LESSON

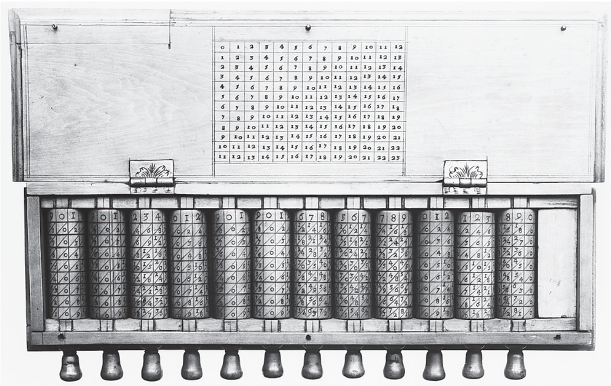
5.7

Properties of Logarithms

Each problem that I solved became a rule which served afterwards to solve other problems.

RENÉ DESCARTES

Before machines and electronics were adapted to do multiplication, division, and raising a number to a power, scientists spent long hours doing computations by hand. Early in the 17th century, Scottish mathematician John Napier (1550–1617) discovered a method that greatly reduced the time and difficulty of these calculations, using a table of numbers that he named logarithms. As you learned in Lesson 5.6, a common logarithm is an exponent—the power of 10 that equals a number—and you already know how to use the multiplication, division, and power properties of exponents. In the next example you will discover some shortcuts and simplifications.



After inventing logarithms, John Napier designed a device for calculating with logarithms in 1617. Later called "Napier's bones," the device used multiplication tables carved on strips of wood or bone. The calculator at left has an entire set of Napier's bones carved on each spindle. You can learn more about Napier's bones and early calculating devices at flourishkh.com.

EXAMPLE

Convert numbers to logarithms to solve these problems.

a. Multiply 183.47 by 19.628 without using the multiplication key on your calculator.

b. Divide 183.47 by 19.628 without using the division key on your calculator.

c. Evaluate $4.70^{2.8}$ without the exponentiation key on your calculator. (You may use the 10^x key.)

▸ Solution

You can do parts a and b by hand. Or you can convert to logarithms and use alternative functions.

a. $183.47 = 10^{\log 183.47}$ and $19.628 = 10^{\log 19.628}$

$$183.47 \cdot 19.628 = 10^{\log 183.47} \cdot 10^{\log 19.628} = 10^{\log 183.47 + \log 19.628} \approx 10^{3.556441}$$
$$\approx 3601.149$$

b. $$\frac{183.47}{19.628} = \frac{10^{\log 183.47}}{10^{\log 19.628}} = 10^{\log 183.47 - \log 19.628} \approx 10^{0.970689} \approx 9.34736$$

c. $4.70 = 10^{\log 4.70}$

$$4.70^{2.8} = (10^{\log 4.70})^{2.8} = 10^{2.8 \log 4.70} \approx 10^{1.8819} \approx 76.2$$

People did these calculations with a table of base-10 logarithms before there were calculators. For example, they looked up log 183.47 and log 19.628 in a table and added them. Then they worked backward in their table to find the **antilog,** or antilogarithm, of that sum.

	0	1	2	3	4
4.0	.6021	.6031	.6042	.6053	.6064
4.1	.6128	.6138	.6149	.6160	.6170
4.2	.6232	.6243	.6253	.6263	.6274
4.3	.6335	.6345	.6355	.6365	.6375
4.4	.6435	.6444	.6454	.6464	.6474
4.5	.6532	.6542	.6551	.6561	.6571
4.6	.6628	.6637	.6646	.6656	.6665
4.7	.6721	.6730	.6739	.6749	.6758
4.8	.6812	.6821	.6830	.6839	.6848
4.9	.6902	.6911	.6920	.6928	.6937

$$\log 4.70 \approx 0.6721$$

	0	1	2	3	4
7.0	.8451	.8457	.8463	.8470	.8476
7.1	.8513	.8519	.8525	.8531	.8537
7.2	.8573	.8579	.8585	.8591	.8597
7.3	.8633	.8639	.8645	.8651	.8657
7.4	.8692	.8698	.8704	.8710	.8716
7.5	.8751	.8756	.8762	.8768	.8774
7.6	.8808	.8814	.8820	.8825	.8831
7.7	.8865	.8871	.8876	.8882	.8887
7.8	.8921	.8927	.8932	.8938	.8943
7.9	.8976	.8982	.8987	.8993	.8998

$$\begin{array}{rl}
\text{antilog } 0.8819 & \approx 7.62 \\
\text{antilog } 1 & = 10 \\
\hline
\text{antilog } 1.8819 & \approx 7.62 \cdot 10 \approx 76.2
\end{array}$$

Can you see why "10 to the power" came to be called the antilog? The antilog of 3 is the same as 10^3, which equals 1000. Later, slide rules were invented to shorten this process, although logarithm tables were still used for more precise calculations.

