LIMITS

To Find a Limit: Summary

GOALS: 1. Understand how the **limit** of a function at a point is **different from** the **value** of the function at the point.
2. Use graphing to determine existence of a limit.
3. **Find limits analytically.**

Homework for Sections:
1.2 Finding Limits Graphically and Numerically
1.3 Evaluating Limits Analytically

1.2 p. 52 Definition of Limit: be aware
skip p. 53, top p. 54
1, [7, 9 graph only], 15-22all, 71

1.3 # 1, 3; 5, 9, 13, ... 25; 15, 27-37, 41-63, 67-73

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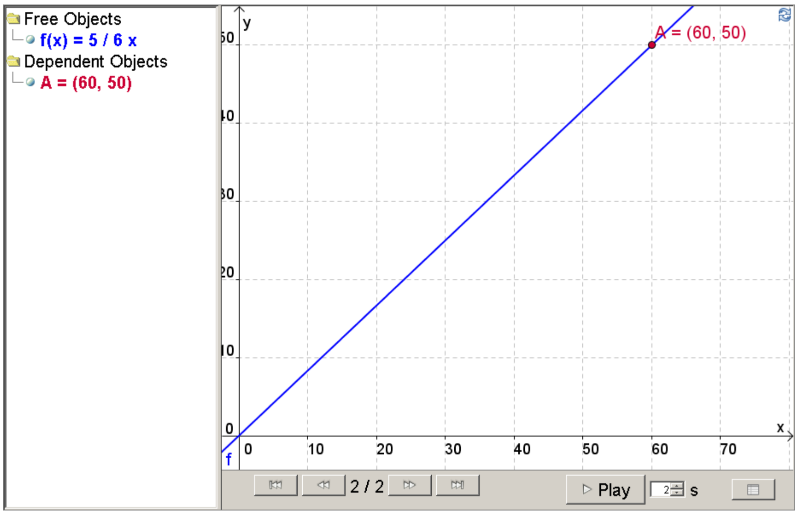
(sinx)/x*

1.2 Finding Limits Graphically and Numerically

Drive exactly 50 m/h.
What distance do you approach as you approach a time of 30 minutes?



geogebra

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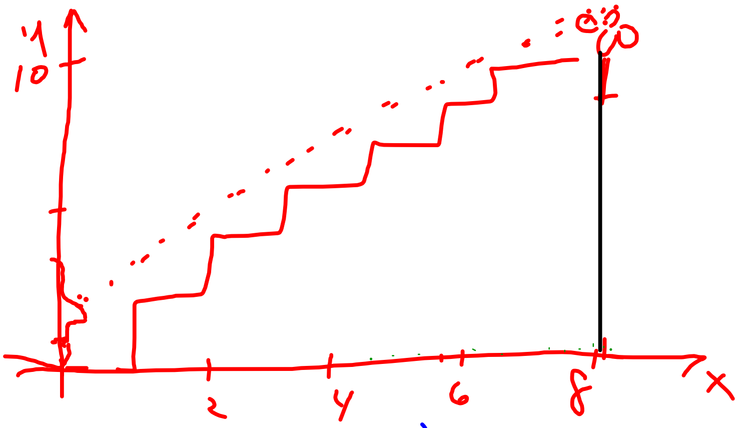
1.2 Finding Limits Graphically and Numerically



Does $f(8)$ exist?

Does $\lim_{x \to 8} f(x)$ exist?

