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Velocity Coefficients For Free Jets From Sharp-Edged Orifices

The viscosity-dependence of the velocity coefficient for a free liquid jet, issuing from a sharp-edged orifice, is predicted by computing the dissipation of energy in the boundary layer on the back of the orifice plate. The prediction is upheld by the only known direct measurements of velocity coefficients. The resulting coefficients are much closer to unity for large orifices than they are generally assumed to be. The influence of surface tension on small jets is also explained.

Objective

The common wisdom of the textbooks has it that the coefficient of velocity for a free jet leaving sharp-edged orifice is about 0.98 and that it is weakly dependent on viscosity. Nothing is normally said about the influence of surface tension. The issue has lain fallow in this state since before WW II.

An increasing use of miniature fluid flows in modern technologies gives us reason to re-examine this issue. Such applications as the IBM ink-jet printer (see e.g. [1]), the use of small free jets to achieve very high heat removal rates (see e.g., [2]), the use of colliding jets to create combustion sprays (see e.g. [3]), and many other configurations create a need to know more about the velocity of small jets.

Our aim is therefore to predict the velocity coefficient, C_v, chiefly for the most basic delivery system – a sharp-edged orifice. To do this we calculate the influence of viscosity, and question the role of surface tension as well.

On Measurements of C_v

Figure 1 shows the configuration of a sharp-edged orifice, and of a Borda mouthpiece. It also defines the terms we use. These include the coefficient of discharge, C_D; the coefficient of contraction, C_c; and the coefficient of velocity, C_v.

By 1908 many detailed measurements of C_D had been made for sharp-edged orifices, and it was well-known that for ideal flows:

$$C_c = \frac{\pi}{\pi + 2} = 0.6110 \qquad (1)$$

Some C_v's had been measured by the ballistic method or by measuring the rise of a vertically oriented jet (see e.g., [4].) Both methods underestimated C_v by including aerodynamic losses. Direct pitot tube measurements were not very accurate. Often C_v was reported as C_D/C_c, where C_c had been obtained with calipers or simply assumed to be 0.611. It was not uncommon to report the *assumption* that C_v was 0.97 or 0.98. This state of affairs is clearly reported in [4].

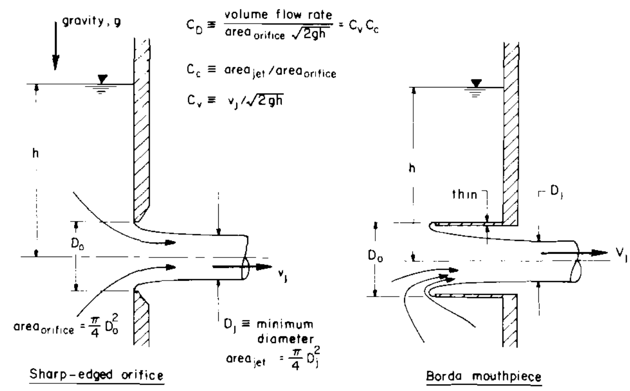

Fig. 1 Configuration and definition of terms

In 1908 Judd and King [5] conducted a remarkably accurate experiment in which they independently measured: C_D with a catch tank, C_c with a micrometer calipers, and C_v with a *differential* pitot tube – one that compared the dynamic pressure, in traverses across the jet, with static pressure upstream. The three measurements satisfied $C_D = C_c C_v$ very closely. For $D_o \geq 2$ in. and $h \geq 10$ ft they obtained $0.99995 \leq C_v \leq 0.99999$, and all their C_v values exceeded 0.9995.

Subsequent measurements of C_D culminated in the work of Medaugh and Johnson [6] who used a 1 in. orifice and found that C_D approached 0.595 at high heads. Unfortunately Judd and King measured C_D closer to 0.61 in their 1 in. orifice. Even though their largest orifice also gave $C_D \simeq 0.595$, the comparison of their 1 in. orifice data with Medaugh and Johnson's results had the unfortunate effect of impugning their otherwise good work.

