Complete Solutions Manual

Linear Algebra
A Modern Introduction

FOURTH EDITION

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ISBN-13: 978-128586960-5
ISBN-10: 1-28586960-5

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Printed in the United States of America
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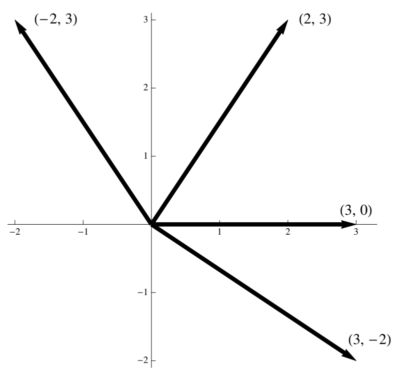
Chapter 1

Vectors

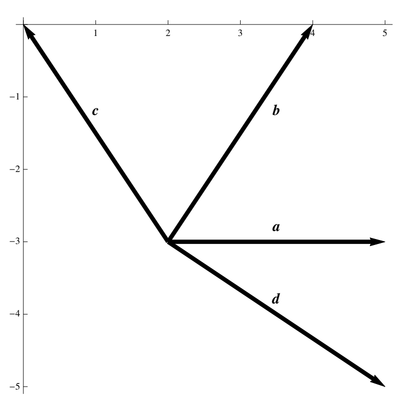
1.1 The Geometry and Algebra of Vectors

1. 

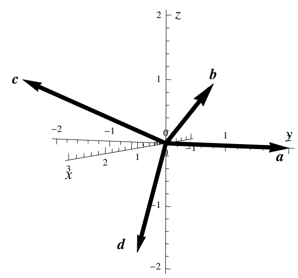
2. Since

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix},$$

plotting those vectors gives



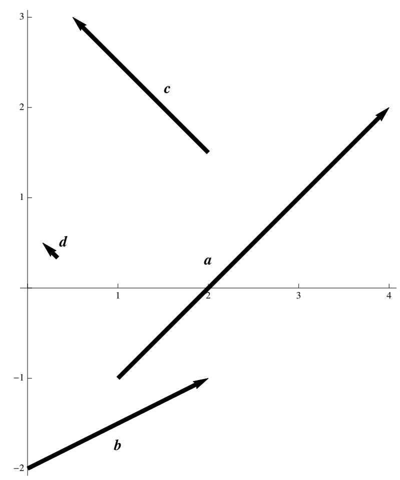
3.



4. Since the heads are all at $(3, 2, 1)$, the tails are at

$$\begin{bmatrix}3\\2\\1\end{bmatrix} - \begin{bmatrix}0\\2\\0\end{bmatrix} = \begin{bmatrix}3\\0\\1\end{bmatrix}, \qquad \begin{bmatrix}3\\2\\1\end{bmatrix} - \begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, \qquad \begin{bmatrix}3\\2\\1\end{bmatrix} - \begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}2\\4\\0\end{bmatrix}, \qquad \begin{bmatrix}3\\2\\1\end{bmatrix} - \begin{bmatrix}-1\\-1\\-2\end{bmatrix} = \begin{bmatrix}4\\3\\3\end{bmatrix}.$$

5. The four vectors $\overrightarrow{AB}$ are



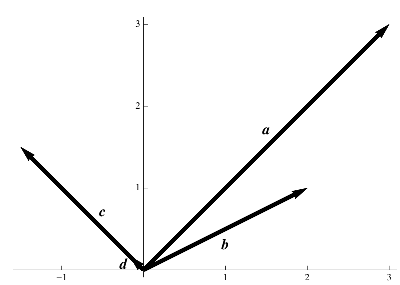
In standard position, the vectors are

(a) $\overrightarrow{AB} = [4 - 1, 2 - (-1)] = [3, 3]$.

(b) $\overrightarrow{AB} = [2 - 0, -1 - (-2)] = [2, 1]$

(c) $\overrightarrow{AB} = \left[\frac{1}{2} - 2, 3 - \frac{3}{2}\right] = \left[-\frac{3}{2}, \frac{3}{2}\right]$

(d) $\overrightarrow{AB} = \left[\frac{1}{6} - \frac{1}{3}, \frac{1}{2} - \frac{1}{3}\right] = \left[-\frac{1}{6}, \frac{1}{6}\right]$.



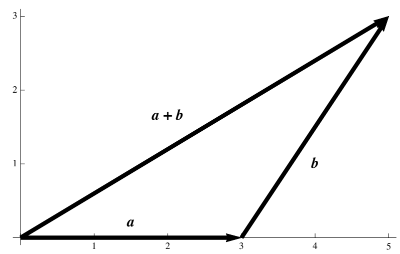
6. Recall the notation that $[a, b]$ denotes a move of a units horizontally and b units vertically. Then during the first part of the walk, the hiker walks 4 km north, so $\mathbf{a} = [0, 4]$. During the second part of the walk, the hiker walks a distance of 5 km northeast. From the components, we get

$$\mathbf{b} = [5\cos 45°,\ 5\sin 45°] = \left[\frac{5\sqrt{2}}{2}, \frac{5\sqrt{2}}{2}\right].$$

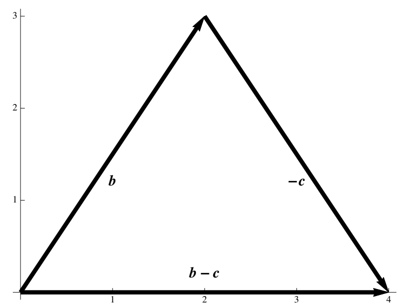
Thus the net displacement vector is

$$\mathbf{c} = \mathbf{a} + \mathbf{b} = \left[\frac{5\sqrt{2}}{2}, 4 + \frac{5\sqrt{2}}{2}\right].$$

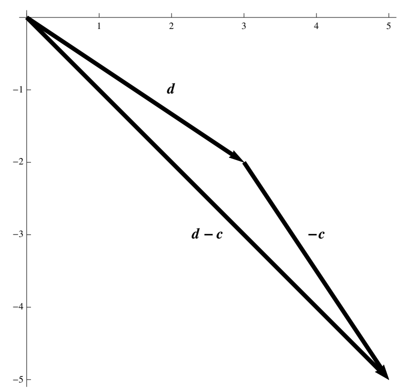
7. $\mathbf{a} + \mathbf{b} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3+2 \\ 0+3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}.$



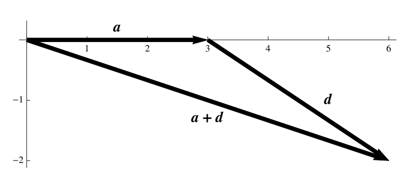
8. $\mathbf{b} - \mathbf{c} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2-(-2) \\ 3-3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}.$



9. $\mathbf{d} - \mathbf{c} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}.$



10. $\mathbf{a} + \mathbf{d} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3+3 \\ 0+(-2) \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$



11. $2\mathbf{a} + 3\mathbf{c} = 2[0, 2, 0] + 3[1, -2, 1] = [2 \cdot 0, 2 \cdot 2, 2 \cdot 0] + [3 \cdot 1, 3 \cdot (-2), 3 \cdot 1] = [3, -2, 3].$

12.

$$\begin{aligned} 3\mathbf{b} - 2\mathbf{c} + \mathbf{d} &= 3[3, 2, 1] - 2[1, -2, 1] + [-1, -1, -2] \\ &= [3 \cdot 3, 3 \cdot 2, 3 \cdot 1] + [-2 \cdot 1, -2 \cdot (-2), -2 \cdot 1] + [-1, -1, -2] \\ &= [6, 9, -1]. \end{aligned}$$

13. $\mathbf{u} = [\cos 60°, \sin 60°] = \left[\frac{1}{2}, \frac{\sqrt{3}}{2}\right]$, and $\mathbf{v} = [\cos 210°, \sin 210°] = \left[-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right]$, so that

$$\mathbf{u}+\mathbf{v} = \left[\frac{1}{2} - \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} - \frac{1}{2}\right], \quad \mathbf{u}-\mathbf{v} = \left[\frac{1}{2} + \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} + \frac{1}{2}\right].$$

14. **(a)** $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

(b) Since $\overrightarrow{OC} = \overrightarrow{AB}$, we have $\overrightarrow{BC} = \overrightarrow{OC} - \mathbf{b} = (\mathbf{b} - \mathbf{a}) - \mathbf{b} = -\mathbf{a}$.

(c) $\overrightarrow{AD} = -2\mathbf{a}$.

(d) $\overrightarrow{CF} = -2\overrightarrow{OC} = -2\overrightarrow{AB} = -2(\mathbf{b} - \mathbf{a}) = 2(\mathbf{a} - \mathbf{b})$.

(e) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = (\mathbf{b} - \mathbf{a}) + (-\mathbf{a}) = \mathbf{b} - 2\mathbf{a}$.

(f) Note that $\overrightarrow{FA}$ and $\overrightarrow{OB}$ are equal, and that $\overrightarrow{DE} = -\overrightarrow{AB}$. Then

$$\overrightarrow{BC} + \overrightarrow{DE} + \overrightarrow{FA} = -\mathbf{a} - \overrightarrow{AB} + \overrightarrow{OB} = -\mathbf{a} - (\mathbf{b} - \mathbf{a}) + \mathbf{b} = \mathbf{0}.$$

15. $2(\mathbf{a} - 3\mathbf{b}) + 3(2\mathbf{b} + \mathbf{a}) \overset{\substack{\text{property e.}\\ \text{distributivity}}}{=} (2\mathbf{a} - 6\mathbf{b}) + (6\mathbf{b} + 3\mathbf{a}) \overset{\substack{\text{property b.}\\ \text{associativity}}}{=} (2\mathbf{a} + 3\mathbf{a}) + (-6\mathbf{b} + 6\mathbf{b}) = 5\mathbf{a}$.

16.

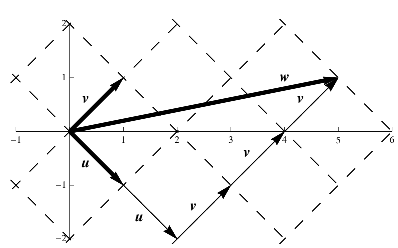
$$\begin{aligned} -3(\mathbf{a} - \mathbf{c}) + 2(\mathbf{a} + 2\mathbf{b}) + 3(\mathbf{c} - \mathbf{b}) &\overset{\substack{\text{property e.}\\ \text{distributivity}}}{=} (-3\mathbf{a} + 3\mathbf{c}) + (2\mathbf{a} + 4\mathbf{b}) + (3\mathbf{c} - 3\mathbf{b}) \\ &\overset{\substack{\text{property b.}\\ \text{associativity}}}{=} (-3\mathbf{a} + 2\mathbf{a}) + (4\mathbf{b} - 3\mathbf{b}) + (3\mathbf{c} + 3\mathbf{c}) \\ &= -\mathbf{a} + \mathbf{b} + 6\mathbf{c}. \end{aligned}$$

17. $\mathbf{x} - \mathbf{a} = 2(\mathbf{x} - 2\mathbf{a}) = 2\mathbf{x} - 4\mathbf{a} \Rightarrow \mathbf{x} - 2\mathbf{x} = \mathbf{a} - 4\mathbf{a} \Rightarrow -\mathbf{x} = -3\mathbf{a} \Rightarrow \mathbf{x} = 3\mathbf{a}$.

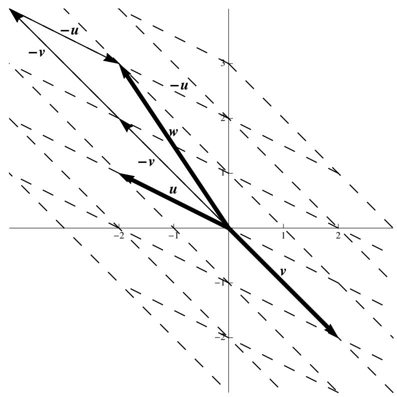
18.

$$\begin{aligned} \mathbf{x} + 2\mathbf{a} - \mathbf{b} = 3(\mathbf{x} + \mathbf{a}) - 2(2\mathbf{a} - \mathbf{b}) = 3\mathbf{x} + 3\mathbf{a} - 4\mathbf{a} + 2\mathbf{b} \quad &\Rightarrow \\ \mathbf{x} - 3\mathbf{x} = -\mathbf{a} - 2\mathbf{a} + 2\mathbf{b} + \mathbf{b} \quad &\Rightarrow \\ -2\mathbf{x} = -3\mathbf{a} + 3\mathbf{b} \quad &\Rightarrow \\ \mathbf{x} = \frac{3}{2}\mathbf{a} - \frac{3}{2}\mathbf{b}. \end{aligned}$$

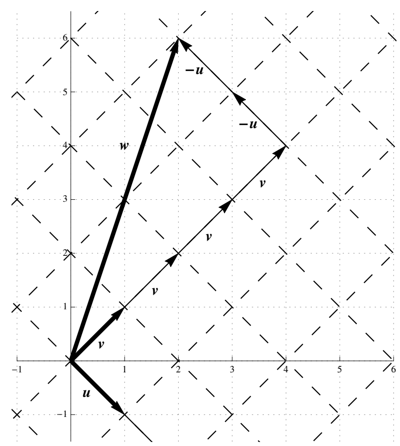
19. We have $2\mathbf{u} + 3\mathbf{v} = 2[1, -1] + 3[1, 1] = [2 \cdot 1 + 3 \cdot 1, 2 \cdot (-1) + 3 \cdot 1] = [5, 1]$. Plots of all three vectors are



