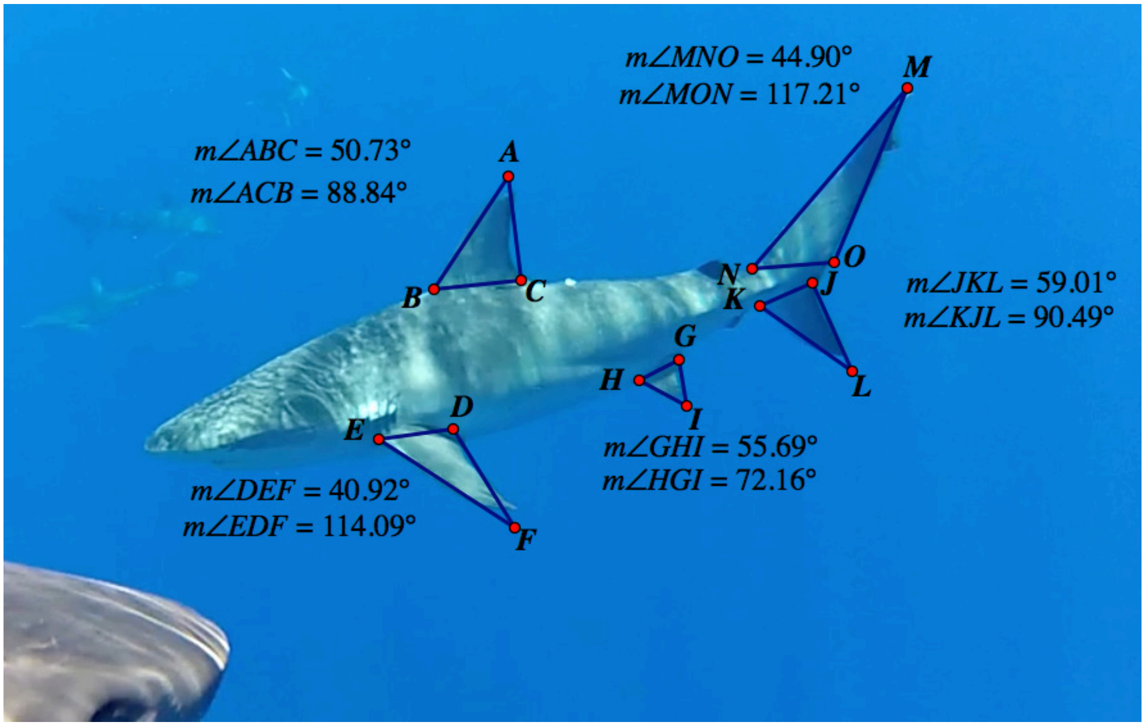
Lesson 9 Geometry, Part I of III



You don't have to be an expert shark scientist or mathematician to notice that the Galapagos sharks pictured here have lots of triangle-shaped patterns on their bodies! Looking closer, what else do you observe? Did you notice that, on all the triangles drawn, the more anterior angles (closer to the head, like angles B, N, K, H and E) are more acute, while the more posterior angles (closer to the tail, like angles C, O, J, G, and D) are more obtuse? Observing patterns like these will direct thoughtful students to consider that the patterns are clues informing us the fins were designed for specific reasons (Lesson 11C). The #1 use of mathematics is as a super-important tool for the study of God's creation, whether that involves enjoying the simple beauty of a shark's triangular patterns, or considering further how those patterns might be important to the shark's swimming movements. *Photo by Dr. Shormann*

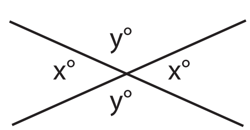
Review: Shormann Algebra 1, Lessons 9, 11, 12, 13, 68

Rules and Definitions (Many of the following rules and definitions come from Vol. 1, Book 1 of Euclid's *Elements*)

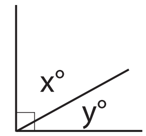
Rules

Angles:

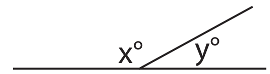
- All right angles equal 90°
- All straight angles equal 180° (two right angles)
- When two line segments intersect, the vertical (opposite) angles are equal.