Or: **Is there a value that y approaches as x → 8 ?**

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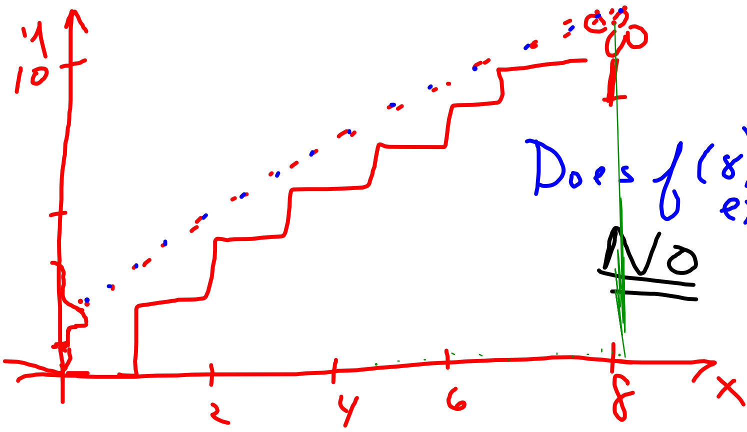
1.2 Finding Limits Graphically and Numerically



Does $f(8)$ exist?

NO

$\lim_{x \to 8} f(x) = \lim_{x \to 8} (\text{height of note}) = 10$

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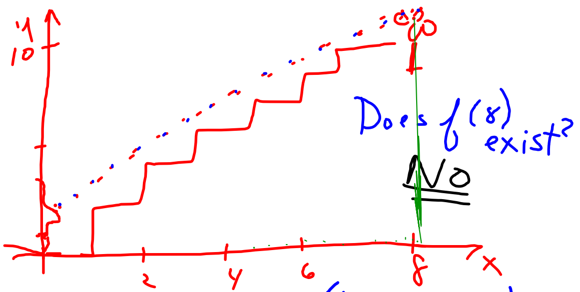
1.2 Finding Limits Graphically and Numerically



$\lim_{x\to 8} f(x) = \lim_{x\to 8}$ (height of note) = 10

Let *f* be a function defined on an open interval containing c, 8 except possibly at c, and let L be a real number. 10

Then $\lim_{x \to c} f(x) = L$ 10 (c = 8)

means that for each $\epsilon > 0\ \exists\ \delta > 0\ \ni$

if $0 < | x-c | < \delta$ then $| f(x)-L | < \epsilon$

x is near 8 eg: x=7.9 then δ =0.1 y is near 10 and ε is small

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1.2 Finding Limits Graphically and Numerically

How do you Find Limits?

1. Numerically - use only to show how the values are changing

$$\lim_{x\to 2} \frac{x-2}{x^2-4} \qquad x \neq \pm 2 \qquad f(2) = \frac{2-2}{4-4} = \frac{0}{0} \quad \text{DNE}$$

indeterminant

x	1.9	1.99	1.999	2.001	2.01	2.1
f(x)						

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1.2 Finding Limits Graphically and Numerically

How do you Find Limits?

1. Numerically - use only to show how the values are changing

$$\lim_{x \to 2} \frac{x-2}{x^2-4} \qquad x \neq \pm 2 \qquad f(2) = \frac{2-2}{4-4} = \frac{0}{0}$$ indeterminant

as x -> +2

x	1.9	1.99	1.999	2.001	2.01	2.1
f(x)	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

f(x) -> 1/4

$$\lim_{x \to 2} \frac{x-2}{x^2-4} = \frac{1}{4}$$

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1.2 Finding Limits Graphically and Numerically

How do you Find Limits?

1. Finding Limits Numerically
   Helpful
   - see where y values are going as x ->c

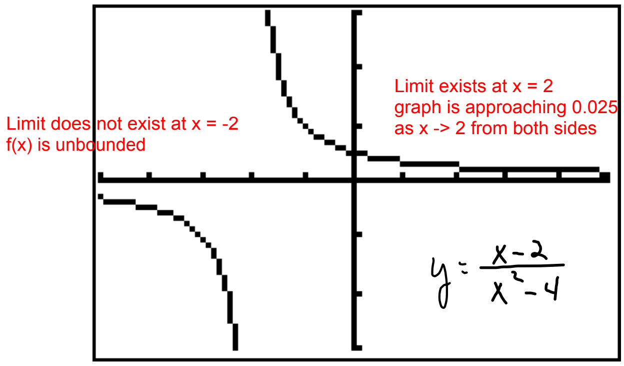
   Not so Helpful
   - tedious
   - distracts from overview of function

as x -> +2

x	1.9	1.99	1.999	2.001	2.01	2.1
f(x)	0.2564	0.2506	0.2501	0.2499	0.2494	0.2439

f(x) -> 1/4

2. Graphical Approach:
   Tells us if limit exists, & suggests a value



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1.2 Finding Limits Graphically and Numerically

When does a Limit NOT exist?

