myweb.ttu.edu/bban

lev.ban@ttu.edu

Real Analysis

Byeong Ho Ban
Mathematics and Statistics
Texas Tech University

Chapter 3. Signed Measures and Differentiation

1.

Proof. □

---

2. *If ν is a signed measure, E is $\nu-null$ iff $|\nu|(E) = 0$. Also, if ν and μ are signed measures, $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.*

Proof. .

Claim 1. *E is ν-null iff $|\nu|(E) = 0$*

Proof. .
$(\Longrightarrow)$
Suppose that E is a $\nu - null$. Since $\nu^+ \perp \nu^-$, there is a partition $\{P, N\}$ of X where N is $\nu^+ - null$ and P is $\nu^- - null$. Observe the below.

$$|\nu|(E) = |\nu|(X \cap E) = \nu^+(P \cap E) + \nu^-(N \cap E)$$

If $|\nu|(E) > 0$, without loss of generality, we can let $\nu^+(P \cap E) > 0$. Then note that

$$\nu(P \cap E) = \nu^+(P \cap E) - \nu^-(P \cap E) = \nu^+(P \cap E) > 0$$

But $P \cap E \subset E$. so it is a contradiction.
$(\Longleftarrow)$
Suppose that $|\nu|(E) = 0$, then since $|\nu|$ is a positive measure, $|\nu|(A) = 0\ \forall A \subset E$. Observe the below.

$$|\nu|(A) = \nu^+(A) + \nu^-(A) = 0 \quad \forall A \subset E$$

It means $\nu^+(A) = \nu^-(A) = 0\ \forall A \subset E$. It also means that

$$\nu(A) = \nu^+(A) - \nu^-(A) = 0 \quad \forall A \subset E$$

Thus, E is a $\nu -$ *null* □

.

Claim 2. *If ν and μ are signed measures, $\nu \perp \mu$ iff $|\nu| \perp \mu$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.*

Proof. .
$\nu \perp \mu$ iff $\exists$ a partition $\{P, N\}$ of X such that P is $\mu - null$ and N is $\nu - null$ iff $\exists$ a partition $\{P, N\}$ of X such that P is $\mu - null$ and N is $|\nu| - null$ iff $|\nu| \perp \mu$ iff $\exists$ a partition $\{P, N\}$ of X such that P is $\mu - null$ and N is $|\nu| - null$ iff $\exists$ a partition $\{P, N\}$ of X such that P is both $\mu - null$ and N is $\nu^+ - null$ and $\nu^- - null$ iff $\nu^+ \perp \mu$ and $\nu^- \perp \mu$. □

□

---

3. *Let ν be a signed measure on $(X, \mathcal{M})$.*
a.$L^1(\nu) = L^1(|\nu|)$
b. *If $f \in L^1(\nu)$, $\left|\int f d\nu\right| \le \int |f| d|\nu|$*
c. *If $E \in \mathcal{M}$, $|\nu|(E) = \sup\left\{\left|\int_E f d\nu\right| : |f| \le 1\right\}$*

Proof. .
a
Obviously, if $f \in L^1(|\nu|)$, then $\int |f| d|\nu| < \infty$, and observe the below

$$\int |f| d\nu = \int |f| d\nu^+ - \int |f| d\nu^- \le \int |f| d\nu^+ + \int |f| d\nu^- = \int |f| d|\nu| < \infty$$

Thus, $f \in L^1(\nu)$.
Conversely, suppose that $f \in L^1(\nu)$, then $\int |f|\, d\nu = \int |f| d\nu^+ - \int |f| d\nu^- < \infty$.
Note that $\nu^+ \perp \nu^-$, so there exists a partition $\{P, N\}$ of X such that P is $\nu - null$ and N is $\mu - null$.
If $f \notin L^1(|\nu|)$, then one of $\int f d\nu^+$ and $\int f d\nu^-$ should be infinite.
Without loss of generality, let $\int f d\nu^+ = \int_P f d\nu^+ = \infty$.
Then

$$\int |f| d\nu \ge \int_P |f| d\nu = \int_P |f| d\nu^+ - \int_P |f| d\nu^- = \int_P |f| d\nu^+ = \infty$$

It is a contradiction.
b
Since $f \in L^1(\nu)$, observe the below.

$$\begin{aligned}
\left|\int f d\nu\right| &= \left|\int f^+ d\nu - \int f^- d\nu\right| \\
&= \left|\left(\int f^+ d\nu^+ - \int f^- d\nu^-\right) - \left(\int f^- d\nu^+ - \int f^- d\nu^-\right)\right| \\
&\le \int f^+ d\nu^+ + \int f^- d\nu^- + \int f^- d\nu^+ + \int f^- d\nu^- \\
&= \int |f| d\nu^+ \int |f| d\nu^- \\
&= \int |f| d|\nu|
\end{aligned}$$

c
For any function $|f| \le 1$, by **b**, observe the below first.

