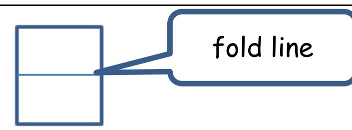
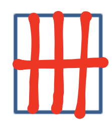
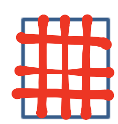
Lesson 4: Geometric Sequences

Opening Exercise

You will need: one sheet of paper

1. A. If we fold a rectangular piece of paper in half multiple times and count the number of rectangles created, what type of sequence are we creating?

B. Use the table below to record the number of rectangles in the paper folding experiment.

Term number (number of folds)	Sequence (number of rectangles)	Sketch of Unfolded Paper
1	2	
2	$4 = 2 \cdot 2$	
3	$8 = 2 \cdot 2 \cdot 2$	
4	$16 = 2 \cdot 2 \cdot 2 \cdot 2$	
5	32	
6	64	
n	2^n	

C. Write a recursive formula for this sequence. Write an explicit formula for this sequence.

Recursive: $f(n+1) = 2 \cdot f(n)$, $n \geq 1$, $f(1) = 2$

Explicit: $f(n) = 2^n$, $n \geq 1$

The sequence in the Opening Exercise is an example of a *geometric sequence*. Unlike arithmetic sequences, there is no common difference. Geometric sequences have a *common ratio*. $= r$

> A geometric sequence is formed by multiplying each term, after the first term, by a non-zero constant.
>
> * The amount by which we multiply each time is called the common ratio of the sequence.

The sequence in the Opening Exercise was fairly simple and you saw something similar to it in Lesson 1. Let's look at a more challenging sequences and determine the pattern for the explicit formula.

2. Consider the sequence: 2, 6, 18, … $(\times 3)$

   A. Write the next three terms.

   2, 6, 18, 54, 162, 486

   B. Why isn't this an arithmetic sequence?

   Because you are not adding/sub.

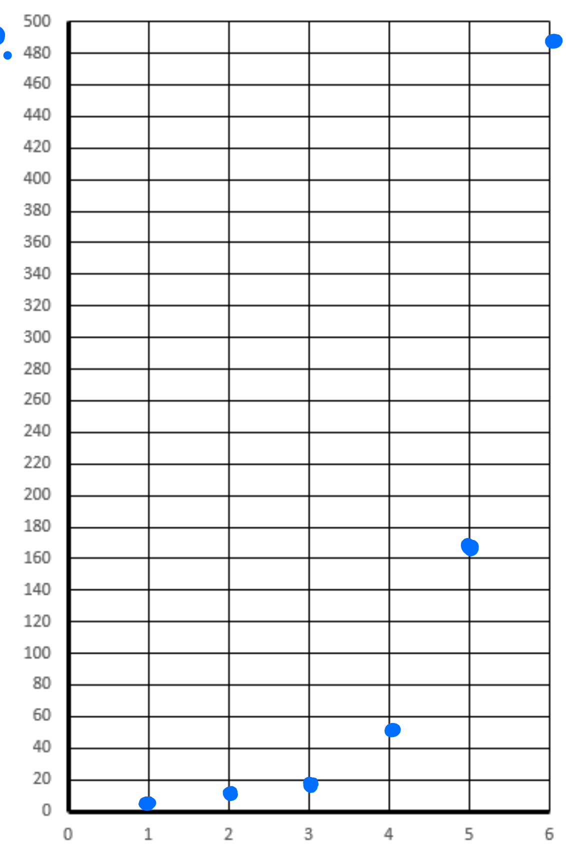
   C. What is the pattern? What is the common ratio?

   Mult. by 3 $= r$

   D. Fill in the table below. Then plot the points on the graph.

n	1	2	3	4	5	6
$f(n)$	2	6	18	54	162	486
Pattern	2	2(3)	2(3)(3)	$2(3)^3$	$2\cdot3^4$	$2\cdot3^5$