Because logarithms are exponents, they must have properties similar to the properties of exponents. In the following investigation you will use your calculator to discover these properties.

Investigation
Properties of Logarithms

Step 1 Use your calculator to complete the table. Record the values to three decimal places.

Log form	Decimal form
Log 2	0.301
Log 3	
Log 5	
Log 6	
Log 8	
Log 9	
Log 10	
Log 12	
Log 15	
Log 16	
Log 25	
Log 27	

Step 2 Look closely at the values for the logarithms in the table. Look for pairs of values that add up to a third value in the table. For example, add log 2 and log 3. Where can you find that sum in the table?

Record the equations that you find in the form $\log 2 + \log 3 = \log \underline{\ ?\ }$. (*Hint*: You should find at least six equations.)

Step 3 Write a conjecture based on your results from Step 2.

Step 4 Use your conjecture to write log 90 as the sum of two logs. Do the same for log 30 and log 72. Then use the table and your calculator to test your conjecture.

Complete the following statement:
$\log a + \log b = \log \underline{\ ?\ }$.

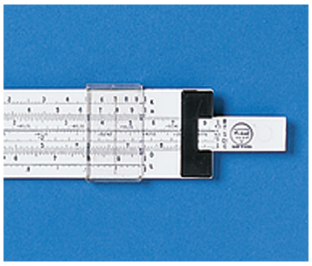
Step 5 Now find pairs of values in the table that subtract to equal another value in the table. Record your results in the form $\log 9 - \log 3 = \log$ _?_. Describe any patterns you see.

Complete the following statement: $\log a - \log b = \log$ _?_.

Step 6 Now find values in the table that can be multiplied by a small integer to give another value in the table, such as $3 \cdot \log 2 = \log$ _?_. Describe any patterns you see. You may want to think about different ways to express numbers such as 25 or 27 using exponents.

Complete the following statement: $b \cdot \log a = \log$ _?_.

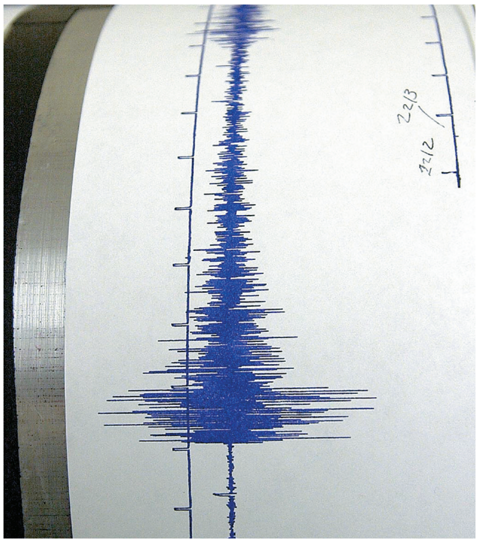
Step 7 How do the properties you recorded in Steps 4–6 relate to the properties of exponents?



Technology CONNECTION

A few years after Napier's discovery, English mathematician William Oughtred (1574–1660) realized that sliding two logarithmic scales next to each other makes calculations easier, and he invented the slide rule. Over the next three centuries, many people made improvements to the slide rule, making it an indispensable tool for engineers and scientists, until computers and calculators became widely available in the 1970s. For more on the history of computational machines, see the links at flourishkh.com.

In this chapter you have learned the properties of exponents and logarithms, summarized on the following page. You can use these properties to solve equations involving exponents. Remember to look carefully at the order of operations and then work step by step to undo each operation.



Developed in 1935 by American scientist Charles F. Richter (1900–1985), the Richter scale measures the magnitude of an earthquake by taking the logarithm of the amplitude of waves recorded by a seismograph, shown at left. Because it is a logarithmic scale, each whole-number increase in magnitude represents an increase in amplitude by a power of 10.