$$\begin{aligned}
\left|\int_E f d\nu\right| &= \left|\int \chi_E f d\nu\right| \\
&\le \int |\chi_E f| d|\nu| \quad (\because \chi_E f \in L^1(\nu)) \\
&\le \int |\chi_E \chi_X| d|\nu| \quad (\because |f| \le 1) \\
&= \int \chi_E d|\nu| = \int_E d|\nu| = |\nu|(E)
\end{aligned}$$

Thus, $|\nu(E)|$ is an upper bound of the set $A = \left\{\left|\int_E f d\nu\right| : |f| \le 1\right\}$.
Let $\epsilon > 0$ be given. Then,

$$|\nu|(E) - \epsilon = \int_E \left(1 - \frac{\epsilon}{|\nu|(E)}\right) d|\nu| \le \int_E 1 d|\nu|$$

Since $1 \in A$, $|\nu|(E)$ is the least upper bound of the set A. □

---

4. *If ν is a signed measure and λ, μ are positive measures such that $\nu = \lambda - \mu$, then $\lambda \ge \nu^+$ and $\mu \ge \nu^-$.*

Proof. .
Note that $\nu^+ \perp \nu^-$, so there is partition $\{P, N\}$ such that P is a $\nu^- - null$ and N is a $\nu^+ - null$.
Now, observe the below.

$$\nu^+(E) = \nu(P \cap E) = \lambda(P \cap E) - \mu(P \cap E) \le \lambda(P \cap E) \le \lambda(E) \quad \forall E \in \mathcal{M}$$
$$\nu^-(E) = -\nu(N \cap E) = -(\lambda(N \cap E) - \mu(N \cap E)) \le \mu(N \cap E) \le \mu(E) \quad \forall E \in \mathcal{M}$$

Then we are done. □

---

5. *If ν_1, ν_2 are signed measures that both omit the value $+\infty$ or $-\infty$, then $|\nu_1 + \nu_2| \le |\nu_1| + |\nu_2|$.*

Proof. .
Let the Jordan decomposition of $\nu_1 + \nu_2 = \mu^+ + \mu^-$. And also, observe the below.

$$\nu_1 + \nu_2 = \nu_1^+ - \nu_1^- + \nu_2^+ - \nu_2^- = (\nu_1^+ + \nu_2^+) - (\nu_1^- + \nu_2^-)$$

Note that $\nu_1^+ + \nu_2^+$ and $\nu_1^- + \nu_2^-$ are positive measures. Therefore, by Exercise 4, $\nu_1^+ + \nu_2^+ \ge \mu^+$ and $\nu_1^- + \nu_2^- \ge \mu^-$.
Therefore,

$$|\nu_1 + \nu_2| = \mu^+ + \mu^- \le (\nu_1^+ + \nu_2^+) + (\nu_1^- + \nu_2^-) = |\nu_1| + |\nu_2|$$

□

6.

Proof. □

7.

Proof. □

---

8. *$\nu \ll \mu$ iff $|\nu| \ll \mu$ iff $\nu^+ \ll \mu$ and $\nu^- \ll \mu$*

Proof. .

Claim 3. $\nu \ll \mu \implies |\nu| \ll \mu$
$(\implies)$
Let $E \in \mathcal{M}$ be a set with $\mu(E) = 0$ and $\{P, N\}$ be a Hahn decomposition of X with respect to ν.
Then observe the below

$$\nu^+(E) = \nu(E \cap P) = 0 \quad (\because 0 \le \mu(E \cap P) \le \mu(E) = 0, \nu \ll \mu)$$
$$\nu^-(E) = -\nu(E \cap N) = 0 \quad (\because 0 \le \mu(E \cap N) \le \mu(E) = 0, \nu \ll \mu)$$

Therefore, $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$, so $|\nu| \ll \mu$

■

Claim 4. *$|\nu| \ll \mu \implies \nu^+ \ll \mu$ and $\nu^- \ll \mu$*
$(\implies)$
Let $E \in \mathcal{M}$ be a set with $\mu(E) = 0$. Since $|\nu| \ll \mu$, $|\nu|(E) = \nu^+(E) + \nu^-(E) = 0$. Since ν^+ and ν^- are positive measure, $\nu^+(E) = \nu^-(E) = 0$. Therefore, $\nu^+ \ll \mu$ and $\nu^- \ll \mu$

■

Claim 5. *$\nu^+ \ll \mu$ and $\nu^- \ll \mu \implies \nu \ll \mu$*
$(\implies)$
Let $E \in \mathcal{M}$ be a set with $\mu(E) = 0$, then $\nu^+(E) = \nu^-(E) = 0$.
Thus, $\nu(E) = \nu^+(E) - \nu^-(E) = 0$. Therefore, $\nu \ll \mu$.