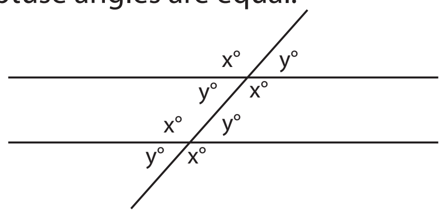
- The sum of complementary angles equals 90°. $x + y = 90°$



-The sum of supplementary angles equals 180°. $x + y = 180°$



- When a transversal intersects two or more parallel lines, all acute angles are equal, and all obtuse angles are equal.



Triangles

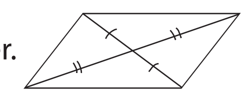
- In any triangle, the sum of the measures of the three angles equals 180°.
- In any triangle, the angles opposite sides of equal lengths have equal measures, and vice-versa.

Polygons

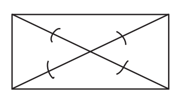
- For any polygon, the number of sides equals the number of vertices.
- For any convex polygon of N sides, the sum of the measures of the interior angles equals (N - 2)180°.
- For any convex polygon of N sides, the sum of the measures of the exterior angles equals 360°.

Diagonals and Quadrilaterals

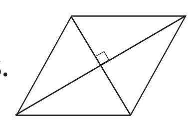
- The diagonals of parallelograms bisect each other.



- The diagonals of rectangles are equal.



- The diagonals of rhombuses are perpendicular bisectors.



- Triangle Congruency Theorems*

Side-angle-side (SAS): If two sides and the included angle in one triangle have the same measures as two sides and the included angle in a second triangle, the triangles are congruent. It is based off of Euclid's proposition 4.

Side-side-side (SSS): If the lengths of the sides in one triangle are equal to the lengths of the sides in a second triangle, the triangles are congruent.
It is based off Euclid's Proposition 8.

Angle-angle-angle-side (AAAS): If the angles in one triangle equal the angles in a second triangle, the triangles are similar. If it is also known that at least one pair of sides opposite the same angle measure are congruent, then the two triangles are also congruent. Also known as ASA (because if two angles are congruent, then so are the third angles), it is based off of Euclid's Proposition 26.

Hypotenuse-Leg (HL): If the lengths of the hypotenuse and a leg in one right triangle equal the lengths of the hypotenuse and a leg in a second right triangle, the right triangles are congruent.

*You may also see some of the previous four theorems described as *postulates* elsewhere. Because Euclid's propositions (Lesson 10)describe most of these, we will refer to them as *theorems*, which is another word for *proposition*.

Third angle theorem: If two angles in one triangle are congruent to two angles in another triangle, the third angles are also congruent.

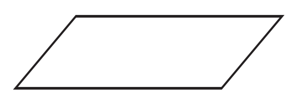
Definitions

- point: that which has no part. Its location is represented by a dot.

- line: A widthless length. Its location is represented on paper by using a pencil and straight edge.

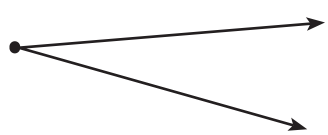
- **line segment**: A line with a start point and end point.

- **plane**: A flat surface having length and width only.
Also called a *Euclidean plane*.

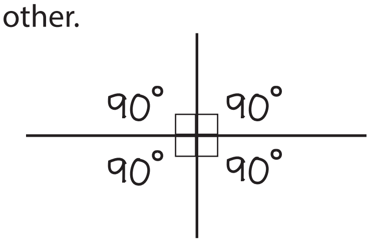


- ray: a line with a starting point but no end point.

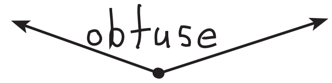
- **angle**: Two rays in the same plane that share a common starting point, and do not overlap.



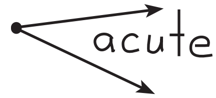
- right angle: Formed when two lines or lines segments intersect, forming 4 adjacent angles that are equal to each other. Lines or line segments forming right angles are **perpendicular** to each other.



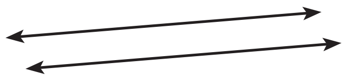
- obtuse angle: An angle greater than a right angle.



- acute angle: An angle less than a right angle.



