Lesson sequence: permutations and combinations

Section A: Introduction and rationale

This learning sequence is designed for a year 10 class undertaking Mathematics Methods Foundation. The lessons are located within a unit on probability, and cover permutations and combinations, i.e. "the concept of ordered samples" and "the concept of unordered samples" (Tasmanian Qualifications Authority [TQA], 2014, p. 7). Students should have just finished learning about the addition and multiplication rules, and must have basic knowledge of algebra. The sequence consists of 10 50 minute lessons – this lesson time is based on the timetable for Montrose Bay High (2015).

Lessons include examples and exercises from Maths Quest 11 (Williams, Karanikolas, Boucher, Roberts, Nolan & Phillips, 2012), but other activities are suggested in conjunction with these. Peer learning and self-assessment are encouraged and actively called for throughout the lesson sequence, as these have been found to increase mathematical communication and engagement in students across various contexts (Andrade & Valtcheva, 2009; Obidoa, Eskay, & Onwubolu, 2013; Warren, Harris, & Miller, 2014). Furthermore, these address 21st century skills such as self-regulation, collaboration and communication (SRI International, 2015). ICT and videos are also used as tools to increase student engagement and participation, and stimulate discussion (Chandra, & Briskey, 2012; June, Yaacob, & Kheng, 2014; Sherer & Shea, 2011).

The rubric is based on course standards given by the Office of Tasmanian Assessment, Standards & Certification (TQA, 2014), and the given assessment task is formative. Rather than an assessment of learning, it is intended as an assessment for learning and assessment as learning (NSW Government, 2015). Using feedback from this, students may improve their knowledge for the end of unit test; the lesson immediately following the given sequence should comprise the end of unit test or revision for the end of unit test, depending on time constraints and student progress.

Section B: Overview of learning sequence

Materials needed:

- "Maths Quest 11" (Williams et al., 2012) (see appendix A) or other source of appropriate questions.
- Calculators.

Learning outcomes:

Students can:

- Explain the definition of permutations and combinations, and recognize the difference between the two.
- Use appropriate mathematical techniques to determine the number of permutations possible within a given set, including:
    - Permutations of sets with unique elements
    - Permutations of sets with repeated elements
    - Permutations of sets with grouped elements
    - Circular permutations
- Use ${}^{n}C_{r}$ to determine the number of combinations possible within a given set.

Notes:

- All 'exercises' refer to those in appendix A, unless otherwise stated. Only a small percentage of the questions will be completed, so it is the teacher's responsibility to select a representative sample.
- Sensitivity must be used when implementing group activities, so that students are placed in appropriate groups. Those with particular learning needs (e.g. students on the autistic spectrum) may be provided with alternatives to certain group activities.

Lesson 1: Introduction and permutations

Materials needed:

- 9-10 sets of unique cards in categories, with 3-4 cards in each category (e.g. 1st, 2nd and 3rd place 'medals'; food types, jobs, colours)

Learning objectives:

- Students can explain what permutations are, and use the multiplication method to determine the number of permutations possible in a set of unique elements.

Summary:

- Explain to students what permutations and combinations are, and how they differ; briefly outline some examples of permutations and ask students if they can think of any. 5-10 mins.
- Students work in groups of 3-5. Each group is given one set of cards, and must determine the number of permutations possible by sharing the cards amongst group members, listing each permutation in their book. Groups of 3 or 4 will have 3 or 4 cards respectively, but groups of 5 should have 2 or 3 cards to limit the total number of permutations possible. For example, a group of 5 might have to answer in how many different ways 2 medals can be awarded to members of the group (given that you cannot give both medals to one person). Prior to beginning the activity, discuss why it is an example of permutations. 20 mins.
- Compile class results. Discuss the relationship between number of students in group, number of cards to be shared and number of combinations. Relate to multiplication rule. 10 mins.
- Do some worked examples as a class, then students work on exercise 12B for remainder of lesson. 10-15 mins.

Lesson 2: Factorials

Learning objectives:

- Students can explain what factorial notation is, and use it to calculate the number of permutations in a set of unique elements.

Summary:

- Students are introduced to factorial notation, and simple algebra (e.g. division) involving factorials. 10 mins.
- Students work on exercises 12B and 12C. 20 mins.
- Watch video (Numberphile, 2013). During calculations, pause to discuss and question students about methods. 20 mins.

Lesson 3: Permutations using ${}^{n}P_{r}$

Learning objectives:

- Students can describe ${}^{n}P_{r}$ notation, and use it to calculate the number of permutations in a set of unique elements.

Summary:

- Explain ${}^{n}P_{r}$ notation, and determining values by hand and on calculator. 10 mins.
- Students work on exercises 12C and 12D. 35 mins.
- Check for understanding – do students understand what permutations are? Do they feel comfortable using the methods covered in the three lessons so far?

Lesson 4: Permutations with restrictions

Materials needed:

- 9-10 sets of objects, as described in activity below.

Learning objectives:

- Students can calculate the number of permutations in a set with repeated elements.
- Students can calculate the number of permutations in a set with grouped elements.