Medaugh and Johnson actually pointed out that C_c is highly susceptible to any minor malformation of an orifice. This explains why Judd and King's smaller orifice gave higher values of both C_c and C_D even though their measurements were accurate.

A careful reading of the literature up to 1940 thus shows

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that for large accurately-shaped sharp-edged orifices under high heads

$$C_c \leqslant 0.595 \quad \text{and} \quad C_v = 1.0000 \tag{2}$$

while the textbooks have reported

$$C_c = 0.611 \quad \text{and} \quad C_v \simeq 0.98$$

We next undertake to make a prediction of C_v that will reproduce Judd and King's data and apply to much smaller orifices as well. We presume that $C_v = fn(D_0, gh, \sigma, \rho, \mu)$, so the prediction should take the dimensionless form:

$$C_v = f(\text{Re, We}) \tag{3}$$

where we use the ideal jet velocity, $\sqrt{2gh}$, to define:

Reynolds No., $\text{Re} \equiv \rho\sqrt{2gh}\, D_0/\mu = \sqrt{2gh}\, D_0/\nu$

Weber No., $\text{We} \equiv \rho(2gh)D_0/\sigma$

and where it remains to be seen whether We really influences C_v.

Mechanical Energy Balance

Our analysis is based on conservation of mechanical energy. By constructing the balance among incoming, outgoing, and dissipated mechanical energy we are able to determine the roles of viscosity and surface tension in retarding a liquid jet issuing from a sharp-edged orifice.

We consider a control volume (C.V.) surrounding a sharp-edged orifice, with liquid entering far above the orifice and exiting at a downstream point in the jet, where contraction has been fully completed. Denoting the volume flowrate as Q and the ambient pressure as p_∞, we have:

rate of mech. energy in

= rate of mech. energy out + rate of visc. dis.

For the circular orifice this takes the form:

$$\underbrace{\rho g h Q}_{(1.)} + \underbrace{p_\infty Q}_{(2a.)} = \underbrace{p_\infty Q}_{(2b.)} + \underbrace{(\rho v_j^2/2)Q}_{(3.)} - \underbrace{\pi D_j \sigma v_j}_{(4a.)} + \underbrace{\pi D_j \sigma v_j}_{(4b.)} + \underbrace{\dot{E}_\mu}_{(5.)} \tag{4}$$

where the significance of the terms is as follows:

(1.) rate of potential energy into C.V.
(2*a*.) rate of work done on C.V. by atmosphere on top.
(2*b*.) rate of work done by C.V. on atmosphere at exit.
(3.) rate of outflow of kinetic energy
(4*a*.) rate of work done on C.V. by surface tension
(4*b*.) rate of outflow of surface energy
(5.) rate of viscous dissipation.

Terms (2*a*. and 4*a*.) cancel (2*b*.) and (4*b*.) so we are left with

$$\rho g h Q = (\rho v_j^2/2)Q + \dot{E}_\mu \tag{4}$$

Thus, no net pdV work is done, and there is also no net effect of surface tension. Using $v_j^2 = C_v^2(2gh)$, we can rearrange equation (4) as

$$C_v = \sqrt{1 - \frac{\dot{E}_\mu}{\dot{E}_f}} \tag{5}$$

where $\dot{E}_f = \rho g h Q$ is a characteristic kinetic energy associated with the liquid efflux (note that the square of the ideal jet velocity is $2gh$). This is the desired expression for C_v. However, before evaluating $\dot{E}_\mu$, we should consider more carefully the surprising disappearance of surface tension.

The Influence of Surface Tension

Our energy accounting shows the clean cancellation of the surface energy outflow and work done by surface tension in the contracted portion of the jet before Rayleigh breakup occurs. Yet net surface energy *is* carried away. We therefore look for the exchange between kinetic energy and surface energy to be made where Rayleigh breakup occurs, but not before.[1] The overall surface energy of the finite unbroken length of a jet stays more-or-less the same once the jet and its breakup length are established; and a continuous exchange of surface tension work with surface energy takes place within it.