- **parallel**: Two lines are considered *parallel* if they never intersect.



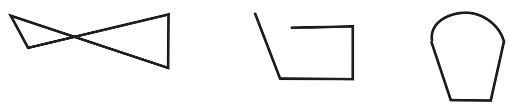
- transversal: A line that intersects two or more parallel lines, and is not perpendicular to those lines.

- **circle:** A plane figure contained by one line called the **circumference**, such that all the straight line segments falling upon it are equal to each other. All line segments have one point in common, called the **center**.

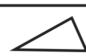
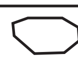
- **diameter:** Any line segment drawn through the center of a circle and terminated in both directions by the circumference. The **radius** equals half the diameter.

- **polygon:** Simple, closed, coplanar geometric figures whose sides are straight lines.

not polygons:



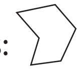
polygons:

Name	Irregular polygon	Regular polygon
Triangle		
Quadrilateral		
Pentagon		
Hexagon		
Heptagon		
Octagon		

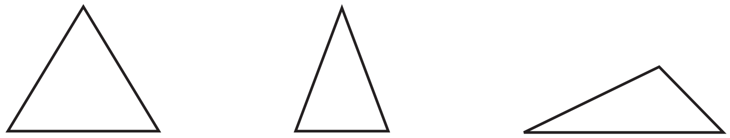
- **regular polygon**: A polygon with sides of equal measure, and interior angles of equal measure. If a polygon is not *regular*, it is *irregular*.

- vertex: Where two polygon sides, or two angle rays, meet. Plural is **vertices**.

- **convex polygon:** A polygon that has no indentations.

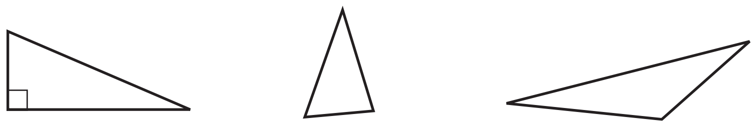
- **concave polygon:** A polygon that has at least one indentation, like this: 

- **triangles by sides:** An **equilateral triangle** has all sides congruent (equal), an **isosceles triangle** has two sides congruent, and a **scalene triangle** has no sides congruent.



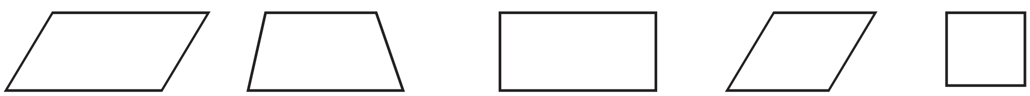
equilateral isosceles scalene

- **triangles by angles**: A **right triangle** has one right angle, an **obtuse triangle** has one angle greater than a right angle, and an **acute triangle** has three acute angles.



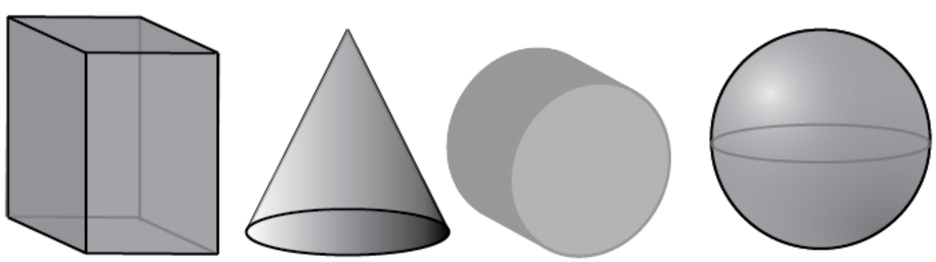
right acute obtuse

- **quadrilaterals**: A **parallelogram** has two pairs of parallel sides; a **trapezoid** has exactly one pair of parallel sides; a **rectangle** is a parallelogram with four right angles; a **rhombus** is an equilateral parallelogram; a **square** is a rhombus with four right angles.



parallelogram trapezoid rectangle rhombus square

- **solid**: Three dimensional figures such as a **polyhedron** formed by four or more polygons that intersect only at their edges; a **cone**, with a circular base and a lateral surface that comes to a point; a **cylinder**, with two parallel circular bases connected by a lateral surface; or a **sphere**, which is the set of points a given distance from a given point called the center.