Properties of Exponents and Logarithms

For $a > 0$, $b > 0$ and all values of m and n, these properties are true:

Definition of Logarithm

If $x = a^m$, then $\log_a x = m$.

Product Property

$a^m \cdot a^n = a^{m+n}$ or $\log_a xy = \log_a x + \log_a y$

Quotient Property

$\dfrac{a^m}{a^n} = a^{m-n}$ or $\log_a \dfrac{x}{y} = \log_a x - \log_a y$

Power Property

$\log_a x^n = n \log_a x$

Power of a Power Property

$(a^m)^n = a^{mn}$

Power of a Product Property

$(ab)^m = a^m b^m$

Power of a Quotient Property

$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$

Change-of-Base Property

$\log_a x = \dfrac{\log_b x}{\log_b a}$

Definition of Rational Exponents

$a^{m/n} = (\sqrt[n]{a})^m$ or $\sqrt[n]{a^m}$

Definition of Negative Exponents

$a^{-n} = \dfrac{1}{a^n}$ or $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n$

EXERCISES

You will need

A graphing calculator for Exercises **9** and **13.**

Practice Your Skills

1. Use the properties of logarithms to rewrite each expression as a single logarithm.

a. $\log 5 + \log 11$ @ **b.** $3 \log 2$ **c.** $\log 28 - \log 7$ @

d. $-2 \cdot \log 6$ **e.** $\log 7 + 2 \cdot \log 3$

2. Rewrite each expression as a sum or difference of logarithms by using the properties of logarithms.

a. Write log 22 as a sum of two logs. @

b. Write log 13 as the difference of two logs. @

c. Write log 39 as a sum of two logs.

d. Write log 7 as the difference of two logs.

3. Use the power property of logarithms to rewrite each expression.

a. $\log 5^x$ @ **b.** $\log x^2$ **c.** $\log \sqrt{3}$ @ **d.** $2 \log 7^x$

4. Determine whether each equation is true or false. If false, rewrite one side of the equation to make it true. Check your answer on your calculator.

a. $\log 3 + \log 7 = \log 21$ @ **b.** $\log 5 + \log 3 = \log 8$ **c.** $\log 16 = 4 \log 2$ @

d. $\log 5 - \log 2 = \log 2.5$ **e.** $\log 9 - \log 3 = \log 6$ **f.** $\log \sqrt{7} = \log \frac{7}{2}$

g. $\log 35 = 5 \log 7$ **h.** $\log \frac{1}{4} = -\log 4$ **i.** $\frac{\log 3}{\log 4} = \log \frac{3}{4}$

j. $\log 64 = 1.5 \log 16$

5. Change the form of each expression below using properties of logarithms or exponents, without looking back in the book. Name each property or definition you use.

a. g^{h+k} **b.** $\log s + \log t$ **c.** $\frac{f^w}{f^v}$ **d.** $\log \frac{h}{k}$ **e.** $(j^s)^t$

f. $\log b^g$ **g.** $\sqrt[n]{k^m}$ **h.** $\frac{\log_s t}{\log_s u}$ **i.** $w^t w^s$ **j.** p^{-h}

Reason and Apply

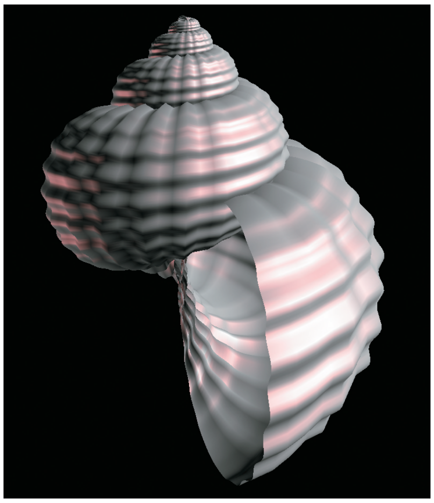
6. **APPLICATION** The half-life of carbon-14, which is used in dating archaeological finds, is 5730 yr.

a. Assume that 100% of the carbon-14 is present at time 0 yr, or $x = 0$. Write the equation that expresses the percentage of carbon-14 remaining as a function of time. (This should be the same equation you found in Lesson 5.6, Exercise 8a.) @

b. Suppose some bone fragments have 25% of their carbon-14 remaining. What is the approximate age of the bones?

c. In the movie *Raiders of the Lost Ark* (1981), a piece of the Ark of the Covenant found by Indiana Jones contained 62.45% of its carbon-14. What year would this indicate that the ark was constructed in?

d. Coal is formed from trees that lived about 100 million years ago. Could carbon-14 dating be used to determine the age of a lump of coal? Explain your answer.