However in the breakup portion, wavy segments are nipped off on the downstream side, creating an unbalanced force on the upstream side until it is too nipped off. The absence of the downstream surface tension force prevents the upstream transfer of surface tension work which allowed the surface energy to be smoothly transported downstream without affecting the jet. The only influence surface tension *can* have on the jet velocity, is that of retarding the droplets during breakup.

[1] We are most grateful to Lloyd M. Trefethen [7] for extremely helpful discussions in which he helped us to see through this paradoxical situation.

Nomenclature

C = constant in $U(r) = Cr^m$
C.V. = control volume
C_c, C_D, C_v = coefficients of contraction, discharge, and velocity defined in Fig. 1
C_{v_d} = coefficient of velocity based on droplet velocity, v_d
D_0, D_j = diameters of orifice and of contracted jet (see Fig. 1)
$\dot{E}_f$ = a characteristic rate of kinetic energy flow in a jet, $\rho g h Q$
$\dot{E}_\mu$ = rate of dissipation of energy as a result of the jet
$f(\eta)$ = dimensionless stream function (see equation (11))
g = gravitational body force per unit mass
h = head
k = constant which defines axisymmetric body shape: $r_0 \propto r^k$
K_1, K_2 = constants defined in equations (16) and (18)
m = constants in $U(r) = Cr^m$
p_∞ = ambient pressure
Q = flow rate (m^3/s in 3-dim case, m^2/s in 2-dim case)
r, y = coordinates along the surface of an axisymmetric body in the direction of flow, and normal to it
r_0 = radius of revolution of an axisymmetric body
Re = Reynolds number, $D_0\sqrt{2gh}/\nu$
$U(r)$ = velocity of flow just outside of a boundary layer
u, v = r and y velocity components
V = volume
v_j, v_d = actual velocity of jet; actual velocity of droplets after Rayleigh breakup
We = Weber number, $\rho D_0(2gh)/\sigma$
α = a positive constant in equations (11) and (12) which takes the form $2k+1$
η = similarity parameter defined by equation (12)
μ, ν = viscosity; kinematic viscosity $= \mu/\rho$
ρ = density of liquid
σ = surface tension

We can clarify this by balancing mechanical energy over a C.V. containing only the portion of the jet undergoing varicose instability. The net rate of energy inflow from upstream is $(\rho g h Q - \dot{E}_\mu)$, and (with the other end of the C.V. beyond the end of the jet) the net outflow is zero. The droplets then store kinetic and surface energies at the rate

$$\left(\frac{1}{2}\rho v_d{}^2 + \pi D_j v_d\right)Q,$$

where v_d is the droplet velocity. No net work is done and we neglect any viscous dissipation by the surface waves. If we let $C_{v_d} \equiv v_d/\sqrt{2gh}$, then some algebra gives

$$C_{v_d} \equiv \sqrt{1 - \frac{8}{\mathrm{We}\sqrt{C_c}} - \frac{\dot{E}_\mu}{\dot{E}_f}} \tag{6}$$

for a circular jet. The same logic gives

$$C_{v_d} = \sqrt{1 - \frac{4}{\mathrm{We}C_c} - \frac{\dot{E}_\mu}{\dot{E}_f}} \tag{7}$$

for a slot jet, where C_c for a slot is D_j/D_0 instead of $(D_j/D_0)^2$. As anticipated, the effect of surface formation is to retard the droplets formed at breakup.

This situation is quite evident when we view the breakup of water bells created by the coaxial collision of two equal jets at modest values of We. (See, e.g., the photographs in [8]). The resulting sheets (or water bells) spread out very thin but they suffer *no reduction* of velocity until surface forces exactly balance momentum. Then the large beads of liquid that form are observed to leave with a much reduced velocity.