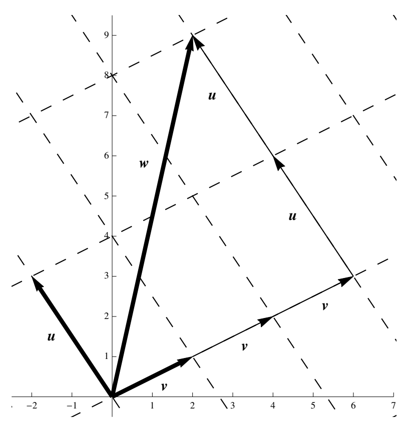
20. We have $-\mathbf{u} - 2\mathbf{v} = -[-2, 1] - 2[2, -2] = [-(-2) - 2 \cdot 2, -1 - 2 \cdot (-2)] = [-2, 3]$. Plots of all three vectors are



21. From the diagram, we see that $\mathbf{w} = -2\mathbf{u} + 4\mathbf{v}$.

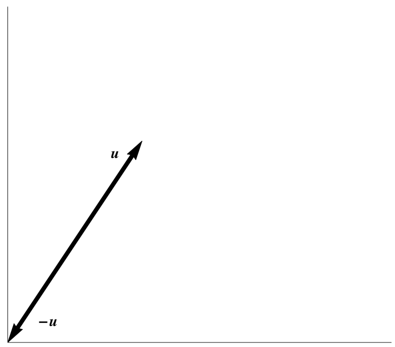


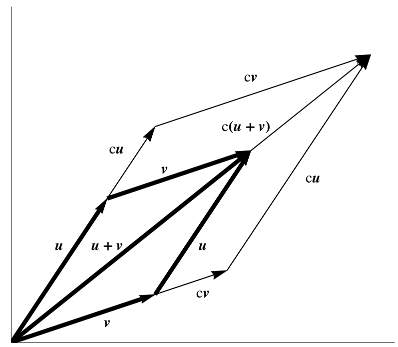
22. From the diagram, we see that $\mathbf{w} = 2\mathbf{u} + 3\mathbf{v}$.



23. Property (d) states that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$. The first diagram below shows $\mathbf{u}$ along with $-\mathbf{u}$. Then, as the diagonal of the parallelogram, the resultant vector is $\mathbf{0}$.

Property (e) states that $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$. The second figure illustrates this.





24. Let $\mathbf{u} = [u_1, u_2, \ldots, u_n]$ and $\mathbf{v} = [v_1, v_2, \ldots, v_n]$, and let c and d be scalars in $\mathbb{R}$.

Property (d):

$$\begin{aligned}\mathbf{u} + (-\mathbf{u}) &= [u_1, u_2, \ldots, u_n] + (-1[u_1, u_2, \ldots, u_n]) \\ &= [u_1, u_2, \ldots, u_n] + [-u_1, -u_2, \ldots, -u_n] \\ &= [u_1 - u_1, u_2 - u_2, \ldots, u_n - u_n] \\ &= [0, 0, \ldots, 0] = \mathbf{0}.\end{aligned}$$

Property (e):

$$\begin{aligned}c(\mathbf{u} + \mathbf{v}) &= c\left([u_1, u_2, \ldots, u_n] + [v_1, v_2, \ldots, v_n]\right) \\ &= c\left([u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n]\right) \\ &= [c(u_1 + v_1), c(u_2 + v_2), \ldots, c(u_n + v_n)] \\ &= [cu_1 + cv_1, cu_2 + cv_2, \ldots, cu_n + cv_n] \\ &= [cu_1, cu_2, \ldots, cu_n] + [cv_1, cv_2, \ldots, cv_n] \\ &= c[u_1, u_2, \ldots, u_n] + c[v_1, v_2, \ldots, v_n] \\ &= c\mathbf{u} + c\mathbf{v}.\end{aligned}$$

Property (f):

$$\begin{aligned}(c + d)\mathbf{u} &= (c + d)[u_1, u_2, \ldots, u_n] \\ &= [(c + d)u_1, (c + d)u_2, \ldots, (c + d)u_n] \\ &= [cu_1 + du_1, cu_2 + du_2, \ldots, cu_n + du_n] \\ &= [cu_1, cu_2, \ldots, cu_n] + [du_1, du_2, \ldots, du_n] \\ &= c[u_1, u_2, \ldots, u_n] + d[u_1, u_2, \ldots, u_n] \\ &= c\mathbf{u} + d\mathbf{u}.\end{aligned}$$

Property (g):

$$\begin{aligned}c(d\mathbf{u}) &= c(d[u_1, u_2, \ldots, u_n]) \\ &= c[du_1, du_2, \ldots, du_n] \\ &= [cdu_1, cdu_2, \ldots, cdu_n] \\ &= [(cd)u_1, (cd)u_2, \ldots, (cd)u_n] \\ &= (cd)[u_1, u_2, \ldots, u_n] \\ &= (cd)\mathbf{u}.\end{aligned}$$

25. $\mathbf{u} + \mathbf{v} = [0, 1] + [1, 1] = [1, 0]$.

26. $\mathbf{u} + \mathbf{v} = [1, 1, 0] + [1, 1, 1] = [0, 0, 1]$.

27. $\mathbf{u}+\mathbf{v}=[1,0,1,1]+[1,1,1,1]=[0,1,0,0]$.

28. $\mathbf{u}+\mathbf{v}=[1,1,0,1,0]+[0,1,1,1,0]=[1,0,1,0,0]$.

29.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

·	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

30.

+	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

31. $2+2+2=6=0$ in $\mathbb{Z}_3$.

32. $2\cdot 2\cdot 2=3\cdot 2=0$ in $\mathbb{Z}_3$.

33. $2(2+1+2)=2\cdot 2=3\cdot 1+1=1$ in $\mathbb{Z}_3$.

34. $3+1+2+3=4\cdot 2+1=1$ in $\mathbb{Z}_4$.

35. $2\cdot 3\cdot 2=4\cdot 3+0=0$ in $\mathbb{Z}_4$.

36. $3(3+3+2)=4\cdot 6+0=0$ in $\mathbb{Z}_4$.

37. $2+1+2+2+1=2$ in $\mathbb{Z}_3$, $2+1+2+2+1=0$ in $\mathbb{Z}_4$, $2+1+2+2+1=3$ in $\mathbb{Z}_5$.

38. $(3+4)(3+2+4+2)=2\cdot 1=2$ in $\mathbb{Z}_5$.

39. $8(6+4+3)=8\cdot 4=5$ in $\mathbb{Z}_9$.

40. $2^{100}=\left(2^{10}\right)^{10}=(1024)^{10}=1^{10}=1$ in $\mathbb{Z}_{11}$.

41. $[2,1,2]+[2,0,1]=[1,1,0]$ in $\mathbb{Z}_3^3$.

42. $2[2,\,2,\,1]=[2\cdot 2,\,2\cdot 2,\,2\cdot 1]=[1,\,1,\,2]$ in $\mathbb{Z}_3^3$.

43. $2([3,1,1,2]+[3,3,2,1])=2[2,0,3,3]=[2\cdot 2,\,2\cdot 0,\,2\cdot 3,\,2\cdot 3]=[0,0,2,2]$ in $\mathbb{Z}_4^4$.
$2([3,1,1,2]+[3,3,2,1])=2[1,4,3,3]=[2\cdot 1,\,2\cdot 4,\,2\cdot 3,\,2\cdot 3]=[2,3,1,1]$ in $\mathbb{Z}_5^4$.

44. $x=2+(-3)=2+2=4$ in $\mathbb{Z}_5$.

45. $x=1+(-5)=1+1=2$ in $\mathbb{Z}_6$

46. $x=2^{-1}=2$ in $\mathbb{Z}_3$.

47. No solution. 2 times anything is always even, so cannot leave a remainder of 1 when divided by 4.

48. $x=2^{-1}=3$ in $\mathbb{Z}_5$.

49. $x=3^{-1}4=2\cdot 4=3$ in $\mathbb{Z}_5$.

50. No solution. 3 times anything is always a multiple of 3, so it cannot leave a remainder of 4 when divided by 6 (which is also a multiple of 3).

51. No solution. 6 times anything is always even, so it cannot leave an odd number as a remainder when divided by 8.

52. $x = 8^{-1}9 = 7 \cdot 9 = 8$ in $\mathbb{Z}_{11}$

53. $x = 2^{-1}(2 + (-3)) = 3(2 + 2) = 2$ in $\mathbb{Z}_5$.

54. No solution. This equation is the same as $4x = 2 - 5 = -3 = 3$ in $\mathbb{Z}_6$. But 4 times anything is even, so it cannot leave a remainder of 3 when divided by 6 (which is also even).

55. Add 5 to both sides to get $6x = 6$, so that $x = 1$ or $x = 5$ (since $6 \cdot 1 = 6$ and $6 \cdot 5 = 30 = 6$ in $\mathbb{Z}_8$).

56. (a) All values. (b) All values. (c) All values.

57. **(a)** All $a \neq 0$ in $\mathbb{Z}_5$ have a solution because 5 is a prime number.

(b) $a = 1$ and $a = 5$ because they have no common factors with 6 other than 1.

(c) a and m can have no common factors other than 1; that is, the *greatest common divisor*, gcd, of a and m is 1.

1.2 Length and Angle: The Dot Product

1. Following Example 1.15, $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = (-1) \cdot 3 + 2 \cdot 1 = -3 + 2 = -1.$

2. Following Example 1.15, $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \end{bmatrix} = 3 \cdot 4 + (-2) \cdot 6 = 12 - 12 = 0.$

3. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 1 = 2 + 6 + 3 = 11.$

4. $\mathbf{u} \cdot \mathbf{v} = 3.2 \cdot 1.5 + (-0.6) \cdot 4.1 + (-1.4) \cdot (-0.2) = 4.8 - 2.46 + 0.28 = 2.62.$

5. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{2} \\ \sqrt{3} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -\sqrt{2} \\ 0 \\ -5 \end{bmatrix} = 1 \cdot 4 + \sqrt{2} \cdot (-\sqrt{2}) + \sqrt{3} \cdot 0 + 0 \cdot (-5) = 4 - 2 = 2.$

6. $\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1.12 \\ -3.25 \\ 2.07 \\ -1.83 \end{bmatrix} \cdot \begin{bmatrix} -2.29 \\ 1.72 \\ 4.33 \\ -1.54 \end{bmatrix} = -1.12 \cdot 2.29 - 3.25 \cdot 1.72 + 2.07 \cdot 4.33 - 1.83 \cdot (-1.54) = 3.6265.$

7. Finding a unit vector $\mathbf{v}$ in the same direction as a given vector $\mathbf{u}$ is called ***normalizing*** the vector $\mathbf{u}$. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5},$$

so a unit vector $\mathbf{v}$ in the same direction as $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

8. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{3^2 + (-2)^2} = \sqrt{9 + 4} = \sqrt{13},$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{13}} \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{13}} \\ -\frac{2}{\sqrt{13}} \end{bmatrix}.$$

9. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14},$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{14}}\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{14}}\\ \frac{2}{\sqrt{14}}\\ \frac{3}{\sqrt{14}}\end{bmatrix}.$$

10. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{3.2^2 + (-0.6)^2 + (-1.4)^2} = \sqrt{10.24 + 0.36 + 1.96} = \sqrt{12.56} \approx 3.544,$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{3.544}\begin{bmatrix}1.5\\0.4\\-2.1\end{bmatrix} \approx \begin{bmatrix}0.903\\-0.169\\-0.395\end{bmatrix}.$$

11. Proceed as in Example 1.19:

$$\|\mathbf{u}\| = \sqrt{1^2 + \left(\sqrt{2}\right)^2 + \left(\sqrt{3}\right)^2 + 0^2} = \sqrt{6},$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\ \sqrt{2}\\ \sqrt{3}\\ 0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\ \frac{\sqrt{2}}{\sqrt{6}}\\ \frac{\sqrt{3}}{\sqrt{6}}\\ 0\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{2}}\\ 0\end{bmatrix} = \begin{bmatrix}\frac{\sqrt{6}}{6}\\ \frac{\sqrt{3}}{3}\\ \frac{\sqrt{2}}{2}\\ 0\end{bmatrix}$$

12. Proceed as in Example 1.19:

$$\begin{aligned}\|\mathbf{u}\| = \sqrt{1.12^2 + (-3.25)^2 + 2.07^2 + (-1.83)^2} &= \sqrt{1.2544 + 10.5625 + 4.2849 + 3.3489}\\ &= \sqrt{19.4507} \approx 4.410,\end{aligned}$$

so a unit vector $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\mathbf{v}\frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{4.410}\begin{bmatrix}1.12 & -3.25 & 2.07 & -1.83\end{bmatrix} \approx \begin{bmatrix}0.254 & -0.737 & 0.469 & -0.415\end{bmatrix}.$$

13. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix}-1\\2\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}-4\\1\end{bmatrix}$, so

$$\mathrm{d}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-4)^2 + 1^2} = \sqrt{17}.$$

14. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix}3\\-2\end{bmatrix} - \begin{bmatrix}4\\6\end{bmatrix} = \begin{bmatrix}-1\\-8\end{bmatrix}$, so

$$\mathrm{d}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-1)^2 + (-8)^2} = \sqrt{65}.$$

15. Following Example 1.20, we compute: $\mathbf{u} - \mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix} - \begin{bmatrix}2\\3\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\\2\end{bmatrix}$, so

$$\mathrm{d}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-1)^2 + (-1)^2 + 2^2} = \sqrt{6}.$$

16. Following Example 1.20, we compute: $\mathbf{u}-\mathbf{v}=\begin{bmatrix}3.2\\-0.6\\-1.4\end{bmatrix}-\begin{bmatrix}1.5\\4.1\\-0.2\end{bmatrix}=\begin{bmatrix}1.7\\-4.7\\-1.2\end{bmatrix}$, so

$$d(\mathbf{u},\mathbf{v})=\|\mathbf{u}-\mathbf{v}\|=\sqrt{1.7^2+(-4.7)^2+(-1.2)^2}=\sqrt{26.42}\approx 5.14.$$

17. **(a)** $\mathbf{u}\cdot\mathbf{v}$ is a real number, so $\|\mathbf{u}\cdot\mathbf{v}\|$ is the norm of a number, which is not defined.