□

9.

Proof. □

10.

Proof. □

11.

Proof. □

---

12. *For $j = 1, 2$ let ν_j, μ_j be $\sigma-$finite measure on $(X_j, \mathcal{M}_j)$ such that $\nu_j \ll \mu_j$.*
Then $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_2}(x_1)\frac{d\nu_2}{d\mu_2}(x_2)$$

Proof. .
Let $E_{n,i} = \left\{x : \frac{d\nu_i}{d\mu_i} < -\frac{1}{n}\right\}$ $(i = 1, 2)$, and observe the below

$$\nu_i(E_{n,i}) = \int_{E_{n,i}} d\nu_i = \int_{E_{n,i}} \frac{d\nu_i}{d\mu_i} d\mu_i < -\frac{1}{n} \int_{E_{n,i}} d\mu_i = -\frac{1}{n}\mu_i(E_{n,i})$$

Since ν_i is positive measure, $\mu_i(E_{n,i}) = 0 = \nu(E_{n,i})$ $\forall n \in \mathbb{N}$.
Also note the below.

$$E = \left\{x : \frac{d\nu_i}{d\mu_i} < 0\right\} = \bigcup_{n \in \mathbb{N}} E_{n,i}$$

By continuous from below

$$\mu(E) = \lim_{n \to \infty} \mu(E_{n,i}) = 0$$

Therefore, $\frac{d\nu_i}{d\mu_i} \geq 0$ $\mu_i - a.e.$ for $i = 1, 2$ which means $\frac{d\nu_i}{d\mu_i} \in L^+(\mu_i)$. Thus by Tonelli's Theorem,

$$\begin{aligned}
\nu_1 \times \nu_2(E) &= \int_E d(\nu_1 \times \nu_2) = \int \chi(E) d(\nu_1 \times \nu_2) \\
&= \int \int \chi(E) d\nu_1 d\nu_2 = \int \left(\int \chi(E) \frac{d\nu_1}{d\mu_1} d\mu_1 \right) \frac{d\nu_2}{d\mu_2} d\mu_2 \\
&= \int \int \chi(E) \frac{d\nu_1}{d\mu_2} \frac{d\nu_2}{d\mu_2} d\mu_1 d\mu_2 \\
&= \int_E \frac{d\nu_1}{d\mu_2} \frac{d\nu_2}{d\mu_2} d(\mu_1 \times \mu_2)
\end{aligned}$$

Therefore, $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and $\frac{d\nu_1}{d\mu_2} \frac{d\nu_2}{d\mu_2} = \frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}$ (by definition of Radon Nikodym derivative) □

13.

Proof. □

---

14. *If ν is an arbitrary signed measure and μ is a $\sigma-$finite measure on $(X, \mathcal{M})$ such that $\nu \ll \mu$, there exists an extended $\mu-$integrable function $f : X \to [-\infty, \infty]$ such that $d\nu = f d\mu$.*
Hint
a. *It suffice to assume that μ is finite and ν is positive.*
b. *With these assumptions, there exists $E \in \mathcal{M}$ that is $\sigma-$finite for ν such that $\mu(E) \geq \mu(F)$ for all sets F that are $\sigma-$finite for ν.*
c. *The Radon-Nikodym theorem applies on E. If $F \cap E = \emptyset$, then either $\nu(F) = \mu(F) = 0$ or $\mu(F) > 0$ and $|\nu(F)| = \infty$*

Proof. .
Suppose that μ is finite and ν is positive.
Let $\mathcal{S} = \{S \in \mathcal{M} : \text{S is } \sigma - finite \text{ for } \nu\}$. And let $\alpha = \sup\{\mu(S) : S \in \mathcal{S}\}$. Then there is a sequence of sets $\{S_n\}_{n\in\mathbb{N}} \subset \mathcal{S}$ such that $\lim_{n\to\infty} \mu(S_n) = \alpha < \infty$.($\because$ μ is finite)
Let $S = \bigcup_{n\in\mathbb{N}} S_n = \bigcup_{n\in\mathbb{N}} \bigcup_{k=1}^n S_k$, then by continuous from below, since S is also $\sigma-$finite,

$$\alpha \geq \mu(S) = \lim_{n\to\infty} \mu(\bigcup_{k=1}^n S_k) = \alpha$$

Therefore, $\mu(S) = \alpha$. Then by Lebesgue-Radon-Nikodym Theorem, $\frac{d\nu}{d\mu}$ exists on S.
Let's define a function F as below.

$$F(x) = f(x) \quad (\text{if } x \in S)$$
$$F(x) = \infty \quad (\text{if } x \notin S)$$

Let $E \in \mathcal{M}$ be given.
If $\mu(E \setminus S) = 0$, then $\nu()$

$$\nu(E) = \nu(E \cap S) + \nu(E \setminus S) = \nu(E \cap S) = \int_{E\cap S} f d\mu = \int_{E\cap S} F d\mu + \int_{E\setminus S} F d\mu = \int_E F d\mu$$