Thus, while equations (6) and (7) apply to the drops formed when the jets break up, they do *not* apply to the unbroken jet, and $C_v \neq fn(\mathrm{We})$. Conversely, when the sum of the rates of creation of surface energy and of viscous loss exceed the rate of supply of potential energy, the radicals in equations (6) and (7) become imaginary, signifying that liquid can no longer escape from the orifice. As this condition approaches, the breakup region moves upstream toward the orifice, we lose the well-defined region of full contraction, the surface forces become increasingly dominant (We decreases), and C_{v_d} finally reaches zero when the jet can no longer flow freely.

To check this limiting behavior, Chen [9] ran the following experiment: He glued standard 0.65 mm, 0.749 mm, and 1.50 mm ASME sharp-edged brass orifices to the bottom of a 20 mm ID vertical graduated tube. Water inflow to the tube was regulated to give a very slowly falling head. When the vertically issuing jet stops flowing freely and starts chugging, we call $C_{v_d} = 0$. At that point, water can only escape by repeatedly wetting the metal outside the hole and oozing out. The only significant "error" in this experiment is that related to identifying the exact point which chugging begins. That uncertainty is about ±10 percent.

The results of the experiments were as follows (we neglect $\dot{E}_\mu$ since there can be no dissipation when there is no flow.):

D_0(mm)	minimum h (mm) for steady flow	$\mathrm{We} = \frac{\rho D_0 (2gh)}{\sigma}$	C_c if $\mathrm{We}\sqrt{C_c} = 8$
0.65	30	7.95	1.01
0.794	25	7.76	1.06
1.50	13	7.73	1.07

The results verify that surface tension throttles the flow as we would expect it to do. The fact that C_c is on the order of unity is consistent with our understanding that contraction is completely suppressed in small enough orifices and slow enough flow rates.

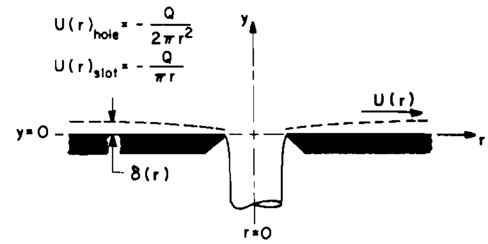

Fig. 2 Potential flows for boundary analyses

The Influence of Viscous Dissipation

We now return to the problem of evaluating $\dot{E}_\mu$, the rate of viscous dissipation of energy, which must be known in order to evaluate *either* C_v or C_{v_d}.

The viscous dissipation is obtained by integrating the incompressible dissipation function, $\mu(\partial u/\partial y)^2$, over and through the volume, V, of the boundary layer (see notation in Fig. 2.). Thus

$$\dot{E}_\mu = \mu \int_V (\partial u/\partial y)^2 dV \tag{8}$$

To evaluate this integral we must find $\partial u/\partial y$ in the boundary layer. The axisymmetric boundary layer equations are

$$\left.\begin{aligned} &\frac{\partial}{\partial r}(r_0 u) + r_0\frac{\partial v}{\partial y} = 0 \\ &u\frac{\partial u}{\partial r} + v\frac{\partial u}{\partial y} = U\frac{dU(r)}{dr} + \nu\frac{\partial^2 u}{\partial y^2} \end{aligned}\right\} \tag{9}$$

where $r_0 = r_0(r)$ is the radius of revolution of body on which the boundary layer lies and r is the coordinate along the surface ($r_0 = r$ for the orifice plate). The pressure gradient term (see Fig. 2) becomes:

$$U\frac{dU}{dr} = \begin{cases} -\dfrac{Q}{2\pi^2}\dfrac{1}{r^5} \text{ for a hole where } Q = Q\dfrac{m^3}{s} \\ -\dfrac{Q}{\pi^2}\dfrac{1}{r^3} \text{ for a slot where } Q = Q\dfrac{m^2}{s} \end{cases} \tag{10}$$

if we use the far-field velocity distribution along the wall.

It is easy to show that the velocity potential at the wall for a two-dimensional slot flow (as given, for example, in [10]) has exactly the far-field form ($U(r) = Q/\pi r$) all the way up to the lip. We have *presumed* that this is also the case for flow through a circular hole.