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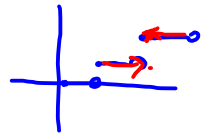
1.2 Finding Limits Graphically and Numerically

When does a Limit NOT exist?

1. $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$

$y = [x]$



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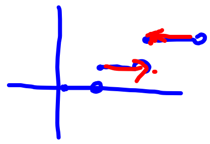
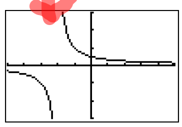


2. lim unbounded



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1.2 Finding Limits Graphically and Numerically

When does a Limit NOT exist?

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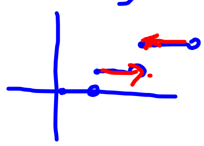

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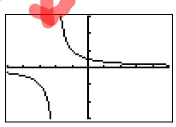
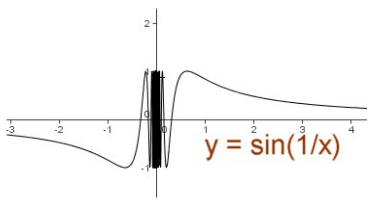


3. f oscillating

on calc:
-.05<x<.05
-1 <y< 1



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1.3 Evaluating Limits Analytically

Find Limits Analytically

$$f(x) = \frac{x-1}{x^2-1}, \quad x \neq \pm 1$$

Investigate limits as $x \Rightarrow +1$ and as $x \Rightarrow -1$

$x \Rightarrow +1$

Subst: $f(1) \rightarrow \frac{1-1}{1-1} = \frac{0}{0}$

Indeterminate Form

1. hole in graph at x=1
2. limit as x-> 1 exists

$x \Rightarrow -1$

$f(-1) \Rightarrow \frac{-2}{0}$ Unbounded DNE

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1.3 Evaluating Limits Analytically

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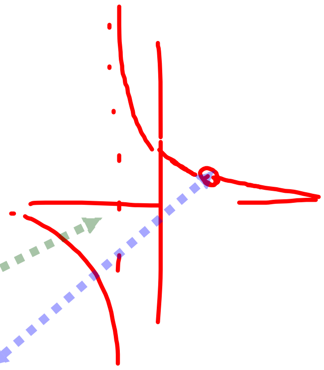
$$f(x) = \frac{x-1}{x^2-1}, \quad x \neq \pm 1$$

$$\text{Subst: } f(1) = \frac{1-1}{1-1} = \frac{0}{0}$$

$$f(x) = \frac{(x-1)}{(x+1)(x-1)} = \frac{1}{x+1} \cdot \frac{x-1}{x-1}$$



as x->-1
f(x) unbounded

as x->+1
f(x) has single
undefined point
(hole)

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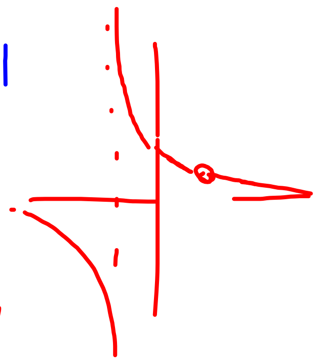
1.3 Evaluating Limits Analytically

Find Limits Analytically

$$f(x) = \frac{x-1}{x^2-1}, \quad x \neq \pm 1$$

$$\text{Subst: } f(1) = \frac{1-1}{1-1} = \frac{0}{0}$$

$$f(x) = \frac{\cancel{(x-1)}^{\,1}}{(x+1)\cancel{(x-1)}_{\,1}} = \frac{1}{x+1} \cdot \frac{x-1}{x-1}$$



$$\frac{x-1}{x^2-1} \stackrel{?}{=} \frac{1}{x+1} \leftarrow \left(1, \tfrac{1}{2}\right)$$