polyhedron (rect. solid) cone cylinder sphere

9A Geometry Fundamentals

Geometry means "earth measure" (geo="earth", metry="measure"). And, as you can see, it has lots of fairly simple Rules and Definitions. However, if you don't remember the rules, or at least where to find them, geometry can become frustrating in a hurry! So, put an effort into memorizing the rules and definitions. Practice Set problems from Lesson 9A are designed to keep your memory fresh (or refreshed). Some of the Rule and Definitions you may only need rarely, like when you do a geometry proof (Lesson 10). Let's do some examples now.

Example 9.1 Points, lines, angles and planes. Define each of the following.

a) A widthless length.
b) Two rays in the same plane that share a common starting point.
c) That which has no part.
d) A flat surface having length and width only.
e) An angle greater than a right triangle.

solution: A quick survey of the Lesson 9 Definitions will reveal the answers. Some may not be defined in the exact same way, but you should be able to deduce them from the information given:

a) line
b) angle
c) point
d) plane
e) obtuse angle

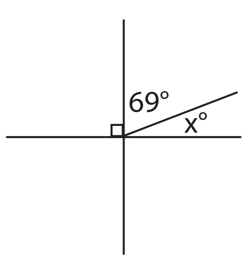
Example 9.2 Circles, polygons, and solids. Define each of the following.

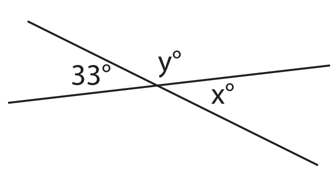
a) Simple, closed, coplanar figures whose sides are straight lines.
b) The set of points a given distance from a given point called the center.
c) Two sides of a polgon meet here.
d) A plane figure contained by one line called the circumference.
e) A name for three-dimensional geometric shapes.

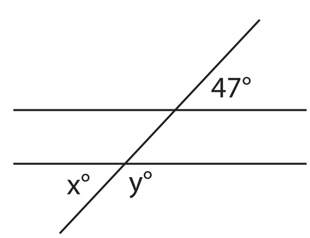
solution: Again, these are simple problems that simply require you to get familiar with the Lesson 9 rules.

a) polygon
b) sphere
c) vertex
d) circle
e) solid

Example 9.3 In each of the following, solve for *x*, or *x* and *y*.

a) 

b) 

c) 

solution: Solving problems like these simply involves recognizing when you have supplementary, complementary, vertical, and/or transversal angles. It's also super-important for you to see how algebra and geometry relate to each other here. You have to use algebra to solve a geometry problem! This should not be a surprise to you to see different math topic in use at the same time. Unity and diversity like this is, after all, an attribute of God (Lesson 1).

a) The right angle box is your clue that 69° and x° are complementary angles. Therefore,

$x + 69 = 90$
$x = 90 - 69 =$ **21°**

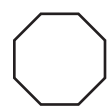
b) Notice that x and 33° are vertical angles, so they are equal. And x and y are supplementary, so they total to 180°.

$x =$ **33°**
$x + y = 180$
$33 + y = 180$
$y = 180 - 33 =$ **147°**

c) Follow the rules for transversals and supplementary angles:

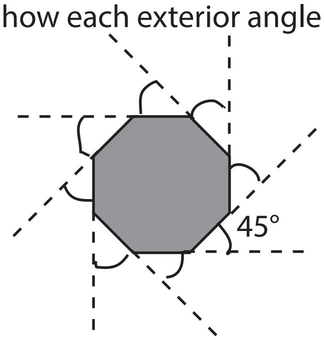
$x =$ **47°**
$x + y = 180$
$47 + y = 180$
$y =$ **133°**

Example 9.4 Find the sum of the interior and exterior angles of the following regular polygon. What do the interior angles equal?



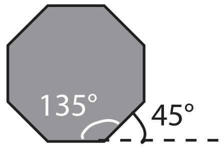
solution: Recognize that the polygon is 1) convex, and 2) a regular octagon (8 equal sides, 8 equal interior angles). Knowing this, we can apply some Lesson 9 polygon rules:

Sum of exterior angles = **360°**. This may seem strange that all polygons have the same sum of exterior angles, but it's true! Looking more closely at the anatomy of the regular octagon shown, we can see how each exterior angle equals 45°, and 45 x 8 = 360.



