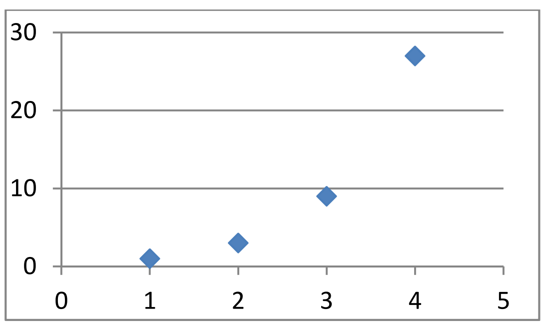
Example 1 (10 minutes)

Now students will look at a geometric sequence. Have students discuss their findings. Some will try to talk about slope, but we want them to see that we no longer have linear data.

What does this mean about our sequence?

- Now look at the sequence $1, 3, 9, 27, \ldots$. What do you notice? How is the sequence different from the sequences above?
  - *We are multiplying each term by* 3 *and not adding or subtracting.*

- Plot this sequence by starting the index of the terms at 1. Is it an arithmetic sequence (i.e., linear sequence)? Is the same number added to each term to get the next?



  - *This sequence is not an arithmetic sequence. The same number is not being added. Each term is being multiplied.*

- A sequence is called geometric if there is a real number r, such that each term in the sequence is a product of the previous term and r. Is this a geometric sequence?
  - *Yes. Each term in the sequence can be found by multiplying or dividing the previous term by 3.*

Ask students to write the definition of a geometric sequence in their own words.

Exercise 1 (15 minutes)

Have students work in groups to identify the following sequences as arithmetic or geometric. Give each group a set of index cards with one sequence on each (all groups should have the same cards). On the board, create two columns with the headings ***Arithmetic*** and ***Geometric***. Students should classify the sequences as arithmetic or geometric, write the reason on the card, write a formula for the sequence, and then tape each card under the correct type of sequence. Some students may have trouble writing geometric sequence formulas, but let them try. Have a class discussion after all groups have placed their cards. Be sure to look at the formulas and make sure students have identified the term they are starting with. You can even revisit the terms recursive and explicit.

Scaffolding:

Teachers can model sequences either by (1) naming the products/differences between terms to help develop the recursive formula and/or (2) naming/showing the operations that lead from the term number to the term to help develop the explicit formula.

Index Card Sequences

1.	$-2, 2, 6, 10, \ldots$	Arithmetic	Add 4	$A(n+1) = A(n) + 4$ for $n \geq 1$ and $A(1) = -2$, or $A(n) = -6 + 4n$ for $n \geq 1$.
2.	$2, 4, 8, 16, \ldots$	Geometric	Multiply by 2	$A(n+1) = A(n) \cdot 2$ for $n \geq 1$ and $\mathrm{A}(1) = 2$, or $A(n) = 2^n$ for $n \geq 1$.
3.	$\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots$	Arithmetic	Add $\frac{1}{2}$	$A(n+1) = A(n) + \frac{1}{2}$ for $n \geq 1$ and $\mathrm{A}(1) = \frac{1}{2}$, or $A(n) = \frac{1}{2}n$ for $n \geq 1$.
4.	$1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$	Geometric	Multiply by $\frac{1}{3}$	$A(n+1) = A(n) \cdot \frac{1}{3}$ for $n \geq 1$ and $\mathrm{A}(1) = 1$, or $A(n) = \left(\frac{1}{3}\right)^{n-1}$ for $n \geq 1$.
5.	$10,\ 1,\ 0.1,\ 0.01,\ 0.001\ldots$	Geometric	Multiply by 0.1 or $\frac{1}{10}$	$A(n+1) = A(n) \cdot (0.1)$ for $n \geq 1$ and $A(1) = 10$, or $A(n) = 10(0.1)^{n-1}$ for $n \geq 1$.
6.	$4, -1, -6, -11, \ldots$	Arithmetic	Add -5 or subtract 5	$A(n+1) = A(n) - 5$ for $n \geq 1$ and $A(1) = 4$, or $A(n) = 9 - 5n$ for $n \geq 1$.

Exercise 2 (Optionally, if time allows)

Exercise 2

Think of a real-world example of an arithmetic or geometric sequence? Describe it and write its formula.

Answers will vary.

Exercise 3 (Optionally, if time allows)

Exercise 3

If we fold a rectangular piece of paper in half multiple times and count the number of rectangles created, what type of sequence are we creating? Can you write the formula?