If $\mu(E \setminus S) > 0$, then

$$\mu(S \cup (E \setminus S)) = \mu(S) + \mu(E \setminus S) > \alpha$$

Thus $S \cup (E \setminus S) \notin \mathcal{S}$. It means $E \setminus S \notin \mathcal{S}$, otherwise $S \cup (E \setminus S)$ is a $\sigma-$finite so it is in $\mathcal{S}$. So $\mu(E \setminus S) = \infty$. Thus,

$$\nu(E) = \infty = \nu(E \setminus S) = \int_{E\setminus S} \infty d\mu = \int_{E\setminus S} F d\mu = \int_E F d\mu$$

Either ways allow us to say that

$$F = \frac{d\nu}{d\mu} \quad \text{on } X$$

Now, let's assume that μ is $\sigma-$finite and ν is positive. Then there is a sequence of disjoint sets $\{E_n\}_{n\in\mathbb{N}}$ such that

$$X = \bigcup_{n\in\mathbb{N}} E_n \ \ and \ \ \mu(E_n) < \infty \ \ \forall n \in \mathbb{N}$$

Thus, there exits $F_n = \frac{d\nu}{d\mu}$ on E_n for $\forall n \in \mathbb{N}$. Then we can define a function as below.

$$F = F_n \quad \text{on } E_n \forall n \in \mathbb{N}$$

Note that F is defined on X.
Lastly, let μ be $\sigma-$finite and ν be signed measure. Due to the arguments above, there exist two functions $F^+ = \frac{d\nu^+}{d\mu}$ and $F^- = \frac{d\nu^-}{d\mu}$, then $F = F^+ - F^- = \frac{d\nu}{d\mu}$ (on X) is the function that we have found. □

15.

Proof. □

---

16. *Suppose that μ, ν are $\sigma-$finite measure on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$. If $f = \frac{d\nu}{d\lambda}$, then $0 \le f < 1$ $\mu - a.e.$ and $\frac{d\nu}{d\mu} = \frac{f}{1-f}$.*

Proof. .
Suppose that μ and ν are $\sigma-$finite measure on $(X, \mathcal{M})$, then by EX12, $\frac{d\nu}{d\mu} \ge 0$ $\mu - a.e.$. Also note that $\lambda = \mu + \nu$ is a $\sigma-$finite, so $\frac{d\nu}{d\lambda} = f \ge 0$ $\lambda - a.e.$. Since $\mu \ll \lambda$, $f \ge 0$ $\mu - a.e.$.
We need to prove that $\mu(E) = 0$ where $E = \{x : f(x) \ge 1\}$.
Observe the below.

$$0 \le \mu(E) = \lambda(E) - \nu(E) = \int_E d\lambda - \int_E d\nu = \int_E d\lambda - \int_E f d\lambda = \int (1-f) d\lambda \le \int_E 0 d\lambda = 0$$

Thus, $\mu(E) = 0$, so $0 \le f < 1$ $\mu - a.e.$
Now, for $\forall E \in \mathcal{M}$ observe the below.

$$\begin{aligned} \int_E \frac{f}{1-f} d\mu &= \int_E \frac{f}{1-f} d\lambda - \int_E \frac{f}{1-f} d\nu \\ &= \int_E \frac{1}{1-f} d\nu - \int_E \frac{f}{1-f} d\nu \\ &= \int_E \frac{1-f}{1-f} d\nu \\ &= \nu(E) \end{aligned}$$

Therefore, $\frac{f}{1-f} = \frac{d\nu}{d\mu}$. □

17.

Proof. □

18.

Proof. □

19.

Proof. □

20.

Proof. □

21.

Proof. □

22.

Proof. □

23.

Proof. □

24.

Proof. □

25.

Proof. □

26.

Proof. □

27.

Proof. □

28. *If $F \in NBV$, let $G(x) = |\mu_F|((-\infty, x])$ by showing that $G = T_F$ via the following steps.*
a. *From the definition of T_F, $T_F \leq G$.*
b. $|\mu_F(E)| \leq \mu_{T_F(E)}$ *when E is an interval, and hence when E is a Borel set.*
c.$|\mu_F| \leq \mu_{T_F}$*, and hence $G \leq T_F$.(use Exercise 21.)*

Proof. .
Suppose that $F \in NBV$.
For any partition $\delta : -\infty = x_0 < x_1 < \cdots < x_n = x$, the relation below holds.

$$G(x) = |\mu_F|((-\infty, x]) = \sum_{i=1}^{n} |\mu_F|((x_{i-1}, x_i]) \geq \sum_{i=1}^{n} |\mu_F((x_{i-1}, x_i])| = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = t_\delta$$