Axisymmetric boundary layer flows for which $U(r) = Cr^m$ are self-similar under the transformation $(r,y) \rightarrow (r,\eta)$, with the stream function:

$$\Psi(r,\eta) = r_0(\pm \nu r U/\alpha)^{1/2} f(\eta) \tag{11}$$

and the similarity variable:

$$\eta = y(\pm \alpha U/\nu r)^{1/2} \tag{12}$$

where: α is an arbitrary constant, greater than zero; we consider $r_0 \propto r^k$ (where k is a constant); and the minus sign applies when the constant, C, is negative. If α is chosen as $2k+1$, we obtain the $f(\eta)$ appropriate to the Falkner-Skan flow for which $U \propto r^{m/\alpha}$ (see e.g., Batchelor [13], Sect. 5.9; White [11], Sect. 4-3, 4-9.)

Equation (8) now becomes

$$\dot{E}_\mu = \mu \int_V [U(r)]^2 \eta_y^2 (f''(\eta))^2 dV \tag{13}$$

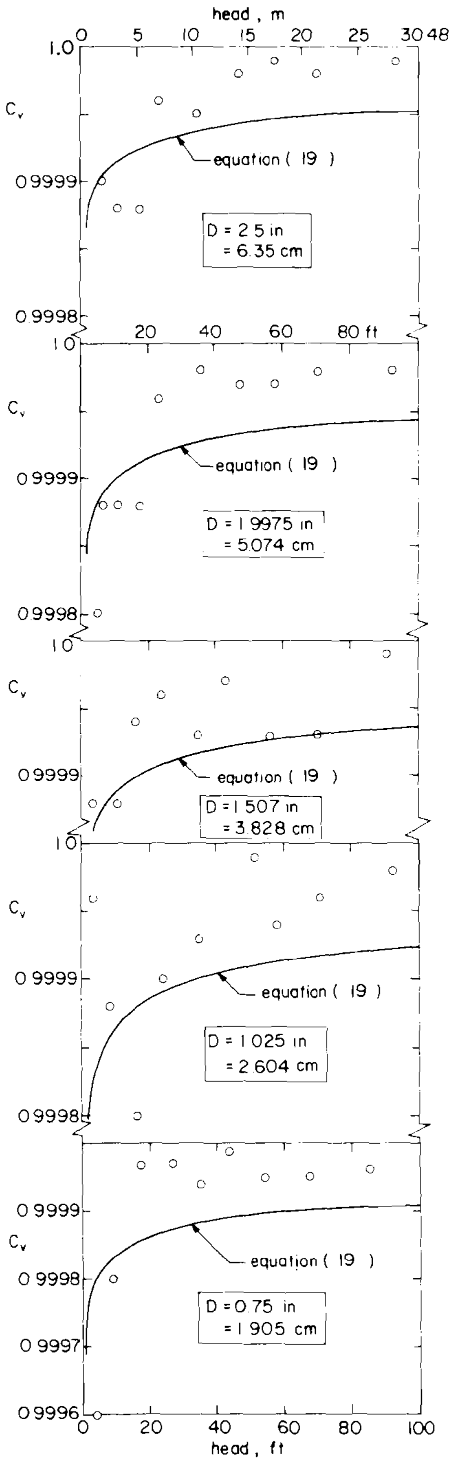

Fig. 3 Comparison of equation (19) with the data of Judd and King

and dV can be transformed with

$$\left.\begin{aligned} &2\pi r dr dy = 2\pi\, |J(r,\eta)|\, r dr d\eta \\ &\qquad \text{for the circular hole} \\ &1\, dr dy = 1\, |J(r,\eta)|\, dr d\eta \\ &\qquad \text{for the slot (per unit depth).} \end{aligned}\right\} \tag{14}$$

and the Jacobian is evaluated as

$$|J(r,\eta)| = |J(r,y)|^{-1} = \frac{1}{\eta_y} \tag{15}$$

Combining equations (13), (14), and (15), and using various preceding definitions, we get

$$\dot{E}_\mu = \dot{E}_f \frac{C_D{}^{3/2}}{\sqrt{\text{Re}}} K_1 \int_0^\infty (f''(\eta))^2 d\eta \tag{16}$$

where

$$\left.\begin{aligned} &K_1 = 0.494872 \quad \text{for the circular hole} \\ &K_1 = 0.457316 \quad \text{for the slot} \end{aligned}\right\} \tag{17}$$