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1.3 Evaluating Limits Analytically

Find Limits Analytically

$f(x) = \dfrac{x-1}{x^2-1}, \ x \neq \pm 1$

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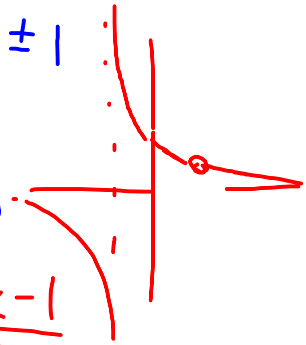

$f(x) = \dfrac{\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \dfrac{1}{x+1} \cdot \dfrac{x-1}{x-1}$

DNE

$\dfrac{x-1}{x^2-1} \overset{?}{=} \dfrac{1}{x+1} \leftarrow (1, \tfrac{1}{2})$

$(1, ?)$

The function on the left, f(x), is not defined at x=1 or x=-1.
The function on the right, 1/(x+1), is not defined at x= -1, but it is defined at x=1, at the point (1, 0.5).
So, these functions are equal, except for the point (1,0.5).
That means that as x -> 1, the y values of both are equal, and they are approaching the same limit, 0.5.

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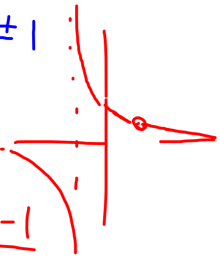
1.3 Evaluating Limits Analytically

Find Limits Analytically

$f(x) = \dfrac{x-1}{x^2-1}, \ x \neq \pm 1$

Subst: $f(1) = \dfrac{1-1}{1-1} = \dfrac{0}{0}$



$f(x) = \dfrac{\cancel{(x-1)}}{(x+1)\cancel{(x-1)}} = \dfrac{1}{x+1} \cdot \dfrac{x-1}{x-1}$

DNE

$\dfrac{x-1}{x^2-1} \overset{?}{=} \dfrac{1}{x+1} \leftarrow (1, \tfrac{1}{2})$

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So, these functions are equal, except for the point (1,0.5).
That means that as x -> 1, the y values of both are equal, and they are approaching the same limit, 0.5.

$\lim_{x \to 1} \dfrac{x-1}{x^2-1} = \lim_{x \to 1} \dfrac{1}{x+1} = \dfrac{1}{2}$

$\lim_{x \to -1} \dfrac{x-1}{x^2-1} = \lim_{x \to -1} \dfrac{1}{x+1} \to \dfrac{1}{0}$ Unbounded

undef.

DNE

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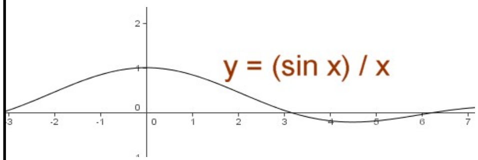
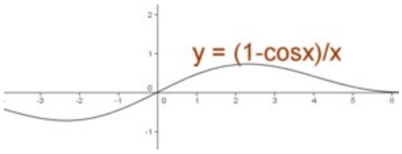
1.3 Evaluating Limits Analytically

Summary: To Find a Limit

1. **Substitute x = c**
   a) If finite number, L, then the limit is L.
   b) If results in form, k / 0 , then f is unbounded and the limit DNE
2. **If indeterminate, use algebra to find a function that is equivalent at all but the undefined point, and substitute again.** $\frac{0}{0}$ $\frac{\infty}{\infty}$
3. **If still indeterminate, consider special limits:**