The spiral shape of this computer-generated shell was created by a logarithmic function.

7. **APPLICATION** This table lists the consecutive notes from middle C to the next C note. This scale is called a chromatic scale and it increases in 12 steps, called half-tones. The frequencies measured in cycles per second, or hertz (Hz), associated with the consecutive notes form a geometric sequence, in which the frequency of the last C note is double the frequency of the first C note.

a. Find a function that will generate the frequencies.

b. Fill in the missing table values.

	Note	Frequency (Hz)
Do	C_4	261.6
	C#	
Re	D	
	D#	
Mi	E	
Fa	F	
	F#	
Sol	G	
	G#	
La	A	
	A#	
Ti	B	
Do	C_5	523.2

Music CONNECTION

If an instrument is tuned to the mathematically simple intervals that make one key sound in tune, it will sound out of tune in a different key. With some adjustments, it will be a well-tempered scale—a scale that is approximately in tune for any key. However, not all music is based on an 8- or 12-note scale. Indian musical compositions are based on a *raga,* a structure of 5 or more notes. There are 72 *melas,* or parent scales, on which all ragas are based.

Anoushka Shankar (b 1981) plays the sitar in the tradition of classical Indian music.

8. Use the properties of logarithms and exponents to solve these equations.

a. $5.1^x = 247$

b. $17 + 1.25^x = 30$

c. $27(0.93^x) = 12$ ⓐ

d. $23 + 45(1.024^x) = 147$ ⓐ

9. **APPLICATION** The altitude of an airplane is calculated by measuring atmospheric pressure on the surface of the airplane. This pressure is exponentially related to the plane's height above Earth's surface. At ground level, the pressure is 14.7 pounds per square inch (abbreviated lb/in^2, or psi). At an altitude of 2 mi, the pressure is reduced to 9.46 lb/in^2.

a. Write an exponential equation for altitude in miles as a function of air pressure. ⓗ

b. Sketch the graph of air pressure as a function of altitude. Sketch the graph of altitude as a function of air pressure. Graph your equation from 9a and its inverse to check your sketches.

c. What is the pressure at an altitude of 12,000 ft? (1 mi = 5280 ft) ⓐ

d. What is the altitude of an airplane if the atmospheric pressure is 3.65 lb/in^2?

Science CONNECTION

Air pressure is the weight of the atmosphere pushing down on objects within the atmosphere, including Earth itself. Air pressure decreases with increasing altitude because there is less air above you as you ascend. A barometer is an instrument that measures air pressure, usually in millibars or inches of mercury, both of which can be converted to lb/in^2, which is the weight of air pressing down on each square inch of surface.

10. **APPLICATION** Carbon-11 decays at a rate of 3.5% per minute. Assume that 100% is present at time 0 min.

a. What percentage remains after 1 min?

b. Write the equation that expresses the percentage of carbon-11 remaining as a function of time. *@*

c. What is the half-life of carbon-11?

d. Explain why carbon-11 is not used for dating archaeological finds.

Review

11. Draw the graph of a function whose inverse is not a function. Carefully describe what must be true about the graph of a function if its inverse is not a function. *h*

12. Find an equation to fit each set of data.

a.

x	y
1	8
4	17
6	23
7	26

b.

x	y
0	2
3	54
4	162
6	1458

13. Describe how each function has been transformed from the parent function $y = 2^x$ or $y = \log x$. Then graph the function.

a. $y = -4 + 3(2)^{x-1}$

b. $y = 2 - \log\left(\frac{x}{3}\right)$

14. Answer true or false. If the statement is false, explain why or give a counterexample.

a. A grade of 86% is always better than being in the 86th percentile. *@*

b. A mean is always greater than a median.

c. If the range of a set of data is 28, then the difference between the maximum and the mean must be 14.

d. The mean for a box plot that is skewed left is to the left of the median. *@*

15. A driver charges \$14 per hour plus \$20 for chauffeuring if a client books directly with her. If a client books her through an agency, the agency charges 115% of what the driver charges plus \$25.

a. Write a function to model the cost of hiring the driver directly. Identify the domain and range.

b. Write a function to model what the agency charges. Identify the domain and range.

c. Give a single function that you can use to calculate the cost of using an agency to hire the driver for h hours.