(b) $\mathbf{u}\cdot\mathbf{v}$ is a scalar, while $\mathbf{w}$ is a vector. Thus $\mathbf{u}\cdot\mathbf{v}+\mathbf{w}$ adds a scalar to a vector, which is not a defined operation.

(c) $\mathbf{u}$ is a vector, while $\mathbf{v}\cdot\mathbf{w}$ is a scalar. Thus $\mathbf{u}\cdot(\mathbf{v}\cdot\mathbf{w})$ is the dot product of a vector and a scalar, which is not defined.

(d) $c\cdot(\mathbf{u}+\mathbf{v})$ is the dot product of a scalar and a vector, which is not defined.

18. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{3\cdot(-1)+0\cdot 1}{\sqrt{3^2+0^2}\sqrt{(-1)^2+1^2}}=-\frac{3}{3\sqrt{2}}=-\frac{\sqrt{2}}{2}.$$

Thus $\cos\theta<0$ (in fact, $\theta=\frac{3\pi}{4}$), so the angle between $\mathbf{u}$ and $\mathbf{v}$ is obtuse.

19. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{2\cdot 1+(-1)\cdot(-2)+1\cdot(-1)}{\sqrt{2^2+(-1)^2+1^2}\sqrt{1^2+(-2)^2+1^2}}=\frac{1}{2}.$$

Thus $\cos\theta>0$ (in fact, $\theta=\frac{\pi}{3}$), so the angle between $\mathbf{u}$ and $\mathbf{v}$ is acute.

20. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{4\cdot 1+3\cdot(-1)+(-1)\cdot 1}{\sqrt{4^2+3^2+(-1)^2}\sqrt{1^2+(-1)^2+1^2}}=\frac{0}{\sqrt{26}\sqrt{3}}=0.$$

Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is a right angle.

21. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Note that we can determine whether θ is acute, right, or obtuse by examining the sign of $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$, which is determined by the sign of $\mathbf{u}\cdot\mathbf{v}$. Since

$$\mathbf{u}\cdot\mathbf{v}=0.9\cdot(-4.5)+2.1\cdot 2.6+1.2\cdot(-0.8)=0.45>0,$$

we have $\cos\theta>0$ so that θ is acute.

22. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Note that we can determine whether θ is acute, right, or obtuse by examining the sign of $\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$, which is determined by the sign of $\mathbf{u}\cdot\mathbf{v}$. Since

$$\mathbf{u}\cdot\mathbf{v}=1\cdot(-3)+2\cdot 1+3\cdot 2+4\cdot(-2)=-3,$$

we have $\cos\theta<0$ so that θ is obtuse.

23. Since the components of both $\mathbf{u}$ and $\mathbf{v}$ are positive, it is clear that $\mathbf{u}\cdot\mathbf{v}>0$, so the angle between them is acute since it has a positive cosine.

24. From Exercise 18, $\cos\theta=-\frac{\sqrt{2}}{2}$, so that $\theta=\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)=\frac{3\pi}{4}=135°$.

25. From Exercise 19, $\cos\theta=\frac{1}{2}$, so that $\theta=\cos^{-1}\frac{1}{2}=\frac{\pi}{3}=60°$.

26. From Exercise 20, $\cos\theta=0$, so that $\theta=\frac{\pi}{2}=90°$ is a right angle.

27. As in Example 1.21, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ and the norms of the two vectors:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 0.9\cdot(-4.5)+2.1\cdot 2.6+1.2\cdot(-0.8)=0.45,\\ \|\mathbf{u}\| &= \sqrt{0.9^2+2.1^2+1.2^2}=\sqrt{6.66},\\ \|\mathbf{v}\| &= \sqrt{(-4.5)^2+2.6^2+(-0.8)^2}=\sqrt{27.65}.\end{aligned}$$

So if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{0.45}{\sqrt{6.66}\sqrt{27.65}}\approx\frac{0.45}{\sqrt{182.817}},$$

so that

$$\theta=\cos^{-1}\left(\frac{0.45}{\sqrt{182.817}}\right)\approx 1.5375\approx 88.09^\circ.$$

Note that it is important to maintain as much precision as possible until the last step, or roundoff errors may build up.

28. As in Example 1.21, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ and the norms of the two vectors:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 1\cdot(-3)+2\cdot 1+3\cdot 2+4\cdot(-2)=-3,\\ \|\mathbf{u}\| &= \sqrt{1^2+2^2+3^2+4^2}=\sqrt{30},\\ \|\mathbf{v}\| &= \sqrt{(-3)^2+1^2+2^2+(-2)^2}=\sqrt{18}.\end{aligned}$$

So if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=-\frac{3}{\sqrt{30}\sqrt{18}}=-\frac{1}{2\sqrt{15}}\quad\text{so that}\quad\theta=\cos^{-1}\left(-\frac{1}{2\sqrt{15}}\right)\approx 1.7\approx 97.42^\circ.$$

29. As in Example 1.21, we begin by calculating $\mathbf{u}\cdot\mathbf{v}$ and the norms of the two vectors:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= 1\cdot 5+2\cdot 6+3\cdot 7+4\cdot 8=70,\\ \|\mathbf{u}\| &= \sqrt{1^2+2^2+3^2+4^2}=\sqrt{30},\\ \|\mathbf{v}\| &= \sqrt{5^2+6^2+7^2+8^2}=\sqrt{174}.\end{aligned}$$

So if θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then

$$\cos\theta=\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}=\frac{70}{\sqrt{30}\sqrt{174}}=\frac{35}{3\sqrt{145}}\quad\text{so that}\quad\theta=\cos^{-1}\left(\frac{35}{3\sqrt{145}}\right)\approx 0.2502\approx 14.34^\circ.$$

30. To show that $\triangle ABC$ is right, we need only show that one pair of its sides meets at a right angle. So let $\mathbf{u}=\overrightarrow{AB}$, $\mathbf{v}=\overrightarrow{BC}$, and $\mathbf{w}=\overrightarrow{AC}$. Then we must show that one of $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{w}$ or $\mathbf{v}\cdot\mathbf{w}$ is zero in order to show that one of these pairs is orthogonal. Then

$$\begin{gathered}\mathbf{u}=\overrightarrow{AB}=[1-(-3),\,0-2]=[4,-2]\,,\quad \mathbf{v}=\overrightarrow{BC}=[4-1,\,6-0]=[3,6]\,,\\ \mathbf{w}=\overrightarrow{AC}=[4-(-3),\,6-2]=[7,4]\,,\end{gathered}$$

and

$$\mathbf{u}\cdot\mathbf{v}=4\cdot 3+(-2)\cdot 6=12-12=0.$$

Since this dot product is zero, these two vectors are orthogonal, so that $\overrightarrow{AB}\perp\overrightarrow{BC}$ and thus $\triangle ABC$ is a right triangle. It is unnecessary to test the remaining pairs of sides.

31. To show that $\triangle ABC$ is right, we need only show that one pair of its sides meets at a right angle. So let $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{BC}$, and $\mathbf{w} = \overrightarrow{AC}$. Then we must show that one of $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{w}$ or $\mathbf{v}\cdot\mathbf{w}$ is zero in order to show that one of these pairs is orthogonal. Then

$$\begin{aligned}\mathbf{u} &= \overrightarrow{AB} = [-3-1,\ 2-1,\ (-2)-(-1)] = [-4,1,-1]\,,\\ \mathbf{v} &= \overrightarrow{BC} = [2-(-3),\ 2-2,\ -4-(-2)] = [5,0,-2],\\ \mathbf{w} &= \overrightarrow{AC} = [2-1,\ 2-1,\ -4-(-1)] = [1,1,-3],\end{aligned}$$

and

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= -4\cdot 5 + 1\cdot 0 - 1\cdot(-2) = -18\\ \mathbf{u}\cdot\mathbf{w} &= -4\cdot 1 + 1\cdot 1 - 1\cdot(-3) = 0.\end{aligned}$$

Since this dot product is zero, these two vectors are orthogonal, so that $\overrightarrow{AB} \perp \overrightarrow{AC}$ and thus $\triangle ABC$ is a right triangle. It is unnecessary to test the remaining pair of sides.

32. As in Example 1.22, the dimensions of the cube do not matter, so we work with a cube with side length 1. Since the cube is symmetric, we need only consider one diagonal and adjacent edge. Orient the cube as shown in Figure 1.34; take the diagonal to be $[1,1,1]$ and the adjacent edge to be $[1,0,0]$. Then the angle θ between these two vectors satisfies

$$\cos\theta = \frac{1\cdot 1 + 1\cdot 0 + 1\cdot 0}{\sqrt{3}\sqrt{1}} = \frac{1}{\sqrt{3}}, \qquad \text{so} \qquad \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 54.74^\circ.$$

Thus the diagonal and an adjacent edge meet at an angle of 54.74°.

33. As in Example 1.22, the dimensions of the cube do not matter, so we work with a cube with side length 1. Since the cube is symmetric, we need only consider one pair of diagonals. Orient the cube as shown in Figure 1.34; take the diagonals to be $\mathbf{u} = [1,1,1]$ and $\mathbf{v} = [1,1,0] - [0,0,1] = [1,1,-1]$. Then the dot product is

$$\mathbf{u}\cdot\mathbf{v} = 1\cdot 1 + 1\cdot 1 + 1\cdot(-1) = 1 + 1 - 1 = 1 \neq 0.$$

Since the dot product is nonzero, the diagonals are not orthogonal.

34. To show a parallelogram is a rhombus, it suffices to show that its diagonals are perpendicular (Euclid). But

$$\mathbf{d}_1\cdot\mathbf{d}_2 = \begin{bmatrix}2\\2\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\3\end{bmatrix} = 2\cdot 1 + 2\cdot(-1) + 0\cdot 3 = 0.$$

To determine its side length, note that since the diagonals are perpendicular, one half of each diagonal are the legs of a right triangle whose hypotenuse is one side of the rhombus. So we can use the Pythagorean Theorem. Since

$$\|\mathbf{d}_1\|^2 = 2^2 + 2^2 + 0^2 = 8, \qquad \|\mathbf{d}_2\|^2 = 1^2 + (-1)^2 + 3^2 = 11,$$

we have for the side length

$$s^2 = \left(\frac{\|\mathbf{d}_1\|}{2}\right)^2 + \left(\frac{\|\mathbf{d}_2\|}{2}\right)^2 = \frac{8}{4} + \frac{11}{4} = \frac{19}{4},$$

so that $s = \frac{\sqrt{19}}{2} \approx 2.18$.

35. Since $ABCD$ is a rectangle, opposite sides BA and CD are parallel and congruent. So we can use the method of Example 1.1 in Section 1.1 to find the coordinates of vertex D: we compute $\overrightarrow{BA} = [1-3, 2-6, 3-(-2)] = [-2,-4,5]$. If $\overrightarrow{BA}$ is then translated to $\overrightarrow{CD}$, where $C = (0,5,-4)$, then

$$D = (0 + (-2),\ 5 + (-4),\ -4 + 5) = (-2,1,1).$$

36. The resultant velocity of the airplane is the sum of the velocity of the airplane and the velocity of the wind:

$$\mathbf{r} = \mathbf{p} + \mathbf{w} = \begin{bmatrix} 200 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -40 \end{bmatrix} = \begin{bmatrix} 200 \\ -40 \end{bmatrix}.$$

37. Let the x direction be east, in the direction of the current, and the y direction be north, across the river. The speed of the boat is 4 mph north, and the current is 3 mph east, so the velocity of the boat is

$$\mathbf{v} = \begin{bmatrix} 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

38. Let the x direction be the direction across the river, and the y direction be downstream. Since $\mathbf{v}t = \mathbf{d}$, use the given information to find $\mathbf{v}$, then solve for t and compute $\mathbf{d}$. Since the speed of the boat is 20 km/h and the speed of the current is 5 km/h, we have $\mathbf{v} = \begin{bmatrix} 20 \\ 5 \end{bmatrix}$. The width of the river is 2 km, and the distance downstream is unknown; call it y. Then $\mathbf{d} = \begin{bmatrix} 2 \\ y \end{bmatrix}$. Thus

$$\mathbf{v}t = \begin{bmatrix} 20 \\ 5 \end{bmatrix} t = \begin{bmatrix} 2 \\ y \end{bmatrix}.$$

Thus $20t = 2$ so that $t = 0.1$, and then $y = 5 \cdot 0.1 = 0.5$. Therefore

(a) Ann lands 0.5 km, or half a kilometer, downstream;

(b) It takes Ann 0.1 hours, or six minutes, to cross the river.

Note that the river flow does not increase the time required to cross the river, since its velocity is perpendicular to the direction of travel.