Sum of interior angles = $(N - 2)180 =$
$(8 - 2)180 = 6(180) =$ **1080°**

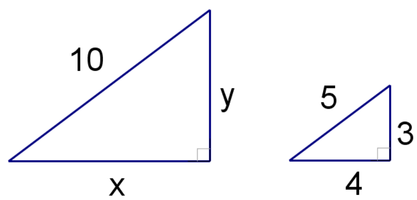
The interior angles equal 1080÷8= **135°**. Notice also how the interior and exterior angles are supplements of each other, which happens whether the polygon is regular or irregular:



9B Triangle Similarity

Similarity: In Lesson 4, you learned about proportion and scale. In geometry, when one shape is a scale model (proportional to) another shape, we say the shapes are *similar*. For example, an interesting triangle is a 3-4-5 right triangle, that just happens to have lengths of 3, 4 and 5 units long. Let's use a 3-4-5 triangle to find some unknown sides of a similar triangle.

Example 9.5 The triangles shown are similar. Find x and y.



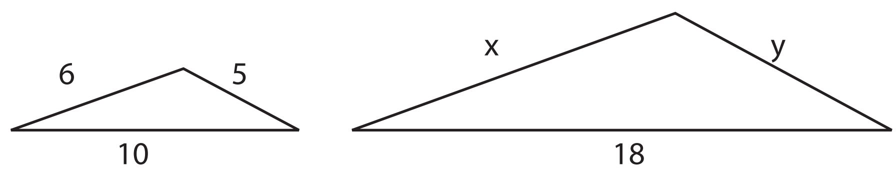
solution: Since the triangles are similar, their corresponding sides are proportional to each other. Notice that 10 and 5 are corresponding sides, as are x and 4, and y and 3. Using an analogy, think "10 is to 5 as x is to 4", or, as a proportion, $\frac{10}{5}=\frac{x}{4}$. Notice that I compared the ratio of known sides to a rato of sides with one unknown. You cannot solve these by comparing ratios of sides that both have an unknown. Think "one equation per unknown." Solve them like you did with the Lesson 4 proportion problems:

$\frac{10}{5}=\frac{x}{4}=$ 10(4) = 5x
40 = 5x
x = **8**

Solve for y in a similar fashion: $\frac{10}{5}=\frac{y}{3}=$ 10(3) = 5y
30 = 5y
y = **6**

It doesn't matter what shape triangles are. As long as they are similar, you can set up proportions to find missing sides.

Example 9.6 The triangles shown are similar. Find x and y.



solution:

$$\frac{10}{18} = \frac{6}{x} = 10x = 6(18)$$
$$10x = 108$$
$$\frac{10}{10}x = \frac{108}{10}$$
$$x = 10.8$$

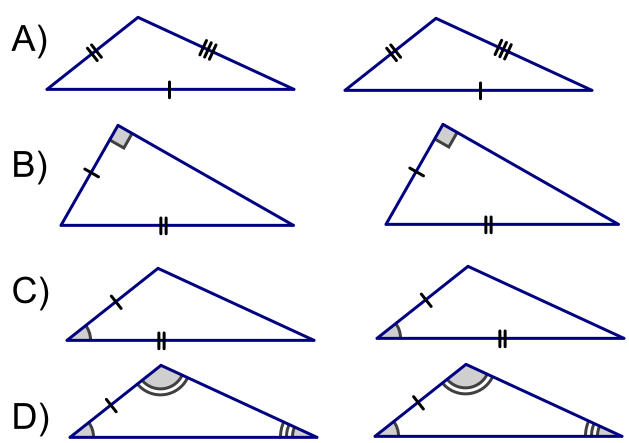
$$\frac{10}{18} = \frac{5}{y} = 10y = 5(18)$$
$$10y = 90$$
$$y = 9$$

9C Triangle Congruency

While similar triangles are identified because they are scale models of each other, congruent triangles (and anything that's congruent, not just triangles) are identified by the fact that you can lay one on top of the other, and they line up exactly. A pair of congruent triangles are like similar triangles whose scale factor equals 1. Two congruent shapes are often compared using $\cong$.