Thus, $|\mu_F|((-\infty, x]) \geq T_F$. Recall that $\mu_F(E) = \mu_P(E) - \mu_N(E)\ \forall E$. Thus, if $\{A_p, A_n\}$ is a Hahn decomposition for μ_F, observe the below.

$$\mu_F^+(E) = \mu_F(E \cap A_p) = \mu_P(E \cap A_p) - \mu_N(E \cap A_p) \leq \mu_P(E)$$
$$\mu_F^-(E) = -\mu_F(E \cap A_n) = -\mu_P(E \cap A_n) + \mu_N(E \cap A_n) \leq \mu_N(E)$$

Therefore,

$$|\mu_F|(E) = \mu_F^+(E) + \mu_F^-(E) \leq \mu_P(E) + \mu_N(E) = \mu_{T_F}(E)$$

It means that

$$T_F(x) = P(x) + N(x) = \mu_P((-\infty, x]) + \mu_N((-\infty, x]) \geq |\mu_F|((-\infty, x]) = G(x)$$

Thus, $T_F(x) = G(x)$. □

---

29. *If $F \in NBV$ is real-valued, then $\mu_F^+ = \mu_P$ and $\mu_F^- = \mu_N$ where P and N are the positive and negative variations of F.(Use Exercise 28.)*

Proof. .
We know that $P = \frac{1}{2}(T + F)$ and $N = \frac{1}{2}(T - F)$ which means $\mu_P = \frac{1}{2}(\mu_T + \mu_F)$ and $\mu_N = \frac{1}{2}(\mu_T - \mu_F)$.
Now, by Exercise 28, observe the below.

$$\mu_P(E) = \frac{1}{2}(\mu_T(E) + \mu_F(E)) = \frac{1}{2}(|\mu_F|(E) + \mu_F(E)) = \frac{1}{2}(2\mu_F^+(E)) = \mu_F^+(E) \quad \forall E$$
$$\mu_N(E) = \frac{1}{2}(\mu_T(E) - \mu_F(E)) = \frac{1}{2}(|\mu_F|(E) - \mu_F(E)) = \frac{1}{2}(2\mu_T^-(E)) = \mu_T^-(E) \quad \forall E$$

□

---

30. *Construct an increasing function on $\mathbb{R}$ whose set of discontinuities is $\mathbb{Q}$.*

Proof. .
Let $\mathbb{Q} = \{q_n\}_{n=1}^{\infty}$ such that $q_n < q_{n+1}\ \forall n \in \mathbb{N}$ and let f be a function defined as below.

$$f = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{[q_n, \infty)} \qquad f_n =^{let} \frac{1}{2^n} \chi_{[q_n, \infty)}$$

Then f is increasing function.

For each irrational point x, since f_n is continuous at x and $|f_n| \leq \frac{1}{2^n}\ \forall n \in \mathbb{N}$, by Weierstrass M test, f is continuous at x.
Let $n \in \mathbb{N}$ and $\delta > 0$ be given. Let $x = q_n + \frac{\delta}{2}$, then $x \in (q_n - \delta, q_n + \delta)$, but $|f(q_n) - f(q_n + \frac{\delta}{2})| \geq \frac{1}{2^{n+1}}$.
Thus, f is discontinuous at every rational number. □

31.

Proof. □

32.

Proof. ∎

---

33. *If F is increasing on $\mathbb{R}$, then $F(b) - F(a) \geq \int_a^b F'(t)dt$.*

Proof. .
Define $F(x) = F(b)\ \forall x \geq b$, and let $f_n(x) = n\left(F(x+\frac{1}{n}) - F(x)\right)$. Since F is increasing, F is measureable and $f_n \geq 0$, so f_n is positively measurable function. Also, recalling that F is observe the below.

$$\lim_{n\to\infty} \frac{F(x+\frac{1}{n}) - F(x)}{\frac{1}{n}} = F'(x)$$

Then by Fatou's Lemma,

$$\int_a^b F'(t)dt = \int_a^b \lim_{n\to\infty} \frac{F(t+\frac{1}{n}) - F(t)}{\frac{1}{n}}dt \leq \liminf_{n\to\infty} \int_a^b \frac{F(t+\frac{1}{n}) - F(t)}{\frac{1}{n}}dt = \liminf_{n\to\infty} n\int_a^b F(t+\frac{1}{n}) - F(t)dt$$

Observe the below

$$n\int_a^b F(t+\frac{1}{n}) - F(t)dt = n\left\{\int_{a+\frac{1}{n}}^{b+\frac{1}{n}} F(t)dt - \int_a^b F(t)dt\right\} = n\left\{\int_b^{b+\frac{1}{n}} F(t)dt - \int_a^{a+\frac{1}{n}} F(t)dt\right\}$$

Since F is increasing,

$$\liminf_{n\to\infty} n\left\{\int_b^{b+\frac{1}{n}} F(t)dt - \int_a^{a+\frac{1}{n}} F(t)dt\right\} \leq \liminf_{n\to\infty} \frac{\int_b^{b+\frac{1}{n}} F(b)dt - \int_a^{a+\frac{1}{n}} F(a)dt}{\frac{1}{n}} = \liminf_{n\to\infty} \frac{F(b)\frac{1}{n} - F(a)\frac{1}{n}}{\frac{1}{n}}$$
$$= F(b) - F(a)$$

Therefore,

$$\int_a^b F'(t)dt \leq F(b) - F(a)$$

∎

34.