39. We want to find the angle between Bert's resultant vector, $\mathbf{r}$, and his velocity vector upstream, $\mathbf{v}$. Let the first coordinate of the vector be the direction across the river, and the second be the direction upstream. Bert's velocity vector directly across the river is unknown, say $\mathbf{u} = \begin{bmatrix} x \\ 0 \end{bmatrix}$. His velocity vector upstream compensates for the downstream flow, so $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So the resultant vector is $\mathbf{r} = \mathbf{u} + \mathbf{v} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ 1 \end{bmatrix}$. Since Bert's speed is 2 mph, we have $\|\mathbf{r}\| = 2$. Thus

$$x^2 + 1 = \|\mathbf{r}\|^2 = 4, \quad \text{so that} \quad x = \sqrt{3}.$$

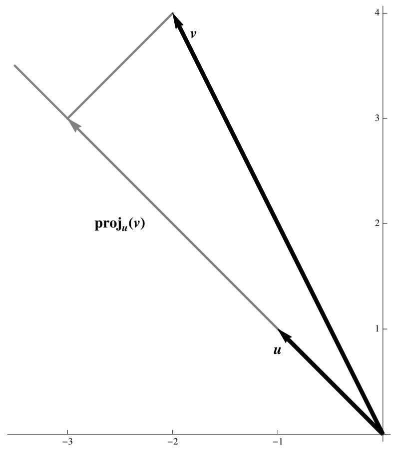
If θ is the angle between $\mathbf{r}$ and $\mathbf{v}$, then

$$\cos\theta = \frac{\mathbf{r} \cdot \mathbf{v}}{\|\mathbf{r}\|\,\|\mathbf{v}\|} = \frac{\sqrt{3}}{2}, \quad \text{so that} \quad \theta = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 60^\circ.$$

40. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u} = \frac{(-1) \cdot (-2) + 1 \cdot 4}{(-1) \cdot (-1) + 1 \cdot 1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}.$$

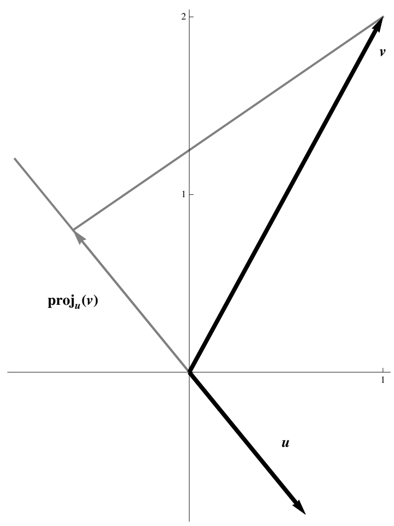
A graph of the situation is (with $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ in gray, and the perpendicular from $\mathbf{v}$ to the projection also drawn)



41. We have

$$\text{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{\frac{3}{5}\cdot 1 + \left(-\frac{4}{5}\cdot 2\right)}{\frac{3}{5}\cdot\frac{3}{5} + \left(-\frac{4}{5}\right)\cdot\left(-\frac{4}{5}\right)}\begin{bmatrix}\frac{3}{5}\\ -\frac{4}{5}\end{bmatrix} = -\frac{1}{1}\mathbf{u} = -\mathbf{u}.$$

A graph of the situation is (with $\text{proj}_{\mathbf{u}}\,\mathbf{v}$ in gray, and the perpendicular from $\mathbf{v}$ to the projection also drawn)



42. We have

$$\text{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{\frac{1}{2}\cdot 2 - \frac{1}{4}\cdot 2 - \frac{1}{2}\cdot(-2)}{\frac{1}{2}\cdot\frac{1}{2} + \left(-\frac{1}{4}\right)\left(-\frac{1}{4}\right) + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)}\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{4}\\ -\frac{1}{2}\end{bmatrix} = \frac{8}{3}\begin{bmatrix}\frac{1}{2}\\ -\frac{1}{4}\\ -\frac{1}{2}\end{bmatrix} = \begin{bmatrix}\frac{4}{3}\\ -\frac{2}{3}\\ -\frac{4}{3}\end{bmatrix}.$$

43. We have

$$\text{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{1\cdot 2 + (-1)\cdot(-3) + 1\cdot(-1) + (-1)\cdot(-2)}{1\cdot 1 + (-1)\cdot(-1) + 1\cdot 1 + (-1)\cdot(-1)}\begin{bmatrix}1\\ -1\\ 1\\ -1\end{bmatrix} = \frac{6}{4}\begin{bmatrix}1\\ -1\\ 1\\ -1\end{bmatrix} = \begin{bmatrix}\frac{3}{2}\\ -\frac{3}{2}\\ \frac{3}{2}\\ -\frac{3}{2}\end{bmatrix} = \frac{3}{2}\mathbf{u}.$$

44. We have

$$\text{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{0.5\cdot 2.1 + 1.5\cdot 1.2}{0.5\cdot 0.5 + 1.5\cdot 1.5}\begin{bmatrix}0.5\\ 1.5\end{bmatrix} = \frac{2.85}{2.5}\begin{bmatrix}0.5\\ 1.5\end{bmatrix} = \begin{bmatrix}0.57\\ 1.71\end{bmatrix} = 1.14\mathbf{u}.$$

45. We have

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{3.01\cdot 1.34 - 0.33\cdot 4.25 + 2.52\cdot(-1.66)}{3.01\cdot 3.01 - 0.33\cdot(-0.33) + 2.52\cdot 2.52}\begin{bmatrix}3.01\\-0.33\\2.52\end{bmatrix}$$

$$= -\frac{1.5523}{15.5194}\begin{bmatrix}3.01\\-0.33\\2.52\end{bmatrix} \approx \begin{bmatrix}-0.301\\0.033\\-0.252\end{bmatrix} \approx -\frac{1}{10}\mathbf{u}.$$

46. Let $\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix}2-1\\2-(-1)\end{bmatrix} = \begin{bmatrix}1\\3\end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix}4-1\\0-(-1)\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$.

(a) To compute the projection, we need

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}3\\1\end{bmatrix} = 6, \qquad \mathbf{u}\cdot\mathbf{u} = \begin{bmatrix}1\\3\end{bmatrix}\cdot\begin{bmatrix}1\\3\end{bmatrix} = 10.$$

Thus

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{3}{5}\begin{bmatrix}1\\3\end{bmatrix} = \begin{bmatrix}\frac{3}{5}\\\frac{9}{5}\end{bmatrix},$$

so that

$$\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v} = \begin{bmatrix}3\\1\end{bmatrix} - \begin{bmatrix}\frac{3}{5}\\\frac{9}{5}\end{bmatrix} = \begin{bmatrix}\frac{12}{5}\\-\frac{4}{5}\end{bmatrix}.$$

Then

$$\|\mathbf{u}\| = \sqrt{1^2+3^2} = \sqrt{10}, \quad \|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\| = \sqrt{\left(\frac{12}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \frac{4\sqrt{10}}{5},$$

so that finally

$$\mathcal{A} = \frac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\| = \frac{1}{2}\sqrt{10}\cdot\frac{4\sqrt{10}}{5} = 4.$$

(b) We already know $\mathbf{u}\cdot\mathbf{v} = 6$ and $\|\mathbf{u}\| = \sqrt{10}$ from part (a). Also, $\|\mathbf{v}\| = \sqrt{3^2+1^2} = \sqrt{10}$. So

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{6}{\sqrt{10}\sqrt{10}} = \frac{3}{5},$$

so that

$$\sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{3}{5}\right)^2} = \frac{4}{5}.$$

Thus

$$\mathcal{A} = \frac{1}{2}\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta = \frac{1}{2}\sqrt{10}\sqrt{10}\cdot\frac{4}{5} = 4.$$

47. Let $\mathbf{u} = \overrightarrow{AB} = \begin{bmatrix}4-3\\-2-(-1)\\6-4\end{bmatrix} = \begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\mathbf{v} = \overrightarrow{AC} = \begin{bmatrix}5-3\\0-(-1)\\2-4\end{bmatrix} = \begin{bmatrix}2\\1\\-2\end{bmatrix}$.

(a) To compute the projection, we need

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix}1\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-2\end{bmatrix} = -3, \qquad \mathbf{u}\cdot\mathbf{u} = \begin{bmatrix}1\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\2\end{bmatrix} = 6.$$

Thus

$$\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = -\frac{3}{6}\begin{bmatrix}1\\-1\\2\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\\frac{1}{2}\\-1\end{bmatrix},$$

so that

$$\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{1}{2} \\ -1 \end{bmatrix}$$

Then

$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}, \quad \|\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}\| = \sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + (-1)^2} = \frac{\sqrt{30}}{2},$$

so that finally

$$\mathcal{A} = \frac{1}{2}\,\|\mathbf{u}\|\,\|\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}\| = \frac{1}{2}\sqrt{6} \cdot \frac{\sqrt{30}}{2} = \frac{3\sqrt{5}}{2}.$$

(b) We already know $\mathbf{u} \cdot \mathbf{v} = -3$ and $\|\mathbf{u}\| = \sqrt{6}$ from part (a). Also, $\|\mathbf{v}\| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$. So

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \frac{-3}{3\sqrt{6}} = -\frac{\sqrt{6}}{6},$$

so that

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 - \left(\frac{\sqrt{6}}{6}\right)^2} = \frac{\sqrt{30}}{6}.$$

Thus

$$\mathcal{A} = \frac{1}{2}\,\|\mathbf{u}\|\,\|\mathbf{v}\|\sin\theta = \frac{1}{2}\sqrt{6} \cdot 3 \cdot \frac{\sqrt{30}}{6} = \frac{3\sqrt{5}}{2}.$$

48. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their dot product $\mathbf{u} \cdot \mathbf{v} = 0$. So we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for k:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} k+1 \\ k-1 \end{bmatrix} = 0 \;\Rightarrow\; 2(k+1) + 3(k-1) = 0 \;\Rightarrow\; 5k - 1 = 0 \;\Rightarrow k = \frac{1}{5}.$$

Substituting into the formula for $\mathbf{v}$ gives

$$\mathbf{v} = \begin{bmatrix} \frac{1}{5} + 1 \\ \frac{1}{5} - 1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ -\frac{4}{5} \end{bmatrix}.$$

As a check, we compute

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{6}{5} \\ -\frac{4}{5} \end{bmatrix} = \frac{12}{5} - \frac{12}{5} = 0,$$

and the vectors are indeed orthogonal.

49. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their dot product $\mathbf{u} \cdot \mathbf{v} = 0$. So we set $\mathbf{u} \cdot \mathbf{v} = 0$ and solve for k:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} k^2 \\ k \\ -3 \end{bmatrix} = 0 \;\Rightarrow\; k^2 - k - 6 = 0 \;\Rightarrow\; (k+2)(k-3) = 0 \;\Rightarrow k = 2, -3.$$

Substituting into the formula for $\mathbf{v}$ gives

$$k = 2: \;\mathbf{v}_1 = \begin{bmatrix} (-2)^2 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}, \qquad k = -3: \;\mathbf{v}_2 = \begin{bmatrix} 3^2 \\ 3 \\ -3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -3 \end{bmatrix}.$$

As a check, we compute

$$\mathbf{u} \cdot \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix} = 1 \cdot 4 - 1 \cdot (-2) + 2 \cdot (-3) = 0, \; \mathbf{u} \cdot \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 9 \\ 3 \\ -3 \end{bmatrix} = 1 \cdot 9 - 1 \cdot 3 + 2 \cdot (-3) = 0$$

and the vectors are indeed orthogonal.

50. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their dot product $\mathbf{u}\cdot\mathbf{v} = 0$. So we set $\mathbf{u}\cdot\mathbf{v} = 0$ and solve for y in terms of x:

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix}3\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix} = 0 \ \Rightarrow\ 3x + y = 0 \ \Rightarrow\ y = -3x.$$

Substituting $y = -3x$ back into the formula for $\mathbf{v}$ gives

$$\mathbf{v} = \begin{bmatrix}x\\-3x\end{bmatrix} = x\begin{bmatrix}1\\-3\end{bmatrix}.$$

Thus any vector orthogonal to $\begin{bmatrix}3\\1\end{bmatrix}$ is a multiple of $\begin{bmatrix}1\\-3\end{bmatrix}$. As a check,

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix}3\\1\end{bmatrix}\cdot\begin{bmatrix}x\\-3x\end{bmatrix} = 3x - 3x = 0 \text{ for any value of } x,$$

so that the vectors are indeed orthogonal.