Let's get familiar with the 4 triangle congruency theorems. We will apply these theorems in Lesson 10 and elsewhere when completing triangle congruency proofs.

Example 9.7 Match each pair of congruent triangles with the congruency theorem that best describes it.



solution: In congruency relationships, "tick marks" are used to identify congruent sides and angles. Note that some sides and angles have one, two, or three tick marks. These are clues to help you identify which congruency theorem matches which triangle pair.

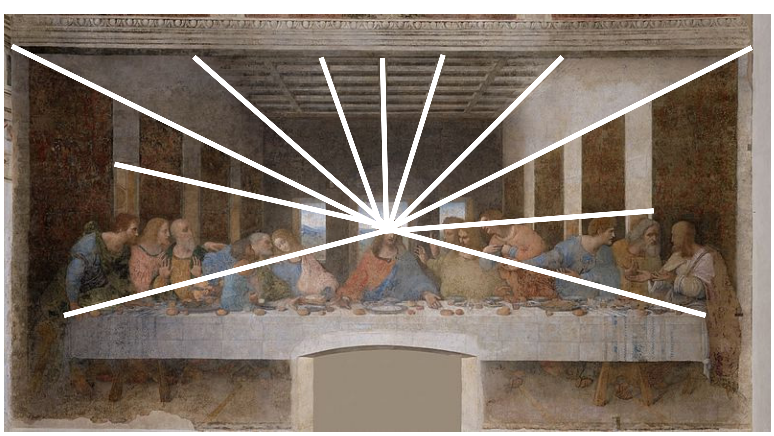
A) SSS, because three congruent sides are identified.
B) HL, because the hypotenuse and one leg are congruent.
C) SAS, because two sides and their included angle are congruent.
D) AAAS, because three angles and one side are congruent.

While the SSS, HL, SAS, and AAAS theorems are primarily used in triangle congruency proofs, the *third angle theorem* is used for both similar and congruent triangle proofs.

9D Geometry in Art and Architecture

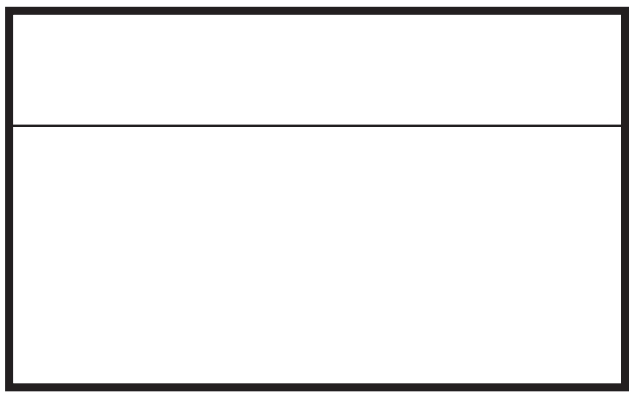
Perspective drawing: Like math, art is also a tool for stuyding God's creation, so it make sense that there should be some connections between math and art. For example, when trying to paint a scene, how does an artist show the reality of our 3D world on a 2D surface? The answer is to use *perspective*. There are several aspects to perspective drawing. We will focus on the aspect of making sizes appear to diminish according to the distance they are from the observer. As an example, we will use Leonardo DaVinci's famous painting, *The Last Supper*(1495-1498). Based on John 13:21, The Last Supper depicts the ensuing discussion amongst the apostles after Jesus predicted his betrayal. Notice how the table appears to be at the "front", even though it was painted on a flat surface.