For the slot, $r_0 = \text{constant} \rightarrow \infty$ and we recover a well-known Jeffrey-Hamel wedge flow (see e.g., [11] Sect. 3–8.7). For the circular hole, $r_0 = r$ and we obtain a nonlinear ordinary *d.e.* in $f(\eta)$. (The latter case is included by Crabtree, Küchmann, and Sowerby [12].)

The dissipation integral was evaluated numerically for both cases, giving

$$K_2 \equiv K_1 \int_0^\infty (f''(\eta))^2 d\eta = \begin{cases} 0.242738 & \text{for the hole} \\ 0.284832 & \text{for the slot} \end{cases} \tag{18}$$

where the estimated accuracy of K_2 is at least 5 significant figures. Thus

$$C_v = \sqrt{1 - \frac{K_2 C_D{}^{3/2}}{\sqrt{\text{Re}}}} \tag{19}$$

An easy calculation shows that 99 percent of the viscous loss occurs within 1.36 diameters of the edge of the circular hole (4.50 diameters for the slot), so that our infinite plate analysis is valid for fairly small plates if they have the appropriate potential flow.

Results

We thus advance equation (19) as the correct expression for C_v for jets leaving slots and orifices, before any air drag or droplet breakup has occurred. The expressions cannot be applied below

$$\text{We} \simeq \begin{cases} 8 & \text{for a circular hole} \\ 4 & \text{for a slot} \end{cases} \tag{20}$$

Equation (19) requires knowledge of C_c, however its influence is secondary. At high values of Re it is adequate to guess $C_c \simeq 0.6$, and even to simplify the computation by taking $C_D = C_c$ under the radical, although we have made no such simplifications here.

Equation (19) is compared with Judd and King's data in Fig. 3. The comparison is good within the variability of the data but that variability is clearly large. We should be aware that Judd and King's C_v data depended on measurements of differential heads on the order of (1/20) in. of water, with manometer deflections on the order of (1/2) in. If we bear in mind that both the prediction and the data focus on $1 - C_v^2$, then we recognize that the prediction lies among the data while the conventional value of $(1 - 0.98^2)$ is high by a factor of 1000. Equation (19) is thus the surest prediction of C_v presently available, and probably is more accurate than any existing data.

It is worth noting that the Borda mouthpiece (see Fig. 1) offers very little way in which any viscous dissipation could occur, since very little of the liquid approaches the hole over a wall. It is well-known (see e.g., [4]) that for a Borda mouthpiece

$$C_c = \frac{1}{2C_v^2} \quad \text{or} \quad C_D = \frac{1}{2C_v} \tag{21}$$

Since C_v must be very close to unity for virtually any Borda flow, we anticipate that C_c and C_D will be equal to 1/2 for a perfectly shaped mouthpiece.

Unfortunately no existing data for the Borda Mouthpiece have accuracy higher than about ± 2 percent thus we cannot verify this prediction. Furthermore C_D and C_c for Borda Mouthpieces, like those for sharp edged orifices, are vulnerable to minor machining defects in the vicinity of the lip.

Conclusions

1. Surface tension does not retard a liquid jet unless it completely stops it (see Conclusion 2). However it will retard the broken-up droplets approximately according to equation (5) or (6).

2. When $We < 8/\sqrt{C_c}$ any circular liquid jet flow will be choked off. When $We \leq 4/C_c$ a slot flow will be choked off.

3. C_v for a sharp-edged circular orifice or for a sharp-edged slot is given by equations (19) and (20).

4. C_v equals unity within 0.1 percent for almost any aperture for which $Re > 10{,}000$.

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