51. As noted in the remarks just prior to Example 1.16, the zero vector $\mathbf{0}$ is orthogonal to all vectors in $\mathbb{R}^2$. So if $\begin{bmatrix}a\\b\end{bmatrix} = \mathbf{0}$, any vector $\begin{bmatrix}x\\y\end{bmatrix}$ will do. Now assume that $\begin{bmatrix}a\\b\end{bmatrix} \neq \mathbf{0}$; that is, that either a or b is nonzero. Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if their dot product $\mathbf{u}\cdot\mathbf{v} = 0$. So we set $\mathbf{u}\cdot\mathbf{v} = 0$ and solve for y in terms of x:

$$\mathbf{u}\cdot\mathbf{v} = \begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix} = 0 \ \Rightarrow\ ax + by = 0.$$

First assume $b \neq 0$. Then $y = -\frac{a}{b}x$, so substituting back into the expression for $\mathbf{v}$ we get

$$\mathbf{v} = \begin{bmatrix}x\\-\frac{a}{b}x\end{bmatrix} = x\begin{bmatrix}1\\-\frac{a}{b}\end{bmatrix} = \frac{x}{b}\begin{bmatrix}b\\-a\end{bmatrix}.$$

Next, if $b = 0$, then $a \neq 0$, so that $x = -\frac{b}{a}y$, and substituting back into the expression for $\mathbf{v}$ gives

$$\mathbf{v} = \begin{bmatrix}-\frac{b}{a}y\\y\end{bmatrix} = y\begin{bmatrix}-\frac{b}{a}\\1\end{bmatrix} = -\frac{y}{a}\begin{bmatrix}b\\-a\end{bmatrix}.$$

So in either case, a vector orthogonal to $\begin{bmatrix}a\\b\end{bmatrix}$, if it is not the zero vector, is a multiple of $\begin{bmatrix}b\\-a\end{bmatrix}$. As a check, note that

$$\begin{bmatrix}a\\b\end{bmatrix}\cdot\begin{bmatrix}rb\\-ra\end{bmatrix} = rab - rab = 0 \text{ for all values of } r.$$

52. (a) The geometry of the vectors in Figure 1.26 suggests that if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$, then $\mathbf{u}$ and $\mathbf{v}$ point in the same direction. This means that the angle between them must be 0. So we first prove

Lemma 1. *For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*,* $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|$ *if and only if the vectors point in the same direction.*

Proof. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|},$$

so that $\cos\theta = 1$ if and only if $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|$. But $\cos\theta = 1$ if and only if $\theta = 0$, which means that $\mathbf{u}$ and $\mathbf{v}$ point in the same direction. □

We can now show

Theorem 2. *For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*,* $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| + \|\mathbf{v}\|$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *point in the same direction.*

Proof. First assume that $\mathbf{u}$ and $\mathbf{v}$ point in the same direction. Then $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|$, and thus

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} && \text{By Example 1.9}\\
&= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 && \text{Since } \mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2 \text{ for any vector } \mathbf{w}\\
&= \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \text{By the lemma}\\
&= (\|\mathbf{u}\| + \|\mathbf{v}\|)^2.
\end{aligned}$$

Since $\|\mathbf{u}+\mathbf{v}\|$ and $\|\mathbf{u}\|+\|\mathbf{v}\|$ are both nonnegative, taking square roots gives $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\|+\|\mathbf{v}\|$. For the other direction, if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\|+\|\mathbf{v}\|$, then their squares are equal, so that

$$\begin{aligned}
(\|\mathbf{u}\| + \|\mathbf{v}\|)^2 &= \|\mathbf{u}\|^2 + 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 \quad \text{and}\\
\|\mathbf{u}+\mathbf{v}\|^2 &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}
\end{aligned}$$

are equal. But $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u}$ and similarly for $\mathbf{v}$, so that canceling those terms gives $2\mathbf{u}\cdot\mathbf{v} = 2\,\|\mathbf{u}\|\,\|\mathbf{v}\|$ and thus $\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|$. Using the lemma again shows that $\mathbf{u}$ and $\mathbf{v}$ point in the same direction. ☐

(b) The geometry of the vectors in Figure 1.26 suggests that if $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$, then $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions. In addition, since $\|\mathbf{u}+\mathbf{v}\| \ge 0$, we must also have $\|\mathbf{u}\| \ge \|\mathbf{v}\|$. If they point in opposite directions, the angle between them must be π. This entire proof is exactly analogous to the proof in part (a). We first prove

Lemma 3. *For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*,* $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\,\|\mathbf{v}\|$ *if and only if the vectors point in opposite directions.*

Proof. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|},$$

so that $\cos\theta = -1$ if and only if $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\,\|\mathbf{v}\|$. But $\cos\theta = -1$ if and only if $\theta = \pi$, which means that $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions. ☐

We can now show

Theorem 4. *For all vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *or* $\mathbb{R}^3$*,* $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$ *if and only if* $\mathbf{u}$ *and* $\mathbf{v}$ *point in opposite directions and* $\|\mathbf{u}\| \ge \|\mathbf{v}\|$*.*

Proof. First assume that $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions and $\|\mathbf{u}\| \ge \|\mathbf{v}\|$. Then $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\,\|\mathbf{v}\|$, and thus

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 &= \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} && \text{By Example 1.9}\\
&= \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2 && \text{Since } \mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2 \text{ for any vector } \mathbf{w}\\
&= \|\mathbf{u}\|^2 - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\| + \|\mathbf{v}\|^2 && \text{By the lemma}\\
&= (\|\mathbf{u}\| - \|\mathbf{v}\|)^2.
\end{aligned}$$

Now, since $\|\mathbf{u}\| \ge \|\mathbf{v}\|$ by assumption, we see that both $\|\mathbf{u}+\mathbf{v}\|$ and $\|\mathbf{u}\| - \|\mathbf{v}\|$ are nonnegative, so that taking square roots gives $\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}\| - \|\mathbf{v}\|$. For the other direction, if $\|\mathbf{u}+\mathbf{v}\| =$

$\|\mathbf{u}\| - \|\mathbf{v}\|$, then first of all, since the left-hand side is nonnegative, the right-hand side must be as well, so that $\|\mathbf{u}\| \geq \|\mathbf{v}\|$. Next, we can square both sides of the equality, so that

$$(\|\mathbf{u}\| - \|\mathbf{v}\|)^2 = \|\mathbf{u}\|^2 - 2\|\mathbf{u}\|\|\mathbf{v}\| + \|\mathbf{v}\|^2 \text{ and}$$
$$\|\mathbf{u}+\mathbf{v}\|^2 = \mathbf{u}\cdot\mathbf{u} + 2\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v}$$

are equal. But $\|\mathbf{u}\|^2 = \mathbf{u}\cdot\mathbf{u}$ and similarly for $\mathbf{v}$, so that canceling those terms gives $2\mathbf{u}\cdot\mathbf{v} = -2\|\mathbf{u}\|\|\mathbf{v}\|$ and thus $\mathbf{u}\cdot\mathbf{v} = -\|\mathbf{u}\|\|\mathbf{v}\|$. Using the lemma again shows that $\mathbf{u}$ and $\mathbf{v}$ point in opposite directions. □

53. Prove Theorem 1.2(b) by applying the definition of the dot product:

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= u_1(v_1+w_1) + u_2(v_2+w_2) + \cdots + u_n(v_n+w_n)\\ &= u_1v_1 + u_1w_1 + u_2v_2 + u_2w_2 + \cdots + u_nv_n + u_nw_n\\ &= (u_1v_1 + u_2v_2 + \cdots + u_nv_n) + (u_1w_1 + u_2w_2 + \cdots + u_nw_n)\\ &= \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w}.\end{aligned}$$

54. Prove the three parts of Theorem 1.2(d) by applying the definition of the dot product and various properties of real numbers:

***Part 1*:** For any vector $\mathbf{u}$, we must show $\mathbf{u}\cdot\mathbf{u} \geq 0$. But

$$\mathbf{u}\cdot\mathbf{u} = u_1u_1 + u_2u_2 + \cdots + u_nu_n = u_1^2 + u_2^2 + \cdots + u_n^2.$$

Since for any real number x we know that $x^2 \geq 0$, it follows that this sum is also nonnegative, so that $\mathbf{u}\cdot\mathbf{u} \geq 0$.

***Part 2*:** We must show that if $\mathbf{u} = \mathbf{0}$ then $\mathbf{u}\cdot\mathbf{u} = 0$. But $\mathbf{u} = 0$ means that $u_i = 0$ for all i, so that

$$\mathbf{u}\cdot\mathbf{u} = 0\cdot 0 + 0\cdot 0 + \cdots + 0\cdot 0 = 0.$$

***Part 3*:** We must show that if $\mathbf{u}\cdot\mathbf{u} = 0$, then $\mathbf{u} = \mathbf{0}$. From part 1, we know that

$$\mathbf{u}\cdot\mathbf{u} = u_1^2 + u_2^2 + \cdots + u_n^2,$$

and that $u_i^2 \geq 0$ for all i. So if the dot product is to be zero, each u_i^2 must be zero, which means that $u_i = 0$ for all i and thus $\mathbf{u} = \mathbf{0}$.

55. We must show $\mathrm{d}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = \|\mathbf{v}-\mathbf{u}\| = \mathrm{d}(\mathbf{v},\mathbf{u})$. By definition, $\mathrm{d}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\|$. Then by Theorem 1.3(b) with $c = -1$, we have $\|-\mathbf{w}\| = \|\mathbf{w}\|$ for any vector $\mathbf{w}$; applying this to the vector $\mathbf{u}-\mathbf{v}$ gives

$$\|\mathbf{u}-\mathbf{v}\| = \|-(\mathbf{u}-\mathbf{v})\| = \|\mathbf{v}-\mathbf{u}\|,$$

which is by definition equal to $\mathrm{d}(\mathbf{v},\mathbf{u})$.

56. We must show that for any vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ that $\mathrm{d}(\mathbf{u},\mathbf{w}) \leq \mathrm{d}(\mathbf{u},\mathbf{v}) + \mathrm{d}(\mathbf{v},\mathbf{w})$. This is equivalent to showing that $\|\mathbf{u}-\mathbf{w}\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\|$. Now substitute $\mathbf{u}-\mathbf{v}$ for x and $\mathbf{v}-\mathbf{w}$ for y in Theorem 1.5, giving

$$\|\mathbf{u}-\mathbf{w}\| = \|(\mathbf{u}-\mathbf{v}) + (\mathbf{v}-\mathbf{w})\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}-\mathbf{w}\|.$$

57. We must show that $\mathrm{d}(\mathbf{u},\mathbf{v}) = \|\mathbf{u}-\mathbf{v}\| = 0$ if and only if $\mathbf{u} = \mathbf{v}$. This follows immediately from Theorem 1.3(a), $\|\mathbf{w}\| = 0$ if and only if $\mathbf{w} = \mathbf{0}$, upon setting $\mathbf{w} = \mathbf{u}-\mathbf{v}$.

58. Apply the definitions:

$$\begin{aligned}\mathbf{u}\cdot c\mathbf{v} &= [u_1, u_2, \ldots, u_n]\cdot[cv_1, cv_2, \ldots, cv_n]\\ &= u_1cv_1 + u_2cv_2 + \cdots + u_ncv_n\\ &= cu_1v_1 + cu_2v_2 + \cdots + cu_nv_n\\ &= c(u_1v_1 + u_2v_2 + \cdots + u_nv_n)\\ &= c(\mathbf{u}\cdot\mathbf{v}).\end{aligned}$$

59. We want to show that $\|\mathbf{u}-\mathbf{v}\| \geq \|\mathbf{u}\| - \|\mathbf{v}\|$. This is equivalent to showing that $\|\mathbf{u}\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}\|$. This follows immediately upon setting $\mathbf{x} = \mathbf{u}-\mathbf{v}$ and $\mathbf{y} = \mathbf{v}$ in Theorem 1.5.

60. If $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$, it does *not* follow that $\mathbf{v} = \mathbf{w}$. For example, since $\mathbf{0}\cdot\mathbf{v} = 0$ for every vector $\mathbf{v}$ in $\mathbb{R}^n$, the zero vector is orthogonal to every vector $\mathbf{v}$. So if $\mathbf{u} = \mathbf{0}$ in the above equality, we know nothing about $\mathbf{v}$ and $\mathbf{w}$. (as an example, $\mathbf{0}\cdot[1,2] = \mathbf{0}\cdot[-17,12]$). Note, however, that $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w}$ implies that $\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{w} = \mathbf{u}(\mathbf{v}-\mathbf{w}) = \mathbf{0}$, so that $\mathbf{u}$ is orthogonal to $\mathbf{v}-\mathbf{w}$.

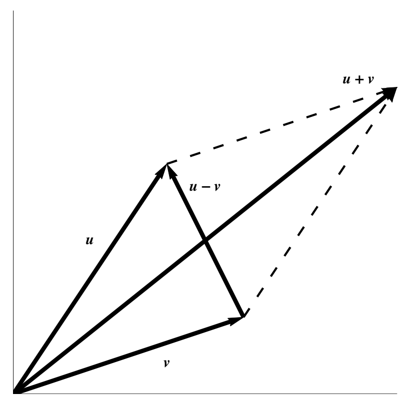
61. We must show that $(\mathbf{u}+\mathbf{v})(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$ for all vectors in $\mathbb{R}^n$. Recall that for any $\mathbf{w}$ in $\mathbb{R}^n$ that $\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2$, and also that $\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u}$. Then

$$(\mathbf{u}+\mathbf{v})(\mathbf{u}-\mathbf{v}) = \mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{v} = \|\mathbf{u}\|^2 + \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - \|\mathbf{v}\|^2 = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2.$$

62. **(a)** Let $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$. Then

$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) + (\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\\ &= (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v}) + (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v})\\ &= \left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) + 2\mathbf{u}\cdot\mathbf{v} + \left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right) - 2\mathbf{u}\cdot\mathbf{v}\\ &= 2\|\mathbf{u}\|^2 + 2\|\mathbf{v}\|^2.\end{aligned}$$

(b) Part (a) tells us that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of its four sides.