Proof. ∎

35.

Proof. ∎

---

36. *Let G be a continuous increasing function on [a.b] and let G(a)=c, G(b)=d.*
a. *If $E \subset [c,d]$ is a Borel set, then $m(E) = \mu_G(G^{-1}(E))$. (First consider the case where E is an interval.)*
b. *If f is a Borel measurable and integrable function on [c,d], then $\int_c^d f(y)dy = \int_a^b f(G(x))dG(x)$. In particular, $\int_c^d f(y)dy = \int_a^b f(G(x))G'(x)dx$ if G is absolutely continuous.*
c. *The validity of (b) may fail if G is merely right continuous rather than continuous.*

Proof. .
a.

Let $A = \{E \subset [c,d] | m(E) = \mu_G(G^{-1}(E))\}$, and let I be an interval such that $\inf I = a < \infty$ and $\sup I = b < \infty$. Note that $G^{-1}(I)$ is an interval since G is continuous. And since G is increasing, $\inf G^{-1}(I) = G^{-1}(a)$ and $\sup G^{-1}(I) = G^{-1}(b)$. Thus, observe the below.

$$\mu_G(G^{-1}(I)) = G(G^{-1}(b)) - G(G^{-1}(a)) = b - a = m(I)$$

Thus, for any interval I, $I \in A$.
Also, note the below.

$$\begin{aligned} E \in A &\implies \mu_G(G^{-1}(E^c)) = \mu_G((G^{-1}(E))^c) = \mu_G([a,b]) - \mu_G(G^{-1}(E)) = m([a,b]) - m(E) = m(E^c) \\ &\implies E^c \in A \end{aligned}$$

If $\{E_n\}_{n\in\mathbb{N}} \subset A$ is a sequence of disjoint sets,

$$\begin{aligned} &\mu_G\left(G^{-1}(\bigcup_{n\in\mathbb{N}} E_n)\right) = \mu_G\left(\bigcup_{n\in\mathbb{N}} G^{-1}(E_n)\right) = \sum_{n\in\mathbb{N}} \mu_G\left(G^{-1}(E_n)\right) = \sum_{n\in\mathbb{N}} m(E_n) = m\left(\bigcup_{n\in\mathbb{N}} E_n\right) \\ &\implies \bigcup_{n\in\mathbb{N}} E_n \in A \end{aligned}$$

Thus, A is an $\sigma-$algebra containing all interval which means $\mathcal{B}_{[c,d]} \subset A$. So we are done.

b.

Observe the below.

$$\int_c^d \chi_E(y)dy = m(E) = \mu_G(G^{-1}(E)) = \int_a^b \chi_{G^{-1}(E)}(x)d\mu_G(x) = \int_a^b \chi_E(G(x))dG(x).$$

where χ_E is a characteristic function and E is in $\mathcal{B}_{[c,d]}$.

Then, if $\phi = \sum_{k=1}^n a_k\chi_{E_k}$ is a positive $\mathcal{B}_{[c,d]}-$measurable simple function,

$$\int_c^d \phi(y)dy = \int_c^d \sum_{k=1}^n a_k\chi_{E_k}(y)dy = \sum_{k=1}^n a_k \int_c^d \chi_{E_k}(y)dy = \sum_{k=1}^n a_k \int_a^b \chi_{E_k}(G(x))dG(x) = \int_a^b \phi(G(x))dG(x).$$

So it is true for any positive $\mathcal{B}_{[c,d]}-$measurable simple function.

Also, due to monotone convergence theorem, it is true for positive $\mathcal{B}_{[c,d]}-$measurable function, f, as we can observe below.

$$\int_c^d f(y)dy = \lim_{n\to\infty} \int_c^d \phi_n(y)dy = \lim_{n\to\infty} \int_a^b \phi_n(G(x))dG(x) = \int_a^b f(G(x))dG(x)$$

Where $\{\phi_n\}_{n\in\mathbb{N}}$ is an increasing sequence of simple functions converging to f.