Summary:

- Recall previous lessons, and methods for determining the number of possible permutations within a given group. Set up activity below, but discuss with the class before they begin: what is different about the permutations they are making now, compared to the permutations they made in lesson 1? What effect do students think this will have on the number of possible permutations? 5-10 mins.
- Students work in small groups. Some groups are given a set of objects in which one object is repeated (e.g. 6 scrabble tiles with 4 'N's, 5 coloured counters of which 3 are red). Other groups are given unique objects but with the restriction that a subset of these must always be side-by-side. For example, they may be given four counters – red, green, yellow and blue – with the restriction that the yellow and green counters must always be side-by-side. Each group must first estimate how many combinations they can make, then use the objects to make as many as they can, listing combinations in their book. Results are discussed in groups, then as a class. 15-10 mins.
- Explain methods for calculating the number of permutations in both cases, including worked examples. Students begin exercise 12E. 25 mins.

Lesson 5: Arrangements in a circle

Learning objectives:

- Students can determine the number of circular permutations possible in a given set, including sets with unique elements, sets with repeated elements and sets with grouped elements.

Summary:

- Students continue exercise 12E. 10 mins.
- Introduce circular permutations using examples such as the example provided in Ronda (2012). Work through examples including unrestricted and restricted circular permutations. 15 mins.
- Students work on exercise 12F. 20 mins.
- Students are asked to write something they've learned about permutations, and something they wish they knew more about – this may be from any of the previous lessons. These are collected by the teacher as students exit the class. 5 mins.

Lesson 6: Permutations revision

See section C.

Lesson 7: Combinations using ${}^{n}C_{r}$

Learning objectives:

- Students can explain what combinations are.
- Students can describe what ${}^{n}C_{r}$ notation means, and use it to calculate the number of combinations possible in a set.

Summary:

- Address any misconceptions apparent from permutations study. Explain definition of combination. Outline some examples, and ask students for suggestions. Explain ${}^{n}C_{r}$ notation and do worked examples. 20 mins.
- Students work through exercise 12G. 20 mins.
- Students play a brief Kahoot quiz. Some questions may be questions not yet assigned from chapter 12 (appendix A), and some describe scenarios which the student must identify as either a permutations or combinations question. 10 mins.

Lesson 8: Winning the lottery

Learning objectives:

- Students can recognise that even events with a very small probability of occurring are not impossible.
- Students demonstrate mathematical and self-regulation skills in completing an assignment.

Summary:

- Warm up: students watch a video (funstufffella, 2012), and discuss: what is right about the video (i.e. the maths)? What is wrong about it (i.e. 'infinitesimally small' chance vs 'impossible')? 10 mins.
- Introduce the assessment task (see section D), and discuss the rubric and criteria. 5 mins.
- Students work on the assessment task. 30 mins.
- Remind students to compare their assessment task to the rubric. Assessment tasks are handed in at the end of class if desired, but otherwise must be completed as homework and handed in at the beginning of lesson 9. Students aiming for an A will almost certainly have to complete the assignment for homework. 5 mins.

Lesson 9: Applications to probability

Learning objectives:

- Students can apply prior knowledge of permutations and combinations to probability, by expressing the probability of a specific permutation or combination occurring.

Summary:

- Discuss expressing permutations and combinations as probabilities. Provide a worked example, then divide the class into small groups. Each group is given worked examples 27, 28 and 29 (see appendix A, p 566-567). For each worked example, they read it and discuss how it has been solved, then create a similar example and solve it. 20 mins.
- Using questions from exercise 12H and questions 2 and 3 from the senior secondary maths relay 2010 (see appendix B), groups compete in a maths relay. In this case, the runner is not excluded from participating in solving questions.

Lesson 10: Self-assessment

Preparation:

Each assignment from lesson 8 should be marked, with feedback, corrections and rubric included on a separate page to the assignment itself.

Learning objectives:

- Students can evaluate their own learning in a realistic way.

Summary:

- Students' assessment tasks are handed back to them without feedback. Students assess their assignment, completing a rubric and correcting any errors they find. The teacher then hands them their rubric, so students can compare their judgement and the teacher's. 10 mins.
- Common misconceptions arising from students' assessment tasks are discussed. 5 mins.
- Students revise probability, including permutations and combinations, in preparation for end of unit test. 40 mins.

Section C: Full lesson plan

Lesson Title: Permutations revision

Objectives:

Students will

- Construct appropriate problems involving permutations.
- Use appropriate mathematical techniques to solve permutation problems.
- Constructively critique permutations problems, to ensure they are well articulated and unambiguous.

This lesson requires students to create their own problem and critically evaluate questions created by others, utilising order thinking according to Bloom's taxonomy. Furthermore, it could be argued that including creativity in mathematics helps students to appreciate the beauty in maths and better enables them to apply it to real world applications (Mann, 2006).