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0$$





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(sinx)/x*

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1.3 Evaluating Limits Analytically

$$\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$$

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1.3 Evaluating Limits Analytically

$$\lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} \rightarrow \frac{0}{0}$$

$$\frac{\sqrt{2+x}-\sqrt{2}}{x} \cdot \frac{\sqrt{2+x}+\sqrt{2}}{\sqrt{2+x}+\sqrt{2}}$$

$$\frac{(2+x)-2}{x(\sqrt{2+x}+\sqrt{2})} = \frac{\cancel{x}^{1}}{\cancel{x}_{1}(\sqrt{2+x}+\sqrt{2})} = \frac{1}{\sqrt{2+x}+\sqrt{2}}$$

$$\therefore \lim_{x\to 0} \frac{\sqrt{2+x}-\sqrt{2}}{x} = \lim_{x\to 0} \frac{1}{\sqrt{2+x}+\sqrt{2}} = \frac{1}{2\sqrt{2}}$$

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1.3 Evaluating Limits Analytically

$$\lim_{x\to 1} (-x^2+1)$$

$$\lim_{x\to 3} \frac{3-x}{x^2-9} \qquad \frac{0}{0}$$

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1.3 Evaluating Limits Analytically

$$\lim_{x\to 3} \frac{3-x}{x^2-9} \to \frac{3-3}{9-9} \to \frac{0}{0}$$ lim exists

$$\lim_{x\to 3} \frac{-(x-3)}{(x+3)(x-3)} = \lim_{x\to 3} \frac{-1}{x+3} = \frac{-1}{6}$$

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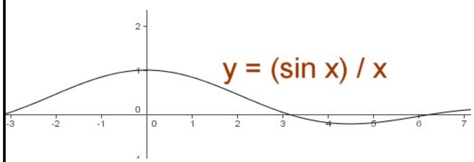
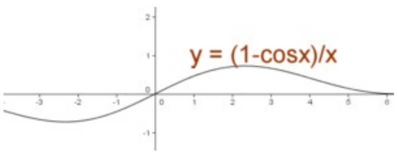
Summary: To Find a Limit

1. **Substitute x = c**
   a) If finite number, L, then the limit is L.
   b) If results in form,   k / 0 , then f is unbounded and the limit DNE
2. **If indeterminate, use algebra to find a function that is equivalent at all but the undefined point, and substitute again.** $\frac{0}{0}$ $\frac{\infty}{\infty}$
3. **If still indeterminate, consider special limits:**

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{1-\cos x}{x} = 0$$





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(sinx)/x*

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1.3 Evaluating Limits Analytically

$$\lim_{x \to 2} \frac{x^3-8}{x-2} \to \frac{0}{0} \quad \therefore \text{ limit exists.}$$

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1.3 Evaluating Limits Analytically

$$\lim_{x \to 2} \frac{x^3-8}{x-2} \to \frac{0}{0} \quad \therefore \text{ limit exists.}$$

$$\frac{\cancel{(x-2)}(x^2+2x+4)}{\cancel{(x-2)}}$$

$$\begin{array}{r|rrrr} 2 & 1 & 0 & 0 & -8 \\ & & 2 & 4 & 8 \\ \hline & 1 & 2 & 4 & \boxed{0} \end{array}$$

$$\lim_{x \to 2} \frac{x^3-8}{x-2} = \lim_{x \to 2} x^2+2x+4 = 12$$

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1.3 Evaluating Limits Analytically

G: $g(x) = \frac{x^2 - x}{x}$ F: simpler funct.

$g(x) = \frac{x(x-1)}{x}$

$= x - 1, \quad x \neq 0$

$\lim_{x \to 0} \frac{x^2 - x}{x} \to \boxed{\frac{0}{0}}$ limit exists.

$= \lim_{x \to 0} (x-1) = 0 - 1 = -1$

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1.3 Evaluating Limits Analytically

$\lim_{x \to 4} \frac{\sqrt{x+5} - 3}{x - 4} \to \frac{3-3}{0} \quad \frac{0}{0}$

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1.3 Evaluating Limits Analytically

$$\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} \to \frac{3-3}{0} \quad \frac{0}{0}$$