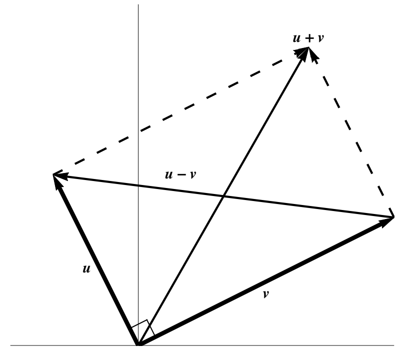
63. Let $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$. Then

$$\begin{aligned}\frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 &= \frac{1}{4}\left[(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v}) - ((\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})\right]\\ &= \frac{1}{4}\left[(\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v} - 2\mathbf{u}\cdot\mathbf{v})\right]\\ &= \frac{1}{4}\left[\left(\|\mathbf{u}\|^2 - \|\mathbf{u}\|^2\right) + \left(\|\mathbf{v}\|^2 - \|\mathbf{v}\|^2\right) + 4\mathbf{u}\cdot\mathbf{v}\right]\\ &= \mathbf{u}\cdot\mathbf{v}.\end{aligned}$$

64. **(a)** Let $\mathbf{u},\mathbf{v} \in \mathbb{R}^n$. Then using the previous exercise,

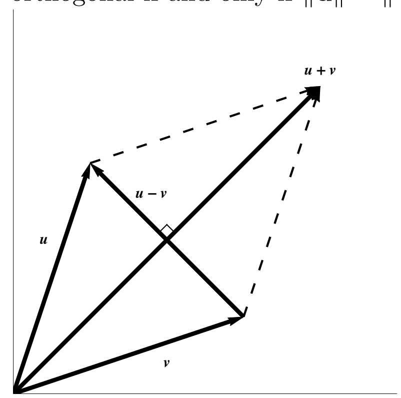
$$\begin{aligned}\|\mathbf{u}+\mathbf{v}\| = \|\mathbf{u}-\mathbf{v}\| &\Leftrightarrow \|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}-\mathbf{v}\|^2\\ &\Leftrightarrow \|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 = 0\\ &\Leftrightarrow \frac{1}{4}\|\mathbf{u}+\mathbf{v}\|^2 - \frac{1}{4}\|\mathbf{u}-\mathbf{v}\|^2 = 0\\ &\Leftrightarrow \mathbf{u}\cdot\mathbf{v} = 0\\ &\Leftrightarrow \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal.}\end{aligned}$$

(b) Part (a) tells us that a parallelogram is a rectangle if and only if the lengths of its diagonals are equal.



65. **(a)** By Exercise 55, $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2$. Thus $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}-\mathbf{v}) = 0$ if and only if $\|\mathbf{u}\|^2 = \|\mathbf{v}\|^2$. It follows immediately that $\mathbf{u}+\mathbf{v}$ and $\mathbf{u}-\mathbf{v}$ are orthogonal if and only if $\|\mathbf{u}\| = \|\mathbf{v}\|$.

(b) Part (a) tells us that the diagonals of a parallelogram are perpendicular if and only if the lengths of its sides are equal, i.e., if and only if it is a rhombus.



66. From Example 1.9 and the fact that $\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2$, we have $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2$. Taking the square root of both sides yields $\|\mathbf{u}+\mathbf{v}\| = \sqrt{\|\mathbf{u}\|^2 + 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{v}\|^2}$. Now substitute in the given values $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = \sqrt{3}$, and $\mathbf{u}\cdot\mathbf{v} = 1$, giving

$$\|\mathbf{u}+\mathbf{v}\| = \sqrt{2^2 + 2\cdot 1 + \left(\sqrt{3}\right)^2} = \sqrt{4+2+3} = \sqrt{9} = 3.$$

67. From Theorem 1.4 (the Cauchy-Schwarcz inequality), we have $|\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|$. If $\|\mathbf{u}\| = 1$ and $\|\mathbf{v}\| = 2$, then $|\mathbf{u}\cdot\mathbf{v}| \le 2$, so we cannot have $\mathbf{u}\cdot\mathbf{v} = 3$.

68. **(a)** If $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$. Then

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{w} = 0 + 0 = 0,$$

so that $\mathbf{u}$ is orthogonal to $\mathbf{v}+\mathbf{w}$.

(b) If $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$, then $\mathbf{u}\cdot\mathbf{v} = \mathbf{u}\cdot\mathbf{w} = 0$. Then

$$\mathbf{u}\cdot(s\mathbf{v}+t\mathbf{w}) = \mathbf{u}\cdot(s\mathbf{v}) + \mathbf{u}\cdot(t\mathbf{w}) = s(\mathbf{u}\cdot\mathbf{v}) + t(\mathbf{u}\cdot\mathbf{w}) = s\cdot 0 + t\cdot 0 = 0,$$

so that $\mathbf{u}$ is orthogonal to $s\mathbf{v}+t\mathbf{w}$.

69. We have

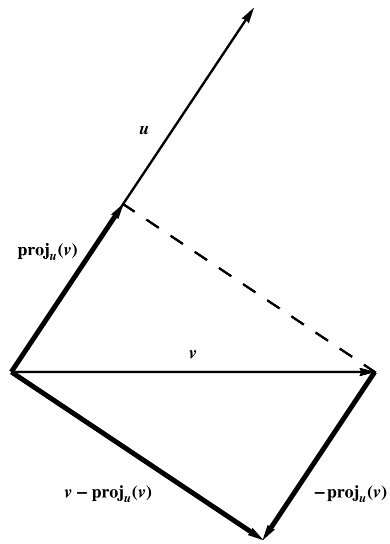
$$\begin{aligned}
\mathbf{u}\cdot(\mathbf{v}-\operatorname{proj}_{\mathbf{u}}\mathbf{v}) &= \mathbf{u}\cdot\left(\mathbf{v}-\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\right)\\
&= \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\\
&= \mathbf{u}\cdot\mathbf{v} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)(\mathbf{u}\cdot\mathbf{u})\\
&= \mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v}\\
&= 0.
\end{aligned}$$

70. **(a)** $\text{proj}_{\mathbf{u}}(\text{proj}_{\mathbf{u}}\,\mathbf{v}) = \text{proj}_{\mathbf{u}}\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\,\text{proj}_{\mathbf{u}}\,\mathbf{u} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \text{proj}_{\mathbf{u}}\,\mathbf{v}$.

(b) Using part (a),

$$\begin{aligned}\text{proj}_{\mathbf{u}}\left(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}\right) = \text{proj}_{\mathbf{u}}\left(\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right) &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\text{proj}_{\mathbf{u}}\,\mathbf{u}\\ &= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = \mathbf{0}.\end{aligned}$$

(c) From the diagram, we see that $\text{proj}_{\mathbf{u}}\,\mathbf{v}\|\mathbf{u}$, so that $\text{proj}_{\mathbf{u}}\left(\text{proj}_{\mathbf{u}}\,\mathbf{v}\right) = \text{proj}_{\mathbf{u}}\,\mathbf{v}$. Also, $(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}) \perp \mathbf{u}$, so that $\text{proj}_{\mathbf{u}}\left(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}\right) = \mathbf{0}$.



71. **(a)** We have

$$\begin{aligned}(u_1^2+u_2^2)(v_1^2+v_2^2) - (u_1v_1+u_2v_2)^2 &= u_1^2v_1^2 + u_1^2v_2^2 + u_2^2v_1^2 + u_2^2v_2^2 - u_1^2v_1^2 - 2u_1v_1u_2v_2 - u_2^2v_2^2\\ &= u_1^2v_2^2 + u_2^2v_1^2 - 2u_1u_2v_1v_2\\ &= (u_1v_2 - u_2v_1)^2.\end{aligned}$$

But the final expression is nonnegative since it is a square. Thus the original expression is as well, showing that $(u_1^2+u_2^2)(v_1^2+v_2^2) - (u_1v_1+u_2v_2)^2 \ge 0$.

(b) We have

$$\begin{aligned}&(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2\\ &\quad= u_1^2v_1^2 + u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_2^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2 + u_3^2v_3^2\\ &\qquad\quad - u_1^2v_1^2 - 2u_1v_1u_2v_2 - u_2^2v_2^2 - 2u_1v_1u_3v_3 - u_3^2v_3^2 - 2u_2v_2u_3v_3\\ &\quad= u_1^2v_2^2 + u_1^2v_3^2 + u_2^2v_1^2 + u_2^2v_3^2 + u_3^2v_1^2 + u_3^2v_2^2\\ &\qquad\quad - 2u_1u_2v_1v_2 - 2u_1v_1u_3v_3 - 2u_2v_2u_3v_3\\ &\quad= (u_1v_2 - u_2v_1)^2 + (u_1v_3 - u_3v_1)^2 + (u_3v_2 - u_2v_3)^2.\end{aligned}$$

But the final expression is nonnegative since it is the sum of three squares. Thus the original expression is as well, showing that $(u_1^2+u_2^2+u_3^2)(v_1^2+v_2^2+v_3^2) - (u_1v_1+u_2v_2+u_3v_3)^2 \ge 0$.

72. **(a)** Since $\text{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$, we have

$$\begin{aligned}\text{proj}_{\mathbf{u}}\,\mathbf{v}\cdot(\mathbf{v} - \text{proj}_{\mathbf{u}}\,\mathbf{v}) &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\cdot\left(\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\right)\\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\left(\mathbf{u}\cdot\mathbf{v} - \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}\cdot\mathbf{u}\right)\\ &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}(\mathbf{u}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v})\\ &= 0,\end{aligned}$$

so that $\operatorname{proj}_{\mathbf{u}} \mathbf{v}$ is orthogonal to $\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}$. Since their vector sum is $\mathbf{v}$, those three vectors form a right triangle with hypotenuse $\mathbf{v}$, so by Pythagoras' Theorem,

$$\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|^2 \le \|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\|^2 + \|\mathbf{v} - \operatorname{proj}_{\mathbf{u}} \mathbf{v}\|^2 = \|\mathbf{v}\|^2 .$$

Since norms are always nonnegative, taking square roots gives $\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\| \le \|\mathbf{v}\|$.

(b)

$$\begin{aligned}
\|\operatorname{proj}_{\mathbf{u}} \mathbf{v}\| \le \|\mathbf{v}\| &\iff \left\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\right\| \le \|\mathbf{v}\| \\
&\iff \left\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\right)\mathbf{u}\right\| \le \|\mathbf{v}\| \\
&\iff \left|\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|^2}\right| \|\mathbf{u}\| \le \|\mathbf{v}\| \\
&\iff \frac{|\mathbf{u}\cdot\mathbf{v}|}{\|\mathbf{u}\|} \le \|\mathbf{v}\| \\
&\iff |\mathbf{u}\cdot\mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\| ,
\end{aligned}$$

which is the Cauchy-Schwarcz inequality.

73. Suppose $\operatorname{proj}_{\mathbf{u}} \mathbf{v} = c\mathbf{u}$. From the figure, we see that $\cos\theta = \frac{c\|\mathbf{u}\|}{\|\mathbf{v}\|}$. But also $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$. Thus these two expressions are equal, i.e.,

$$\frac{c\,\|\mathbf{u}\|}{\|\mathbf{v}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} \Rightarrow c\,\|\mathbf{u}\| = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|} \Rightarrow c = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{u}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}.$$

74. The basis for induction is the cases $n = 1$ and $n = 2$. The $n = 1$ case is the assertion that $\|\mathbf{v}_1\| \le \|\mathbf{v}_2\|$, which is obviously true. The $n = 2$ case is the Triangle Inequality, which is also true.

Now assume the statement holds for $n = k \ge 2$; that is, for any $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$,

$$\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k\| \le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k\| .$$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}$ be any vectors. Then

$$\begin{aligned}
\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k + \mathbf{v}_{k+1}\| &= \|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + (\mathbf{v}_k + \mathbf{v}_{k+1})\| \\
&\le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k + \mathbf{v}_{k+1}\|
\end{aligned}$$

using the inductive hypothesis. But then using the Triangle Inequality (or the case $n = 2$ in this theorem), $\|\mathbf{v}_k + \mathbf{v}_{k+1}\| \le \|\mathbf{v}_k\| + \|\mathbf{v}_{k+1}\|$. Substituting into the above gives

$$\begin{aligned}
\|\mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_k + \mathbf{v}_{k+1}\| &\le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k + \mathbf{v}_{k+1}\| \\
&\le \|\mathbf{v}_1\| + \|\mathbf{v}_2\| + \cdots + \|\mathbf{v}_k\| + \|\mathbf{v}_{k+1}\| ,
\end{aligned}$$

which is what we were trying to prove.

Exploration: Vectors and Geometry

1. As in Example 1.25, let $\mathbf{p} = \overrightarrow{OP}$. Then $\mathbf{p} - \mathbf{a} = \overrightarrow{AP} = \frac{1}{3}\overrightarrow{AB} = \frac{1}{3}(\mathbf{b} - \mathbf{a})$, so that $\mathbf{p} = \mathbf{a} + \frac{1}{3}(\mathbf{b} - \mathbf{a}) = \frac{1}{3}(2\mathbf{a} + \mathbf{b})$. More generally, if P is the point $\frac{1}{n}$ of the way from A to B along $\overrightarrow{AB}$, then $\mathbf{p} - \mathbf{a} = \overrightarrow{AP} = \frac{1}{n}\overrightarrow{AB} = \frac{1}{n}(\mathbf{b} - \mathbf{a})$, so that $\mathbf{p} = \mathbf{a} + \frac{1}{n}(\mathbf{b} - \mathbf{a}) = \frac{1}{n}((n-1)\mathbf{a} + \mathbf{b})$.