As for the Borel measurable integrable function f, we need to firstly observe the below.

$$\int_a^b |f(G(x))|dG(x) = \int_c^d |f(y)|dy < \infty$$

From here, by definition of integration of integrable function, we get the below.

$$\int_c^d f(y)dy = \int_c^d f^+(y)dy - \int_c^d f^-(y)dy = \int_a^b f^+(G(x))dG(x) - \int_a^b f^-(G(x))dG(x) = \int_a^b f(G(x))dG(x)$$

What if G is absolutely continuous?

Note that $\mu_G([a,x]) = \int_a^x G'(x)dx$ and $G'(x) \in L^+$ since G is increasing. By the homework question 14 from chapter 2,

$$\int_c^d f(y)dy = \int_a^b f(G(x))dG(x) = \int_a^b f(G(x))G'(x)dx$$

c.

Think of the situation when $f(x) = y$ and $G(x) = \chi_{[0,\infty)}$. Then f is continuous so Borel measurable on $[0,1]$ and G is increasing right continuous function, but since $G(-1) = 0$ and $G(1) = 1$

$$\int_0^1 ydy = \frac{1}{2} \neq 1 = 1 - 0 = G(1) - G(0^-) = \mu_G([0,1]) = \int_{-1}^1 G(x)dG(x) = \int_{-1}^1 f(G(x))dG(x)$$

□

37.

Proof. □

38.

Proof. □

---

39. *If $\{F_j\}$ is a sequence of nonnegative increasing function on [a,b] such that $F(x) = \sum_{n=1}^{\infty} F_j(x) < \infty$ for all $x \in [a,b]$, then $F'(x) = \sum_{j=1}^{\infty} F_j'(x)$ for a.e. $x \in [a,b]$.(It suffices to assume $F_j \in NBV$. Consider the measures μ_{F_j})*

Proof. .
We can always assume that $F_j \in NBV$ by redefine the function as below.

$$\hat{F_j}(x) = F_j(x^+)$$

then $\hat{F_j}' = F_j' a.e.$ and let $\overline{F_j} = \hat{F_j}(x) - \hat{F_j}(a)$ and

$$\begin{aligned} \overline{F_j}(x) &= 0 && \forall x < a \\ \overline{F_j}(x) &= \overline{F_j}(b) && \forall x > b \end{aligned}$$

Then $\overline{F_j} \in NBV$ and $\overline{F_j}' = \hat{F_j}' = F_j'$.

Assume that $F_j \in NBV$, then there is a Lebesgue decomposition of μ_{F_j} and μ_F with respect to Lebesgue measure, m, as below.

$$\mu_{F_j} = \lambda_j + \rho_j$$

where $\lambda_j \perp m$, and $\rho_j \ll m$. Let's define two measures as below.

$$\rho = \sum_{j=1}^{\infty} \rho_j \qquad \lambda = \sum_{j=1}^{\infty} \lambda_j$$

Let's prove that $\lambda \perp m$.

First suppose that $\{A_j, B_j\}$ is a partition for λ_j where A_j is λ_j-null and B_j is $m-$null. Then $\bigcup_{j=1}^{\infty} B_j$ is $m-$null since

$$m\left(\bigcup_{j=1}^{\infty} B_j\right) \leq \sum_{j=1}^{\infty} m(B_j) = 0$$

Automatically, $\bigcap_{j=1}^{\infty} A_j$ is $\lambda-$null set and $\left\{\bigcap_{j=1}^{\infty} A_j, \bigcup_{j=1}^{\infty} B_j\right\}$ is a partition. So $\lambda \perp m$.

Let's prove that $\rho \ll m$.

Observe the below.

$$\rho((-\infty, x]) = \sum_{j=1}^{\infty} \rho_j((-\infty, x]) = \sum_{j=1}^{\infty} \int_{-\infty}^{x} \frac{d\rho_j}{dm} dm = \lim_{n\to\infty} \sum_{j=1}^{n} \int_{-\infty}^{x} \frac{d\rho_j}{dm} dm = \lim_{n\to\infty} \int_{-\infty}^{x} \sum_{j=1}^{n} \frac{d\rho_j}{dm} dm$$

Since F_j are increasing functions, so since $F_j' \geq 0$, by monotone convergence theorem,

$$\rho((-\infty, x]) = \lim_{n\to\infty} \int_{-\infty}^{x} \sum_{j=1}^{n} \frac{d\rho_j}{dm} dm = \int_{-\infty}^{x} \lim_{n\to\infty} \sum_{j=1}^{n} \frac{d\rho_j}{dm} dm = \int_{-\infty}^{x} \sum_{j=1}^{\infty} \frac{d\rho_j}{dm} dm$$

Thus, $\rho \ll m$.
Now observe the below.

$$\mu_F((x, y]) = \sum_{j=1}^{\infty} \mu_{F_j}((x, y]) = \sum_{j=1}^{\infty} \lambda((x, y]) + \sum_{j=1}^{\infty} \rho_j((x, y]) = \lambda((x, y]) + \rho((x, y])$$