Now, look at the drawing again, this time with perspective lines drawn over it:

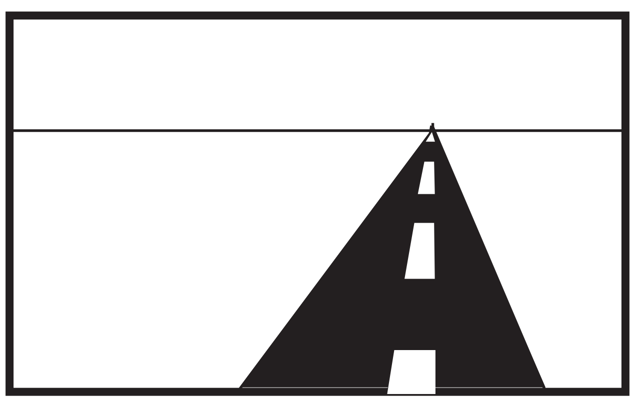


When we include the perspective lines, we start to see how DaVinci made this painting. Obviously, he made Jesus the center of the painting, with all perspective lines drawn relative to His head. This is called *one-point perspective*. The point of origin of all perspective lines is called the *vanishing point*. Now try making your own perspective drawing!

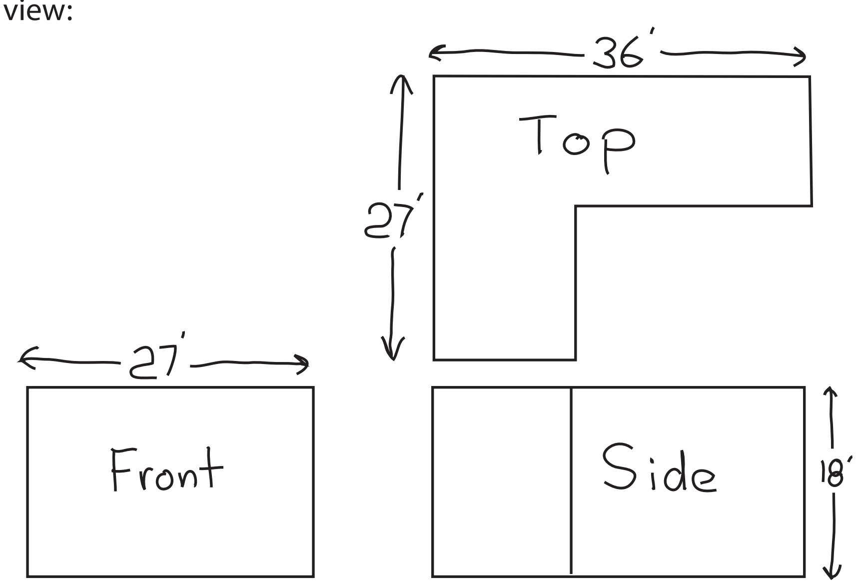
Example 9.8 Using the space below, make a one-point perspective drawing of a highway. The thin, horizontal line represents the horizon. All perspective lines should originate from a vanishing point placed somewhere on the horizon line (Note *The Last Supper* perspective lines also originate from a vanishing point on the horizon).



solution: Your vanishing point does not have to start in the same place. Just make sure it starts on the horizon. Try drawing the center stripes, too! Also, you can use computer graphics software to create your drawing instead of pencil/paper.

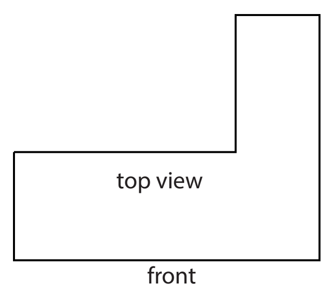


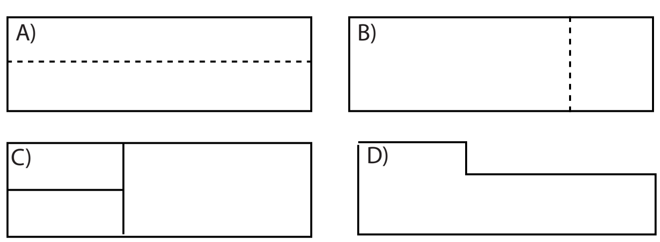
<u>Architectural drawing</u>: This normally refers to the art and science of drawing a habitable structure. Like artists, architects (and engineers) must work with 2D paper representations of 3D objects. An architectural drawing will often contain at least three drawings, including a top view, side view, and frontal view:



The “ ‘ “ marks are shorthand for “feet”. For example, 27' = 27 feet. You will solve problems like the following to practice navigating an architectural drawing.