2. Use the notation that the vector $\overrightarrow{OX}$ is written $\mathbf{x}$. Then from exercise 1, we have $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{c})$ and $\mathbf{q} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$, so that

$$\overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \frac{1}{2}(\mathbf{b} + \mathbf{c}) - \frac{1}{2}(\mathbf{a} + \mathbf{c}) = \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}\overrightarrow{AB}.$$

3. Draw $\overrightarrow{AC}$. Then from exercise 2, we have $\overrightarrow{PQ} = \frac{1}{2}\overrightarrow{AB} = \overrightarrow{SR}$. Also draw $\overrightarrow{BD}$. Again from exercise 2, we have $\overrightarrow{PS} = \frac{1}{2}\overrightarrow{BD} = \overrightarrow{QR}$. Thus opposite sides of the quadrilateral $PQRS$ are equal. They are also parallel: indeed, $\triangle BPQ$ and $\triangle BAC$ are similar, since they share an angle and $BP : BA = BQ : BC$. Thus $\angle BPQ = \angle BAC$; since these angles are equal, $PQ \| AC$. Similarly, $SR \| AC$ so that $PQ \| SR$. In a like manner, we see that $PS \| RQ$. Thus $PQRS$ is a parallelogram.

4. Following the hint, we find $\mathbf{m}$, the point that is two-thirds of the distance from A to P. From exercise 1, we have

$$\mathbf{p} = \frac{1}{2}(\mathbf{b}+\mathbf{c}), \text{ so that } \mathbf{m} = \frac{1}{3}(2\mathbf{p}+\mathbf{a}) = \frac{1}{3}\left(2\cdot\frac{1}{2}(\mathbf{b}+\mathbf{c})+\mathbf{a}\right) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Next we find $\mathbf{m}'$, the point that is two-thirds of the distance from B to Q. Again from exercise 1, we have

$$\mathbf{q} = \frac{1}{2}(\mathbf{a}+\mathbf{c}), \text{ so that } \mathbf{m}' = \frac{1}{3}(2\mathbf{q}+\mathbf{b}) = \frac{1}{3}\left(2\cdot\frac{1}{2}(\mathbf{a}+\mathbf{c})+\mathbf{b}\right) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Finally we find $\mathbf{m}''$, the point that is two-thirds of the distance from C to R. Again from exercise 1, we have

$$\mathbf{r} = \frac{1}{2}(\mathbf{a}+\mathbf{b}), \text{ so that } \mathbf{m}'' = \frac{1}{3}(2\mathbf{r}+\mathbf{c}) = \frac{1}{3}\left(2\cdot\frac{1}{2}(\mathbf{a}+\mathbf{b})+\mathbf{c}\right) = \frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c}).$$

Since $\mathbf{m} = \mathbf{m}' = \mathbf{m}''$, all three medians intersect at the centroid, G.

5. With notation as in the figure, we know that $\overrightarrow{AH}$ is orthogonal to $\overrightarrow{BC}$; that is, $\overrightarrow{AH}\cdot\overrightarrow{BC} = 0$. Also $\overrightarrow{BH}$ is orthogonal to $\overrightarrow{AC}$; that is, $\overrightarrow{BH}\cdot\overrightarrow{AC} = 0$. We must show that $\overrightarrow{CH}\cdot\overrightarrow{AB} = 0$. But

$$\begin{aligned}
\overrightarrow{AH}\cdot\overrightarrow{BC} = 0 &\Rightarrow (\mathbf{h}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{c}) = 0 \Rightarrow \mathbf{h}\cdot\mathbf{b}-\mathbf{h}\cdot\mathbf{c}-\mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} = 0\\
\overrightarrow{BH}\cdot\overrightarrow{AC} = 0 &\Rightarrow (\mathbf{h}-\mathbf{b})\cdot(\mathbf{c}-\mathbf{a}) = 0 \Rightarrow \mathbf{h}\cdot\mathbf{c}-\mathbf{h}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{a} = 0.
\end{aligned}$$

Adding these two equations together and canceling like terms gives

$$0 = \mathbf{h}\cdot\mathbf{b}-\mathbf{h}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c} = (\mathbf{h}-\mathbf{c})\cdot(\mathbf{b}-\mathbf{a}) = \overrightarrow{CH}\cdot\overrightarrow{AB},$$

so that these two are orthogonal. Thus all the altitudes intersect at the orthocenter H.

6. We are given that $\overrightarrow{QK}$ is orthogonal to $\overrightarrow{AC}$ and that $\overrightarrow{PK}$ is orthogonal to $\overrightarrow{CB}$, and must show that $\overrightarrow{RK}$ is orthogonal to $\overrightarrow{AB}$. By exercise 1, we have $\mathbf{q} = \frac{1}{2}(\mathbf{a}+\mathbf{c})$, $\mathbf{p} = \frac{1}{2}(\mathbf{b}+\mathbf{c})$, and $\mathbf{r} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$. Thus

$$\begin{aligned}
\overrightarrow{QK}\cdot\overrightarrow{AC} = 0 &\Rightarrow (\mathbf{k}-\mathbf{q})\cdot(\mathbf{c}-\mathbf{a}) = 0 \Rightarrow \left(\mathbf{k}-\frac{1}{2}(\mathbf{a}+\mathbf{c})\right)\cdot(\mathbf{c}-\mathbf{a}) = 0\\
\overrightarrow{PK}\cdot\overrightarrow{CB} = 0 &\Rightarrow (\mathbf{k}-\mathbf{p})\cdot(\mathbf{b}-\mathbf{c}) = 0 \Rightarrow \left(\mathbf{k}-\frac{1}{2}(\mathbf{b}+\mathbf{c})\right)\cdot(\mathbf{b}-\mathbf{c}) = 0.
\end{aligned}$$

Expanding the two dot products gives

$$\begin{aligned}
\mathbf{k}\cdot\mathbf{c}-\mathbf{k}\cdot\mathbf{a}-\frac{1}{2}\mathbf{a}\cdot\mathbf{c}+\frac{1}{2}\mathbf{a}\cdot\mathbf{a}-\frac{1}{2}\mathbf{c}\cdot\mathbf{c}+\frac{1}{2}\mathbf{a}\cdot\mathbf{c} = 0\\
\mathbf{k}\cdot\mathbf{b}-\mathbf{k}\cdot\mathbf{c}-\frac{1}{2}\mathbf{b}\cdot\mathbf{b}+\frac{1}{2}\mathbf{b}\cdot\mathbf{c}-\frac{1}{2}\mathbf{c}\cdot\mathbf{b}+\frac{1}{2}\mathbf{c}\cdot\mathbf{c} = 0.
\end{aligned}$$

Add these two together and cancel like terms to get

$$0 = \mathbf{k}\cdot\mathbf{b}-\mathbf{k}\cdot\mathbf{a}-\frac{1}{2}\mathbf{b}\cdot\mathbf{b}+\frac{1}{2}\mathbf{a}\cdot\mathbf{a} = \left(\mathbf{k}-\frac{1}{2}(\mathbf{b}+\mathbf{a})\right)\cdot(\mathbf{b}-\mathbf{a}) = (\mathbf{k}-\mathbf{r})\cdot(\mathbf{b}-\mathbf{a}) = \overrightarrow{RK}\cdot\overrightarrow{AB}.$$

Thus $\overrightarrow{RK}$ and $\overrightarrow{AB}$ are indeed orthogonal, so all the perpendicular bisectors intersect at the circumcenter.

7. Let O, the center of the circle, be the origin. Then $\mathbf{b} = -\mathbf{a}$ and $\|\mathbf{a}\|^2 = \|\mathbf{c}\|^2 = r^2$ where r is the radius of the circle. We want to show that $\overrightarrow{AC}$ is orthogonal to $\overrightarrow{BC}$. But

$$\begin{aligned}\overrightarrow{AC} \cdot \overrightarrow{BC} &= (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{b}) \\ &= (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} + \mathbf{a}) \\ &= \|\mathbf{c}\|^2 + \mathbf{c} \cdot \mathbf{a} - \|\mathbf{a}\|^2 - \mathbf{a} \cdot \mathbf{c} \\ &= (\mathbf{a} \cdot \mathbf{c} - \mathbf{c} \cdot \mathbf{a}) + (r^2 - r^2) = 0.\end{aligned}$$

Thus the two are orthogonal, so that $\angle ACB$ is a right angle.

8. As in exercise 5, we first find $\mathbf{m}$, the point that is halfway from P to R. We have $\mathbf{p} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$ and $\mathbf{r} = \frac{1}{2}(\mathbf{c} + \mathbf{d})$, so that

$$\mathbf{m} = \frac{1}{2}(\mathbf{p} + \mathbf{r}) = \frac{1}{2}\left(\frac{1}{2}(\mathbf{a} + \mathbf{b}) + \frac{1}{2}(\mathbf{c} + \mathbf{d})\right) = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}).$$

Similarly, we find $\mathbf{m}'$, the point that is halfway from Q to S. We have $\mathbf{q} = \frac{1}{2}(\mathbf{b} + \mathbf{c})$ and $\mathbf{s} = \frac{1}{2}(\mathbf{a} + \mathbf{d})$, so that

$$\mathbf{m}' = \frac{1}{2}(\mathbf{q} + \mathbf{s}) = \frac{1}{2}\left(\frac{1}{2}(\mathbf{b} + \mathbf{c}) + \frac{1}{2}(\mathbf{a} + \mathbf{d})\right) = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}).$$

Thus $\mathbf{m} = \mathbf{m}'$, so that $\overrightarrow{PR}$ and $\overrightarrow{QS}$ intersect at their mutual midpoints; thus, they bisect each other.

1.3 Lines and Planes

1. **(a)** The normal form is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, or $\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \left(\mathbf{x} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = 0$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, we get

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = 3x + 2y = 0.$$

The general form is $3x + 2y = 0$.

2. **(a)** The normal form is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, or $\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \left(\mathbf{x} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 0$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, we get

$$\begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} x - 1 \\ y - 2 \end{bmatrix} = 3(x - 1) - 4(y - 2) = 0.$$

Expanding and simplifying gives the general form $3x - 4y = -5$.

3. **(a)** In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and expanding the vector form from part (a) gives $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 - t \\ 3t \end{bmatrix}$, which yields the parametric form $x = 1 - t$, $y = 3t$.

4. **(a)** In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} -4 \\ 4 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and expanding the vector form from part (a) gives the parametric form $x = -4 + t$, $y = 4 + t$.

5. **(a)** In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and expanding the vector form from part (a) gives the parametric form $x = t$, $y = -t$, $z = 4t$.

6. **(a)** In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}$.

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and expanding the vector form from part (a) gives $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 + 2t \\ 5t \\ -2 \end{bmatrix}$, which yields the parametric form $x = 3 + 2t$, $y = 5t$, $z = -2$.

7. **(a)** The normal form is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, or $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \left(\mathbf{x} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = 0.$

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we get

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y - 1 \\ z \end{bmatrix} = 3x + 2(y - 1) + z = 0.$$

Expanding and simplifying gives the general form $3x + 2y + z = 2$.

8. **(a)** The normal form is $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, or $\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \cdot \left(\mathbf{x} - \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \right) = 0.$

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we get

$$\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} \right) = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x - 3 \\ y \\ z + 2 \end{bmatrix} = 2(x - 3) + 5y = 0.$$

Expanding and simplifying gives the general form $2x + 5y = 6$.

9. **(a)** In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, or

$$\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}.$$

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and expanding the vector form from part (a) gives

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2s - 3t \\ s + 2t \\ 2s + t \end{bmatrix}$$

which yields the parametric form the parametric form $x = 2s - 3t$, $y = s + 2t$, $z = 2s + t$.

10. **(a)** In vector form, the equation of the line is $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, or

$$\mathbf{x} = \begin{bmatrix} 6 \\ -4 \\ -3 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

(b) Letting $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and expanding the vector form from part (a) gives the parametric form $x = 6-t$, $y = -4 + s + t$, $z = -3 + s + t$.

11. Any pair of points on ℓ determine a direction vector, so we use P and Q. We choose P to represent the point on the line. Then a direction vector for the line is $\mathbf{d} = \overrightarrow{PQ} = (3, 0) - (1, -2) = (2, 2)$. The vector equation for the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} + t\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

12. Any pair of points on ℓ determine a direction vector, so we use P and Q. We choose P to represent the point on the line. Then a direction vector for the line is $\mathbf{d} = \overrightarrow{PQ} = (-2, 1, 3) - (0, 1, -1) = (-2, 0, 4)$. The vector equation for the line is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, or $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}$.

13. We must find two direction vectors, $\mathbf{u}$ and $\mathbf{v}$. Since P, Q, and R lie in a plane, we compute We get two direction vectors

$$\begin{aligned} \mathbf{u} &= \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = (4, 0, 2) - (1, 1, 1) = (3, -1, 1) \\ \mathbf{v} &= \overrightarrow{PR} = \mathbf{r} - \mathbf{p} = (0, 1, -1) - (1, 1, 1) = (-1, 0, -2). \end{aligned}$$

Since $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples of each other, they will serve as direction vectors (if they were parallel to each other, we would have not a plane but a line). So the vector equation for the plane is $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, or $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$.

14. We must find two direction vectors, $\mathbf{u}$ and $\mathbf{v}$. Since P, Q, and R lie in a plane, we compute We get two direction vectors

$$\begin{aligned} \mathbf{u} &= \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = (1, 0, 1) - (1, 1, 0) = (0, -1, 1) \\ \mathbf{v} &= \overrightarrow{PR} = \mathbf{r} - \mathbf{p} = (0, 1, 1) - (1, 1, 0) = (-1, 0, 1). \end{aligned}$$

Since $\mathbf{u}$ and $\mathbf{v}$ are not scalar multiples of each other, they will serve as direction vectors (if they were parallel to each other, we would have not a plane but a line). So the vector equation for the plane is $\mathbf{x} = \mathbf{p} + s\mathbf{u} + t\mathbf{v}$, or $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

15. The parametric and associated vector forms $\mathbf{x} = \mathbf{p} + t\mathbf{d}$ found below are not unique.

(a) As in the remarks prior to Example 1.20, we start by letting $x = t$. Substituting $x = t$ into $y = 3x - 1$ gives $y = 3t - 1$. So we get parametric equations $x = t$, $y = 3t - 1$, and corresponding vector form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(b) In this case since the coefficient of y is 2, we start by letting $x = 2t$. Substituting $x = 2t$ into $3x + 2y = 5$ gives $3 \cdot 2t + 2y = 5$, which gives $y = -3t + \frac{5}{2}$. So we get parametric equations $x = 2t$, $y = \frac{5}{2} - 3t$, with corresponding vector equation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{2} \end{bmatrix} + t\begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

Note that the equation was of the form $ax + by = c$ with $a = 3$, $b = 2$, and that a direction vector was given by $\begin{bmatrix} b \\ -a \end{bmatrix}$. This is true in general.