   $2(3)^0$ $2(3)^1$ $2(3)^2$



   E. What type of graph do the points make?

   Exponential pattern

   F. Write the explicit formula for this sequence based on the pattern in the table.

   $f(n) = 2(3)^{n-1}, n \geq 1$

3. We can now say that . . .

   arithmetic sequences are modeled by linear functions

   geometric sequences are modeled by Exponential functions

4. To find the general term, $f(n)$, of a geometric sequence you need the first term, $f(1)$, and the common ratio, r.

*

$$f(n) = f(1) \cdot r^{n-1}$$

$a_1 \cdot r^{n-1}$

$$r = \frac{a_{n+1}}{a_n}$$

5. Find the common ratio for the following geometric sequences:

   A. 1, 5, 25, 125, 625, …

   $r = 5$

   B. 9, -3, 1, $\frac{-1}{3}$, $\frac{1}{9}$, …

   $r = -\frac{1}{3}$

6. Write the first six terms of the geometric sequence with first term 6 and common ratio -2.

   6, -12, 24, -48, 96, -192

7. Write a formula for the n^{th} term of each sequence below. Then find $f(8)$.

   A. 12, 6, 3, $\frac{3}{2}$, ……..

   $f(n) = f(1) \cdot r^{n-1}$

   $f(n) = 12 \cdot \left(\frac{1}{2}\right)^{n-1}$

   $f(8) = 12 \cdot \left(\frac{1}{2}\right)^{7}$

   $= \cancel{12}^{3} \cdot \frac{1}{\cancel{128}_{32}} = \frac{3}{32}$

   B. 3, 6, 12, 24, 48, ……

   $f(n) = 3 \cdot (2)^{n-1}$

   $f(8) = 3(2)^7$

   $= 3(128)$

   $= 384$

Lesson Summary

A *geometric sequence* is a sequence in which each term after the first is obtained by multiplying the preceding term by a fixed nonzero constant. The amount by which we multiply each time is called the *common ratio* of the sequence.

Geometric sequences can be modeled by exponential functions.

The common ratio, r, is found by dividing any term after the first term by the term that directly precedes it.

General Term of a Geometric Sequence

The nth term (the general term) of a geometric sequence with first term $f(1)$ and common ratio r is

$$f(n) = f(1)r^{n-1}$$

Example: Find $f(8)$ of the geometric sequence when $f(1) = -4$ and the common ratio is -2.

$$f(n) = -4 \bullet (-2)^{n-1}$$

$$\begin{aligned} f(8) &= -4 \bullet (-2)^{8-1} \\ &= -4 \bullet (-2)^{7} \\ &= -4 \bullet (-128) \\ &= 512 \end{aligned}$$

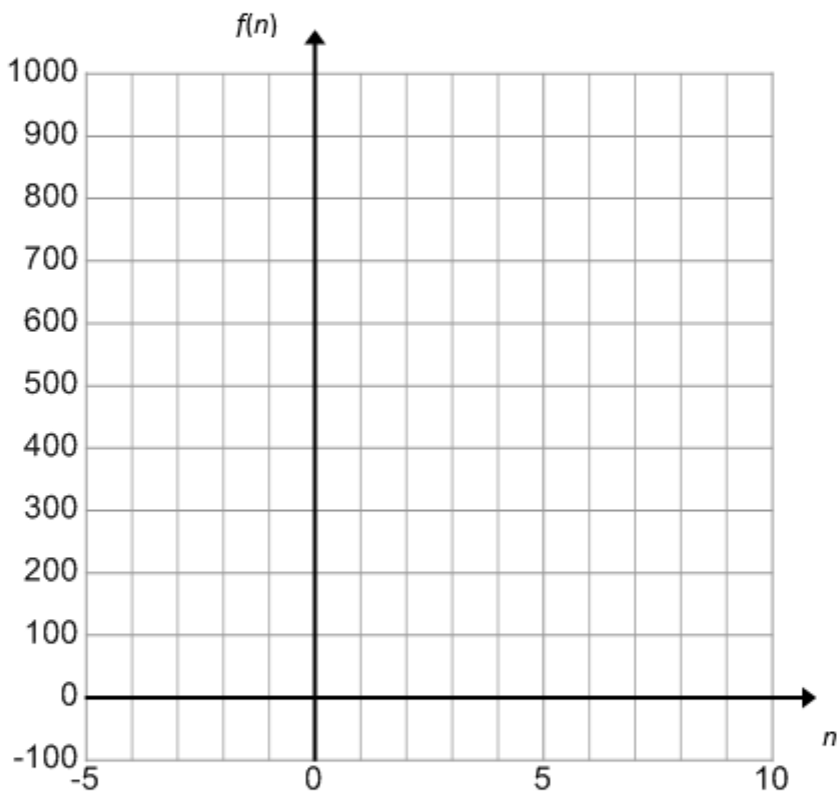
Homework Problem Set

1. Consider a sequence that follows a times 5 pattern: $1, 5, 25, 125, \ldots$.

   a. Write a formula for the n^{th} term of the sequence. Be sure to specify what value of n your formula starts with.