Example 9.9 The top view of the architectural drawing is shown below. Which of the following choices matches what a view of the building’s front would look like? The dashed lines represent edges hidden from view.





solution: An important part of architectural drawings is showing all edges, both visible and hidden. Hidden edges are represented using dashed lines. Therefore, **Choice B** makes the most sense, as it includes a vertical dashed line to represent the vertical edge that occurs in the back of the building when viewed from the front.

Practice Set 9

(subscripts tell you which lesson each problem came from)

Use your best judgement as to when you should and shouldn't use a calculator.

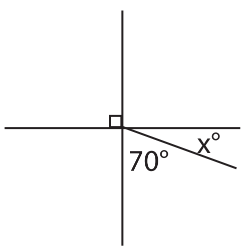
1_9. Match the following terms with the statements about points, lines, and angles. (transversal, acute, 90°, 180°, 360°)

A) The measure of a right angle.
B) The total measure of an angle and its supplement.
C) The measure of the sum of a polygon's exterior angles.
D) An angle that is less than 90°.
E) A non-perpendicular line that intersects two or more parallel lines.

2_9. Match the following terms with the statements about circles, polygons, and solids. (concave, cone, curved, diameter, scalene)

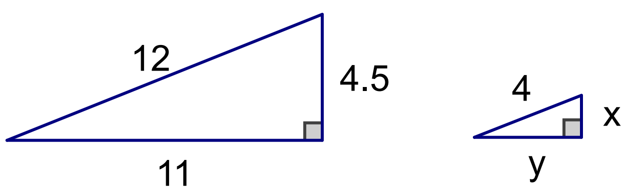
A) A line segment drawn through the center of a circle that terminates on the circumference on both sides.
B) Polygons do not have any ______________ lines.
C) A ______________ polygon has at least one indentation.
D) A ______________ triangle has no congruent sides.
E) A ______________ has a circular base and a lateral surface that comes to a point.

3_9. Find x.



4_9. What do each of the interior angles of a regular hexagon equal?

5_9. The right triangles shown below are similar. Find *x* and *y*. Type answers rounded to 1 d.p.

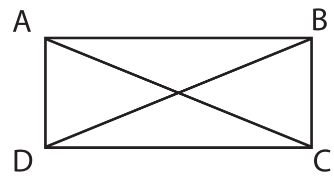


6_9. Add to your one-point perspective highway drawing from Example 9.8. Try to add a rectangular building on the left side of the road. Start by drawing a rectangle, then drawing line segments from its vertices to your vanishing point. Then, answer the following question: Who painted *The Last Supper*?

A) Monet B) Mangelsen C) DaVinci D) Aivazovsky

7_9. In a Euclidean plane, the sum of the measures of a triangle's interior angles equals ______.

8_9. In the rectangle shown, diagonal AC equals 9 inches. What does diagonal BD equal?



9_9. True or False. All rhombuses are also squares.

10_8. Find the roots in the following 2nd degree polynomial. Begin by rearranging and setting y = 0. $3x + y = x^2 - 28$

11_8. Use the completing the square algorithm to find the roots. Factor an x out first, then use the algorithm on the remaining quadratic relationship. $y = x^3 + 4x^2 - 3x$

12_8. Simplify. $\sqrt{16x^4}$

13_8. Simplify. $\dfrac{h^{a-1}k^{2a}}{h^2k^{a-4}}$

For Problems 14 & 15, use substitution or elimination (your choice) to solve the systems.

14_7. $\begin{cases} x + 4y = 13 \\ -2x + 3y = 18 \end{cases}$

15_7. $\begin{cases} 2x + 5y = 9 \\ 3x - 3y = 3 \end{cases}$

16_7. Factor the trinomial. Factor x first, then factor the quadratic into two binomials.
$x^3 + 14x^2 + 13x$

17_7. Solve the non-linear system of equations. $\begin{cases} y = x^2 - 4x + 7 \\ y = x + 3 \end{cases}$

18_6. Expand. $(y^2 + 2xyz)(x^2 + 2x + 1)$

19_5. Two of the main things we do in algebra are _______________ (moving things around) and canceling.

20_5. Simplify by adding like terms. $\dfrac{3x^3y^2}{x^2y} + \dfrac{2x}{y^{-1}} - xy$