We are creating a geometric sequence because each time we fold we double the number of rectangles. $R(n) = 2^n$, where n is the number of times we have folded the paper.

Closing (3 minutes)

- Explain the difference between arithmetic and geometric sequences.

Students may say something like, "*Arithmetic sequences grow additively; geometric sequences grow multiplicatively*." However, such a statement is only partially true. The sequence may be decaying or decreasing instead of growing. Prompt students to get more specific so as to include both increasing and decreasing sequences in their description of the differences.

- Why are arithmetic sequences sometimes called linear sequences?
  - *If we graph an arithmetic sequence as points $(n, A(n))$ on the Cartesian plane, the points lie on a line. That's why we sometimes call arithmetic sequences linear sequences.*

Lesson Summary

Two types of sequences were studied:

Arithmetic Sequence: A sequence is called *arithmetic* if there is a real number d such that each term in the sequence is the sum of the previous term and d.

Geometric Sequence: A sequence is called geometric if there is a real number r such that each term in the sequence is a product of the previous term and r.

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 3: Arithmetic and Geometric Sequences

Exit Ticket

1. Write the first 3 terms in the following sequences. Identify them as arithmetic or geometric.
   a. $A(n+1) = A(n) - 5$ for $n \geq 1$ and $A(1) = 9$.

   b. $A(n+1) = \frac{1}{2}A(n)$ for $n \geq 1$ and $A(1) = 4$.

   c. $A(n+1) = A(n) \div 10$ for $n \geq 1$ and $A(1) = 10$.

2. Identify each sequence as arithmetic or geometric. Explain your answer, and write an explicit formula for the sequence.
   a. $14, 11, 8, 5, \ldots$

   b. $2, 10, 50, 250, \ldots$

   c. $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \ldots$

Exit Ticket Sample Solutions

1. Write the first 3 terms in the following sequences. Identify them as arithmetic or geometric.

   a. $A(n+1) = A(n) - 5$ for $n \geq 1$ and $A(1) = 9$.

   $9, 4, -1$ *Arithmetic*

   b. $A(n+1) = \frac{1}{2}A(n)$ for $n \geq 1$ and $A(1) = 4$.

   $4, 2, 1$ *Geometric*

   c. $A(n+1) = A(n) \div 10$ for $n \geq 1$ and $A(1) = 10$.

   $10, 1, \frac{1}{10}$ *or* $10, 1, 0.1$ *Geometric*

2. Identify each sequence as arithmetic or geometric and explain your answer. Write an explicit formula for the sequence.

   a. $14, 11, 8, 5, \ldots$ *Arithmetic* -3 *pattern* $17 - 3\mathrm{n}$*, where* n *starts at* 1

   b. $2, 10, 50, 250, \ldots$ *Geometric* $\times 5$ *pattern* $2(5^{\mathrm{n}-1})$*, where* n *starts at* 1

   c. $-\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, -\frac{7}{2}, \ldots$ *Arithmetic* -1 *pattern* $\frac{1}{2} - \mathrm{n}$*, where* n *starts at* 1

Problem Set Sample Solutions

For Problems 1–4, list the first five terms of each sequence, and identify them as arithmetic or geometric.

1. $A(n+1) = A(n) + 4$ for $n \geq 1$ and $A(1) = -2$

   $-2, 2, 6, 10, 14$ *Arithmetic*

2. $A(n+1) = \frac{1}{4} \cdot A(n)$ for $n \geq 1$ and $A(1) = 8$

   $8, 2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}$ *Geometric*

3. $A(n+1) = A(n) - 19$ for $n \geq 1$ and $A(1) = -6$

   $-6, -25, -44, -63, -82$ *Arithmetic*

4. $A(n+1) = \frac{2}{3}A(n)$ for $n \geq 1$ and $A(1) = 6$

   $6, 4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}$ *Geometric*

For Problems 5–8, identify the sequence as arithmetic or geometric, and write a recursive formula for the sequence. Be sure to identify your starting value.