   b. Using the formula, find the 10^{th} term of the sequence.

   c. Graph the terms of the sequence as ordered pairs $(n, f(n))$ on a coordinate plane.



2. A radioactive substance decreases in the amount of grams by one-third each year. If the starting amount of the substance in a rock is $1{,}452$ g, write a recursive formula and an explicit formula for a sequence that models the amount of the substance left after the end of each year.

3. Write the first five terms of each geometric sequence.

A. $f(1) = 20$, $r = \frac{1}{2}$

B. $f(1) = 4$, $r = 3$

Use the formula for the general term (n^{th} term) of a geometric sequence to find the indicated term of each sequence with the given first term, $f(1)$, and common ratio, r.

4. Find $f(8)$ when $f(1) = 6$, $r = 2$.

5. Find $f(12)$ when $f(1) = 5$, $r = -2$.

6. Find $f(22)$ when $f(1) = 1000$, $r = -\frac{1}{2}$.

7. Find $f(15)$ when $f(1) = 9000$, $r = -\frac{1}{3}$

Write a formula for the n^{th} term of each geometric sequence. Then use the formula to find $f(7)$.

8. 3, 12, 48, 192, ……

9. $18, 6, 2, \frac{2}{3}, \ldots\ldots$

Find the first 5 terms of the following function.

10. $f(n) = -f(n-1) + 2;\ \ f(1) = -3;\ n \geq 2$

11. $f(n) = \frac{f(n-1)}{3} - 1;\ \ f(1) = 3;\ n \geq 2$

Write a formula for the general term (the n^{th} term) of each geometric sequence. Then use the formula for $f(n)$ to find $f(9)$.

12. $5, -1, \frac{1}{5}, -\frac{1}{25}, \ldots.$

13. $0.07, 0.007, 0.0007, 0.00007, \ldots$

14. A mine worker discovers an ore sample containing 500 mg of radioactive material. It is discovered that the radioactive material has a half life of 1 day. (This means that each day, half of the material decays, and only half is left.) Find the amount of radioactive material in the sample at the beginning of the 7^{th} day.

15. A culture of bacteria doubles every 2 hours. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours?

16. You complain that the hot tub in your hotel suite is not hot enough. The hotel tells you that they will increase the temperature by 10% each hour. If the current temperature of the hot tub is 75º F, what will be the temperature of the hot tub after 3 hours, to the nearest tenth of a degree?

CHALLENGE PROBLEMS

17. Find the common ratio and an explicit form in each of the following geometric sequences.

    a. $4, 12, 36, 108, \ldots$

    b. $162, 108, 72, 48, \ldots$

    c. $\frac{4}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{6}, \ldots$

    d. $xz, x^2z^3, x^3z^5, x^4z^7, \ldots$

18. The first term in a geometric sequence is 54, and the 5^{th} term is $\frac{2}{3}$. Find an explicit form for the geometric sequence.

19. If $2, a, b, -54$ forms a geometric sequence, find the values of a and b.

20. Find the explicit form $f(n)$ of a geometric sequence if $f(3) - f(1) = 48$ and $\frac{f(3)}{f(1)} = 9$.

$n = 1$

$f(1) \to a$

- Sequence
  - Arithmetic
    - +, − by d (common diff)
    - explicit: $f(n) = f(1) + d(n-1)$, $n \geq 1$
    - recursive: $f(n+1) = f(n) + d$, $n \geq 1$, $f(1) = ?$
  - Geometric
    - mult. by r (common ratio)
    - explicit: $f(n) = f(1) \cdot r^{n-1}$, $n \geq 1$
    - recursive: $f(n+1) = r \cdot f(n)$, $n \geq 1$, $f(1) = ?$
  - Neither
    - i.e. Fibonacci sequence.