Since μ_F and m are $\sigma-$finite, the Lebesgue decomposition is unique.
Recall that $F_j'(x) = \frac{d\rho_j}{dm}$ $m-a.e$, then

$$F'(x) = \frac{d\rho}{dm} = \sum_{j=1}^{\infty} \frac{d\rho_j}{dm} = \sum_{j=1}^{\infty} F'(x) \qquad m-a.e.$$

□

---

40. *Let F denote the Cantor function on [0,1](see ∮1.5), and set F(x)=0 for* $x < 0$ *and F(x)=1 for* $x > 1$. *Let* $\{[a_n, b_n]\}$ *be an enumeration of the closed subintervals of [0,1] with rational endpoints,and let* $F_n(x) = F((x-a_n)/(b_n-a_n))$. *Then* $G = \sum_{n=1}^{\infty} \frac{F_n}{2^n}$ *is continuous and strictly increasing on [0,1], and* $G' = 0$ *a.e.(Use Ex 39.)*

Proof. .
Observe that $F_n(x) \leq 1$ $\forall x \in [0,1]$, so by Weierstrass M-test, the series,$\sum_{n=1}^{\infty} 2^{-n} F_n$ converges uniformly on $[0,1]$. Since each F_n is continuous, G is also continuous.

Let $a, b \in [0,1]$ such that $a < b$. Then there exists k such that $a < a_k < b_k < b$. Observe the below.

$$G(b) - G(a) = \sum_{j=1}^{\infty} 2^{-n} \{F_n(b) - F_n(a)\} \geq \frac{1}{2^k} \{F_k(b) - F_k(a)\} = \frac{1}{2^k} > 0$$

Thus, G is strictly increasing on $[0,1]$.
By Exercise 39, since F_n is a sequence of nonnegative increasing functions,

$$G'(x) = \sum_{j=1}^{\infty} 2^{-n} F_n'(x) = 0 \qquad m-a.e.$$

since $F_n' = 0$ $m-a.e.$

□

---

41. *Let* $A \subset [0,1]$ *be a Borel set such that* $0 < m(A \cap I) < m(I)$ *for every subinterval I of [0,1](Exercise 33, Chapter 1)*
a. *Let* $F(x) = m([0,x] \cap A)$. *Then F is absolutely continuous and strictly increasing on [0,1], but F'=0 on a set of positive measure.*
b. *Let* $G(x) = m([0,x] \cap A) - m([0,x] \setminus A)$. *Then G is absolutely continuous on [0,1], but G is not monotone on any subinterval of [0,1].*

Proof. .
a.
Note the below.

$$F(x) = m([0,x] \cap A) = \int_0^x \chi_A dm$$

Then clearly, F is absolutely continuous.

And for any $a, b \in [0,1]$ with $a < b$, observe the below.

$$F(b) - F(a) = m(A \cap [a,b]) > 0$$

by given condition. So F is strictly increasing.
However, observe the below.

$$F'(x) = \chi_A(x) \quad m-a.e.$$

Thus, $F'(x) = 0$ on $[0,1] \cap A^c$. And following relation from the given condition shows it is a set of positive measure.

$$m([0,1] \cap A^c) = m([0,1]) - m([0,1] \cap A) > 0$$

b.
Note the below.

$$G(x) = \int_{[0,x]} \chi_A dm - \int_{[0,1]} \chi_{A^c} dm = \int_{[0,1]} \chi_A - \chi_{A^c} dm$$

Thus, G is absolutely continuous on $[0,1]$ and $G' = \chi_A - \chi_{A^c}\ m - a.e.$
And observe the below.

$$G'(x) = \chi_A(x) - \chi_{A^c}(x) = 1 - 0 = 1 \qquad \text{on } a.e.[0,1] \cap A$$
$$G'(x) = \chi_A(x) - \chi_{A^c}(x) = 0 - 1 = -1 \qquad \text{on } a.e.[0,1] \cap A^c$$

Now, for any subinterval $I \in [0,1]$, observe the below.

$$m(I \cap A \cap [0,1]) = m(I \cap A) > 0$$
$$m(I \cap A^c \cap [0,1]) = m([0,1] \cap I) - m([0,1] \cap A \cap [0,1]) = m(I) - m(A \cap I) > 0$$

Therefore, G is not monotone on any subinterval $I \subset [0,1]$ □

42.

Proof. □

43.

Proof. □

44.

Proof. □

45.

Proof. □

46.

Proof. □

47.

Proof. □

48.

Proof. □

49.

Proof. □

50.

Proof. □

51.

Proof. □

52.

Proof. □

53.

Proof. □

54.

Proof. □

55.

Proof. □

56.

Proof. □

57.

Proof. □

58.

Proof. □

59.

Proof. □

60.

Proof. □

61.

Proof. □

62.

Proof. □

63.

Proof. □

64.

Proof. □