5. $14, 21, 28, 35, \ldots$

   $f(n+1) = f(n) + 7$ *for* $n \geq 1$ *and* $f(1) = 14$ *Arithmetic*

6. $4, 40, 400, 4000, \ldots$

   $f(n+1) = 10f(n)$ *for* $n \geq 1$ *and* $f(1) = 4$ *Geometric*

7. $49,\ 7,\ \frac{1}{7},\frac{1}{49},\ldots$

$f(n+1)=\frac{1}{7}f(n)$ for $n\geq 1$ and $f(1)=49$ *Geometric*

8. $-101,\ -91,\ -81,\ -71,\ldots$

$f(n+1)=f(n)+10$ for $n\geq 1$ and $f(1)=-101$ *Arithmetic*

9. **The local football team won the championship several years ago, and since then, ticket prices have been increasing $\$20$ per year. The year they won the championship, tickets were $\$50$. Write a recursive formula for a sequence that will model ticket prices. Is the sequence arithmetic or geometric?**

$T(n)=50+20n$, where n is the number of years since they won the championship; $n\geq 1$ ($n\geq 0$ is also acceptable). Arithmetic.

Or,

$T(n+1)=T(n)+20$, where n is the number of years since the year they won the championship; $n\geq 1$ and $T(1)=70$ ($n\geq 0$ and $T(0)=50$ is also acceptable). Arithmetic.

10. **A radioactive substance decreases in the amount of grams by one third each year. If the starting amount of the substance in a rock is $1,452$ g, write a recursive formula for a sequence that models the amount of the substance left after the end of each year. Is the sequence arithmetic or geometric?**

$A(n+1)=\frac{2}{3}A(n)$ or $A(n+1)=2A(n)\div 3$, where n is the number of years since the measurement started, $A(0)=1,452$

Geometric

Since the problem asked how much radioactive substance was left, students must take the original amount, divide by 3 or multiply by $\frac{1}{3}$, then subtract that portion from the original amount. An easier way to do this is to just multiply by the amount remaining. If $\frac{1}{3}$ is eliminated, $\frac{2}{3}$ remains.

11. **Find an explicit form $f(n)$ for each of the following arithmetic sequences (assume a is some real number and x is some real number):**

   a. $-34,\ -22,\ -10,\ 2,\ldots$

   $f(n)=-34+12(n-1)=12n-46$, where $n\geq 1$

   b. $\frac{1}{5},\ \frac{1}{10},\ 0,\ -\frac{1}{10},\ldots$

   $f(n)=\frac{1}{5}-\frac{1}{10}(n-1)=\frac{3}{10}-\frac{1}{10}n$, where $n\geq 1$

   c. $x+4,\ x+8,\ x+12,\ x+16,\ldots$

   $f(n)=x+4+4(n-1)=x+4n$, where $n\geq 1$

   d. $a,\ 2a+1,\ 3a+2,\ 4a+3,\ldots$

   $f(n)=a+(a+1)(n-1)=a+an-a+n-1=an+n-1$, where $n\geq 1$

12. Consider the arithmetic sequence $13, 24, 35, ...$
    a. Find an explicit form for the sequence in terms of n.

       $f(n) = 13 + 11(n-1) = 11n + 2$, *where* $n \geq 1$

    b. Find the 40^{th} term.

       $f(40) = 442$

    c. If the n^{th} term is 299, find the value of n.

       $299 = 11n + 2 \rightarrow n = 27$

13. If $-2, a, b, c, 14$ forms an arithmetic sequence, find the values of a, b, and c.

    $14 = -2 + (5-1)d$ $\quad$ $a = -2 + 4 = 2$

    $16 = 4d$ $\quad$ $b = 2 + 4 = 6$

    $d = 4$ $\quad$ $c = 6 + 4 = 10$

14. $3 + x, 9 + 3x, 13 + 4x, ...$ is an arithmetic sequence for some real number x.
    a. Find the value of x.

       The difference between term 1 and term 2 can be expressed as $(9 + 3x) - (3 + x) = 6 + 2x$

       The difference between term 2 and term 3 can be expressed as $(13 + 4x) - (9 + 3x) = 4 + x$

       Since the sequence is known to be arithmetic, the difference between term 1 and term 2 must be equal to the difference between term 2 and term 3. Thus, $6 + 2x = 4 + x$*, and* $x = -2$*; therefore, the sequence is* $1, 3, 5, \ldots$

    b. Find the 10^{th} term of the sequence.

       $f(n) = 1 + 2(n-1) = 2n - 1$, *where* $n \geq 1$

       $f(10) = 19$

15. Find an explicit form $f(n)$ of the arithmetic sequence where the 2^{nd} term is 25 and the sum of the 3^{rd} term and 4^{th} term is 86.