Preparation:

Before the lesson, the teacher should determine student groupings for the activity below. It is suggested that students are divided into groups of 3 where possible, and include a range of abilities. Depending on numbers, one or two groups may have 4 students; in this case, students will need to be capable of getting through all discussion sessions within the specified time. During class, the teacher should listen to group discussions and ask questions to stimulate thought. Students' questions and answers may be collected at the end of class, so that the teacher may look through the work and get an idea of how well students are understanding the material, and in which are(s) they need more work.

Introduction:

Recall permutations work covered so far. Explain that, as revision work for permutations, students will be creating questions for other class members. These questions should be creative where possible, and be written so that other maths students will understand what it being asked of them. Spend approximately 5 mins on this.

Summary:

- Each student creates two questions about permutations – at least one of these must include permutations with restrictions and/or circular permutations. They provide two copies of each question, and one copy of model answers. 10 mins.
- Students exchange questions within their assigned groups. After this process, every student should have four questions, none of which are written by them. Students complete the questions they have, with working out. They also note whether the question was articulated well; if it was not, they state which part was ambiguous or confusing, and how it could be improved. If they have time to spare, they should make general comments about the questions, e.g. how creative and/or challenging they were. 15 mins.
- Question by question, students discuss commonalities and differences in their working out. They also briefly discuss comments about the question, and how it could be improved. Each discussion consists of three students (the writer and the two who answered the question), and lasts for approximately 5-7 mins. 20 mins.

Section D: Assessment task

Tattslotto is a gambling game in which people buy a ticket in the hopes of winning prize money. Each ticket is considered one entry (i.e. one chance of winning), and consists of a selection of six numbers. Below is a section of the Tattslotto rules (Tattersall's Sweeps Pty Ltd, 2015).

2 OBJECTIVE

2.1 The objective of the player participating in TattsLotto is to correctly select the Winning Numbers from the numbers 1 to 45 inclusive in any one Standard Entry.

3 DRAWING OF TATTSLOTTO

3.1 In each Draw, 8 balls will be drawn from a Drawing Device containing forty-five balls numbered from 1 to 45 inclusive.

3.2 The first 6 drawn balls drawn from the Drawing Device show the Winning Numbers and the 7th and 8th drawn balls show the Supplementary Numbers.

3.3 A Winning Number or Supplementary Number in a Draw is deemed drawn when a numbered ball rests in the display section forming part of the Drawing Device.

3.4 A Draw is deemed complete when 8 numbered balls (and not fewer or more than 8) are drawn from the Drawing Device in accordance with the Rules.

4 DIVISION OF PRIZES

4.1 In each Draw of TattsLotto there shall be six (6) Prize divisions as follows:

Division One:	6 Winning Numbers in any one Standard Entry
Division Two:	5 of the 6 Winning Numbers plus either of the Supplementary Numbers in any one Standard Entry
Division Three:	5 of the 6 Winning Numbers in any one Standard Entry
Division Four:	4 of the 6 Winning Numbers in any one Standard Entry
Division Five:	3 of the 6 Winning Numbers plus either or both of the Supplementary Numbers in any one Standard Entry
Division Six	Either 1 or 2 of the 6 Winning Numbers plus both of the Supplementary Numbers in any one Standard Entry

Using calculations to support your answers, determine the following:

1. Charlie has bought a Tattslotto ticket, and says his chance of winning is one in a million. Comment on his statement, assuming "winning" means getting a division one prize.
2. Charlie has picked 6 random numbers for his ticket, but Rob has picked the numbers 1, 2, 3, 4, 5 and 6 for his ticket. Compare Charlie's and Rob's chances of winning.
3. Is Charlie more likely to win a division one prize or a division two prize?

Hint: Probability of getting 5 out of 6 winning numbers $= \frac{Number\ of\ ways\ to\ choose\ 5\ numbers\ out\ of\ 6}{Number\ of\ ways\ to\ choose\ 6\ numbers\ out\ of\ 45}$

4. ***Extension:*** Which prize division is Charlie most likely to win?

Section E: Rubric

	A standard	**B standard**	**C standard**
Criteria 1: Communicate mathematical ideas and information	Working out is shown. The answer is made clear and put in context. Mathematical symbols are used correctly. Explanations clearly articulated, using correct mathematical terminology.	Working out is shown. The answer is made clear. Mathematical symbols are usually used correctly. Explanations articulated well, including some mathematical terminology.	Some working out shown. Final answer shown. Mathematical symbols are generally used correctly, but may have one or two errors. Explanations articulated in a way that is able to be understood.
Criteria 3: Plan, organise and complete mathematical activities	The student: Returns assignment on time, with all questions complete and mostly accurate. Uses strategies to successfully solve complex problems (e.g. breaks a task into sub-tasks).	The student: Returns assignment on time, with all questions attempted. Uses strategies to successfully complete straightforward problems.	The student: Returns assignment on time, mostly complete. Uses given strategies to complete straightforward problems.
Criteria 8: Demonstrate an understanding of experimental and theoretical probabilities	Each question is completed, with few or no errors.	Questions 1-3 are completed with few or no errors. Question 4 has been attempted, but is either incomplete or has significant errors.	Explanations and calculations are mostly correct for questions 1-3.

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