$(a+b)(a-b)$

a^2-b^2

$$\frac{(\sqrt{x+5}-3)}{(x-4)} \cdot \frac{(\sqrt{x+5}+3)}{(\sqrt{x+5}+3)}$$

$$\frac{(x+5)-9}{(x-4)[\sqrt{x+5}+3]} = \frac{(x-4)}{(x-4)[\sqrt{x+5}+3]} = \frac{1}{\sqrt{x+5}+3}$$

$$\lim_{x\to 4} \frac{\sqrt{x+5}-3}{x-4} = \lim_{x\to 4} \frac{1}{\sqrt{x+5}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

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1.3 Evaluating Limits Analytically

$$\lim_{x\to 0} \frac{\tan x}{\boxed{\tan 2x}} \to \frac{0}{0}$$

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1.3 Evaluating Limits Analytically

$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\sin 2x = 2\sin x \cos x$$

$$\lim_{x\to 0} \frac{\tan x}{\tan 2x} \to \frac{0}{0}$$

$$\frac{\frac{\sin x}{\cos x}}{\frac{\sin 2x}{\cos 2x}} = \frac{\sin x}{\cos x}\cdot\frac{\cos 2x}{\sin 2x}$$

$$= \frac{\cancel{\sin x}}{\cos x}\cdot\frac{\cos 2x}{\cancel{\sin 2x}\ 2\cancel{\sin x}\cos x} = \frac{\cos 2x}{2\cos^2 x}$$

$$= \lim_{x\to 0} \frac{\cos 2x}{2\cos^2 x} = \frac{1}{2}$$

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1.3 Evaluating Limits Analytically

$$\lim_{\Delta x\to 0} \frac{2(x+\Delta x)-2x}{\Delta x} \to \frac{2(x+0)-2x}{0}$$

$$\frac{2x-2x}{0} \quad \frac{0}{0}$$

$\therefore$ limit exists

$$= \lim_{\Delta x\to 0} \frac{2x+2\Delta x-2x}{\Delta x}$$

$$= \lim_{\Delta x\to 0} \frac{2\Delta x}{\Delta x} = \lim_{\Delta x\to 0} 2$$

$$= 2$$

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1.3 Evaluating Limits Analytically

$\lim_{x \to 2} x+3$ 5

$\lim_{x \to 2} 2x+3$ 7

$\lim_{x \to 2} \boxed{2}(x+3)$ 10

$\lim_{x \to 2} [(2x+3)+(x-1)]$

$4+3+(2-1)$

$7 + 1 = 8$

$\lim_{x \to 2} (2x+3) + \lim_{x \to 2}(x-1)$

$7 + 1 = 8$

1.3 Evaluating Limits Analytically

G: $\lim_{x \to c} f(x) = 3$

$\lim_{x \to c} g(x) = 2$

a) $\lim_{x \to c} [5(g(x))] = 5 \lim_{x \to c} g(x)$

$= 5(2) = 10$

b) $\lim_{x \to c} [f(x) + g(x)] = 3 + 2 = 5$

c) $\lim_{x \to c} f(x)\, g(x) = 3(2) = 6$

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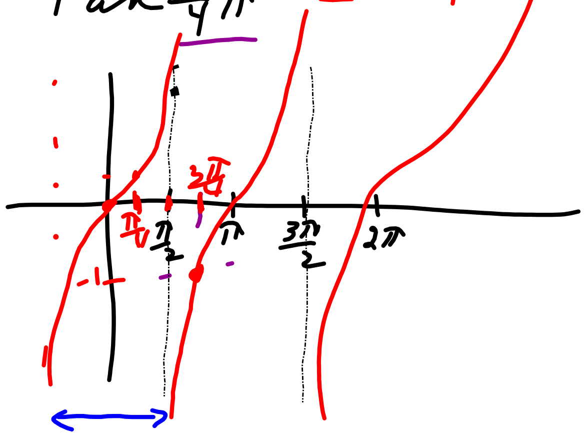
1.3 Evaluating Limits Analytically

$$\lim_{x \to 3} \tan\left(\frac{\pi}{4}x\right) \quad -1$$

$$\to \tan\frac{3}{4}\pi = -1$$



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