    $a, 25, b, c$ $\quad$ $25 = a + (2-1)d$, $25 = a + d$ $\quad$ $b = 25 + d$, $b = a + 2d$ $\quad$ $c = 25 + 2d$, $c = a + 3d$

    $b + c = (a + 2d) + (a + 3d) = 2a + 5d = 86$

    $a + d = 25$

    Solving this system: $d = 12$, $a = 13$, *so* $f(n) = 13 + 12(n-1)$, *where* $n \geq 1$.

    $b = 13 + 2(12) = 37$ $\quad$ $c = 13 + 3(12) = 49$

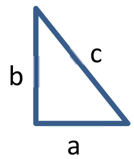
    OR

    $b + c = (25 + d) + (25 + 2d) = 50 + 3d = 86 \rightarrow d = 12$

    $25 = a + 12 \rightarrow a = 13$ $\quad$ $b = 25 + 12 \rightarrow b = 37$ $\quad$ $c = 25 + 2(12) \rightarrow c = 49$

    So, $f(n) = 13 + 12(n-1)$.

16. (Challenge) In the right triangle figure below, the lengths of the sides a cm, b cm, c cm of the right triangle form a finite arithmetic sequence. If the perimeter of the triangle is 18 cm, find the values of a, b, and c.



$a + b + c = 18$ $\quad$ $b = a + d$ $\quad$ $c = a + 2d$

$$a + (a+d) + (a + 2d) = 18$$
$$3a + 3d = 18$$
$$a + d = 6 = b$$

Now, do not forget that it is a right triangle, so the Pythagorean Theorem must apply: $a^2 + b^2 = c^2$.

Since we know that $b = 6$*, the perimeter equation becomes* $a + c = 12$ *once* b *is substituted. So, substituting* $c = 12 - a$ *and* b *into the Pythagorean Theorem equation gives us* $a^2 + 36 = (12 - a)^2$*, which gives us the following answer:* $a = \frac{9}{2}, b = 6, c = \frac{15}{2}$.

17. Find the common ratio and an explicit form in each of the following geometric sequences:

a. $4, 12, 36, 108, \ldots$

$r = 3$ $\quad$ $f(n) = 4(3)^{n-1}$, *where* $n \geq 1$

b. $162, 108, 72, 48, \ldots$

$r = \frac{108}{162} = \frac{2}{3}$ $\quad$ $f(n) = 162\left(\frac{2}{3}\right)^{n-1}$, *where* $n \geq 1$

c. $\frac{4}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \ldots$

$r = \frac{1}{2}$ $\quad$ $f(n) = \left(\frac{4}{3}\right)\left(\frac{1}{2}\right)^{n-1} = \left(\frac{4}{3}\right)(2)^{1-n}$, *where* $n \geq 1$

d. $xz, x^2z^3, x^3z^5, x^4z^7, \ldots$

$r = xz^2$ $\quad$ $f(n) = xz(xz^2)^{n-1}$, *where* $n \geq 1$

18. The first term in a geometric sequence is 54, and the 5^{th} term is $\frac{2}{3}$. Find an explicit form for the geometric sequence.

$\frac{2}{3} = 54(r)^4$ $\quad$ $\frac{1}{81} = r^4$ $\quad$ $r = \frac{1}{3}$ *or* $-\frac{1}{3}$ $\quad$ $f(n) = 54\left(\frac{1}{3}\right)^{n-1}$

19. If $2, a, b, -54$ forms a geometric sequence, find the values of a and b.

$a = 2r,$ $\quad$ $b = 2(r)^2$

$$-54 = 2(r)^3$$
$$-27 = r^3$$
$$-3 = r \quad \textit{so } a = -6 \textit{ and } b = 18$$

20. Find the explicit form $f(n)$ of a geometric sequence if $f(3) - f(1) = 48$ and $\frac{f(3)}{f(1)} = 9$.

$$f(3) = f(1)(r)^2$$
$$\frac{f(3)}{f(1)} = r^2 = 9$$
$$r = 3 \textit{ or } -3$$

$$f(1)r^2 - f(1) = 48$$
$$f(1)(r^2 - 1) = 48$$
$$f(1)(8) = 48$$
$$f(1) = 6$$

$f(n) = 6(3)^{n-1}$, *where* $n \geq 1$ *or* $f(n) = 6(-3)^{n-1}$, *where* $n \geq 1$.