16. Note that $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$ is the line that passes through $\mathbf{p}$ (when $t = 0$) and $\mathbf{q}$ (when $t = 1$). We write $\mathbf{d} = \mathbf{q} - \mathbf{p}$; this is a direction vector for the line through $\mathbf{p}$ and $\mathbf{q}$.

(a) As noted above, the line $\mathbf{p} + t\mathbf{d}$ passes through P at $t = 0$ and through Q at $t = 1$. So as t varies from 0 to 1, the line describes the line segment $\overline{PQ}$.

(b) As shown in **Exploration: Vectors and Geometry**, to find the midpoint of $\overline{PQ}$, we start at P and travel half the length of $\overline{PQ}$ in the direction of the vector $\overrightarrow{PQ} = \mathbf{q} - \mathbf{p}$. That is, the midpoint of $\overline{PQ}$ is the head of the vector $\mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p})$. Since $\mathbf{x} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$, we see that this line passes through the midpoint at $t = \frac{1}{2}$, and that the midpoint is in fact $\mathbf{p} + \frac{1}{2}(\mathbf{q} - \mathbf{p}) = \frac{1}{2}(\mathbf{p} + \mathbf{q})$.

(c) From part (b), the midpoint is $\frac{1}{2}([2, -3] + [0, 1]) = \frac{1}{2}[2, -2] = [1, -1]$.

(d) From part (b), the midpoint is $\frac{1}{2}([1, 0, 1] + [4, 1, -2]) = \frac{1}{2}[5, 1, -1] = \left[\frac{5}{2}, \frac{1}{2}, -\frac{1}{2}\right]$.

(e) Again from **Exploration: Vectors and Geometry**, the vector whose head is $\frac{1}{3}$ of the way from P to Q along $\overline{PQ}$ is $\mathbf{x}_1 = \frac{1}{3}(2\mathbf{p} + \mathbf{q})$. Similarly, the vector whose head is $\frac{2}{3}$ of the way from P to Q along $\overline{PQ}$ is also the vector one third of the way from Q to P along $\overline{QP}$; applying the same formula gives for this point $\mathbf{x}_2 = \frac{1}{3}(2\mathbf{q} + \mathbf{p})$. When $\mathbf{p} = [2, -3]$ and $\mathbf{q} = [0, 1]$, we get

$$\mathbf{x}_1 = \frac{1}{3}(2[2, -3] + [0, 1]) = \frac{1}{3}[4, -5] = \left[\frac{4}{3}, -\frac{5}{3}\right]$$
$$\mathbf{x}_2 = \frac{1}{3}(2[0, 1] + [2, -3]) = \frac{1}{3}[2, -1] = \left[\frac{2}{3}, -\frac{1}{3}\right].$$

(f) Using the formulas from part (e) with $\mathbf{p} = [1, 0, -1]$ and $\mathbf{q} = [4, 1, -2]$ gives

$$\mathbf{x}_1 = \frac{1}{3}(2[1, 0, -1] + [4, 1, -2]) = \frac{1}{3}[6, 1, -4] = \left[2, \frac{1}{3}, -\frac{4}{3}\right]$$
$$\mathbf{x}_2 = \frac{1}{3}(2[4, 1, -2] + [1, 0, -1]) = \frac{1}{3}[9, 2, -5] = \left[3, \frac{2}{3}, -\frac{5}{3}\right].$$

17. A line ℓ_1 with slope m_1 has equation $y = m_1x + b_1$, or $-m_1x + y = b_1$. Similarly, a line ℓ_2 with slope m_2 has equation $y = m_2x + b_2$, or $-m_2x + y = b_2$. Thus the normal vector for ℓ_1 is $\mathbf{n}_1 = \begin{bmatrix} -m_1 \\ 1 \end{bmatrix}$, and the normal vector for ℓ_2 is $\mathbf{n}_2 = \begin{bmatrix} -m_2 \\ 1 \end{bmatrix}$. Now, ℓ_1 and ℓ_2 are perpendicular if and only if their normal vectors are perpendicular, i.e., if and only if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$. But

$$\mathbf{n}_1 \cdot \mathbf{n}_2 = \begin{bmatrix} -m_1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -m_2 \\ 1 \end{bmatrix} = m_1m_2 + 1,$$

so that the normal vectors are perpendicular if and only if $m_1m_2 + 1 = 0$, i.e., if and only if $m_1m_2 = -1$.

18. Suppose the line ℓ has direction vector $\mathbf{d}$, and the plane $\mathscr{P}$ has normal vector $\mathbf{n}$. Then if $\mathbf{d} \cdot \mathbf{n} = 0$ ($\mathbf{d}$ and $\mathbf{n}$ are orthogonal), then the line ℓ is parallel to the plane $\mathscr{P}$. If on the other hand $\mathbf{d}$ and $\mathbf{n}$ are parallel, so that $\mathbf{d} = \mathbf{n}$, then ℓ is perpendicular to $\mathscr{P}$.

(a) Since the general form of $\mathscr{P}$ is $2x + 3y - z = 1$, its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. Since $\mathbf{d} = 1\mathbf{n}$, we see that ℓ is perpendicular to $\mathscr{P}$.

(b) Since the general form of $\mathscr{P}$ is $4x - y + 5z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$. Since

$$\mathbf{d} \cdot \mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} = 2 \cdot 4 + 3 \cdot (-1) - 1 \cdot 5 = 0,$$

ℓ is parallel to $\mathscr{P}$.

(c) Since the general form of $\mathscr{P}$ is $x - y - z = 3$, its normal vector is $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Since

$$\mathbf{d} \cdot \mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 2 \cdot 1 + 3 \cdot (-1) - 1 \cdot (-1) = 0,$$

ℓ is parallel to $\mathscr{P}$.

(d) Since the general form of $\mathscr{P}$ is $4x + 6y - 2z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$. Since

$$\mathbf{d} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = \frac{1}{2}\mathbf{n},$$

ℓ is perpendicular to $\mathscr{P}$.

19. Suppose the plane $\mathscr{P}_1$ has normal vector $\mathbf{n}_1$, and the plane $\mathscr{P}$ has normal vector $\mathbf{n}$. Then if $\mathbf{n}_1 \cdot \mathbf{n} = 0$ ($\mathbf{n}_1$ and $\mathbf{n}$ are orthogonal), then $\mathscr{P}_1$ is perpendicular to $\mathscr{P}$. If on the other hand $\mathbf{n}_1$ and $\mathbf{n}$ are parallel, so that $\mathbf{n}_1 = c\mathbf{n}$, then $\mathscr{P}_1$ is parallel to $\mathscr{P}$. Note that in this exercise, $\mathscr{P}_1$ has the equation $4x - y + 5z = 2$, so that $\mathbf{n}_1 = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$.

(a) Since the general form of $\mathscr{P}$ is $2x + 3y - z = 1$, its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$. Since

$$\mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = 4 \cdot 2 - 1 \cdot 3 + 5 \cdot (-1) = 0,$$

the normal vectors are perpendicular, and thus $\mathscr{P}_1$ is perpendicular to $\mathscr{P}$.

(b) Since the general form of $\mathscr{P}$ is $4x - y + 5z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$. Since $\mathbf{n}_1 = \mathbf{n}$, $\mathscr{P}_1$ is parallel to $\mathscr{P}$.

(c) Since the general form of $\mathscr{P}$ is $x - y - z = 3$, its normal vector is $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Since

$$\mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 4 \cdot 1 - 1 \cdot (-1) + 5 \cdot (-1) = 0,$$

the normal vectors are perpendicular, and thus $\mathscr{P}_1$ is perpendicular to $\mathscr{P}$.

(d) Since the general form of $\mathscr{P}$ is $4x + 6y - 2z = 0$, its normal vector is $\mathbf{n} = \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix}$. Since

$$\mathbf{n}_1 \cdot \mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 6 \\ -2 \end{bmatrix} = 4 \cdot 4 - 1 \cdot 6 + 5 \cdot (-2) = 0,$$

the normal vectors are perpendicular, and thus $\mathscr{P}_1$ is perpendicular to $\mathscr{P}$.

20. Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine $\mathbf{p}$ and $\mathbf{d}$. The general equation of the given line is $2x - 3y = 1$, so its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Our line is perpendicular to that line, so it has direction vector $\mathbf{d} = \mathbf{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Furthermore, since our line passes through the point $P = (2, -1)$, we have $\mathbf{p} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Thus the vector form of the line perpendicular to $2x - 3y = 1$ through the point $P = (2, -1)$ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

21. Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine $\mathbf{p}$ and $\mathbf{d}$. The general equation of the given line is $2x - 3y = 1$, so its normal vector is $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$. Our line is parallel to that line, so it has direction vector $\mathbf{d} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (note that $\mathbf{d} \cdot \mathbf{n} = 0$). Since our line passes through the point $P = (2, -1)$, we have $\mathbf{p} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, so that the vector equation of the line parallel to $2x - 3y = 1$ through the point $P = (2, -1)$ is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

22. Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine $\mathbf{p}$ and $\mathbf{d}$. A line is perpendicular to a plane if its direction vector $\mathbf{d}$ is the normal vector $\mathbf{n}$ of the plane. The general equation of the given plane is $x - 3y + 2z = 5$, so its normal vector is $\mathbf{n} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$. Thus the direction vector of our line is $\mathbf{d} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}$. Furthermore, since our line passes through the point $P = (-1, 0, 3)$, we have $\mathbf{p} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}$. So the vector form of the line perpendicular to $x - 3y + 2z = 5$ through $P = (-1, 0, 3)$ is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}.$$

23. Since the vector form is $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we use the given information to determine $\mathbf{p}$ and $\mathbf{d}$. Since the given line has parametric equations

$$x = 1 - t, \quad y = 2 + 3t, \quad z = -2 - t, \quad \text{it has vector form} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}.$$

So its direction vector is $\begin{bmatrix}-1\\3\\-1\end{bmatrix}$, and this must be the direction vector **d** of the line we want, which is parallel to the given line. Since our line passes through the point $P=(-1,0,3)$, we have $\mathbf{p}=\begin{bmatrix}-1\\0\\3\end{bmatrix}$. So the vector form of the line parallel to the given line through $P=(-1,0,3)$ is

$$\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-1\\0\\3\end{bmatrix}+t\begin{bmatrix}-1\\3\\-1\end{bmatrix}.$$

24. Since the normal form is $\mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{p}$, we use the given information to determine **n** and **p**. Note that a plane is parallel to a given plane if their normal vectors are equal. Since the general form of the given plane is $6x-y+2z=3$, its normal vector is $\mathbf{n}=\begin{bmatrix}6\\-1\\2\end{bmatrix}$, so this must be a normal vector of the desired plane as well. Furthermore, since our plane passes through the point $P=(0,-2,5)$, we have $\mathbf{p}=\begin{bmatrix}0\\-2\\5\end{bmatrix}$. So the normal form of the plane parallel to $6x-y+2z=3$ through $(0,-2,5)$ is

$$\begin{bmatrix}6\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}6\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}0\\-2\\5\end{bmatrix}\quad\text{or}\quad\begin{bmatrix}6\\-1\\2\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=12.$$

25. Using Figure 1.34 in Section 1.2 for reference, we will find a normal vector **n** and a point vector **p** for each of the sides, then substitute into $\mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{p}$ to get an equation for each plane.

(a) Start with $\mathscr{P}_1$ determined by the face of the cube in the xy-plane. Clearly a normal vector for $\mathscr{P}_1$ is $\mathbf{n}=\begin{bmatrix}1\\0\\0\end{bmatrix}$, or any vector parallel to the x-axis. Also, the plane passes through $P=(0,0,0)$, so we set $\mathbf{p}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. Then substituting gives

$$\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}0\\0\\0\end{bmatrix}\quad\text{or}\quad x=0.$$

So the general equation for $\mathscr{P}_1$ is $x=0$. Applying the same argument above to the plane $\mathscr{P}_2$ determined by the face in the xz-plane gives a general equation of $y=0$, and similarly the plane $\mathscr{P}_3$ determined by the face in the xy-plane gives a general equation of $z=0$.

Now consider $\mathscr{P}_4$, the plane containing the face parallel to the face in the yz-plane but passing through $(1,1,1)$. Since $\mathscr{P}_4$ is parallel to $\mathscr{P}_1$, its normal vector is also $\begin{bmatrix}1\\0\\0\end{bmatrix}$; since it passes through $(1,1,1)$, we set $\mathbf{p}=\begin{bmatrix}1\\1\\1\end{bmatrix}$. Then substituting gives

$$\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\end{bmatrix}\quad\text{or}\quad x=1.$$

So the general equation for $\mathscr{P}_4$ is $x=1$. Similarly, the general equations for $\mathscr{P}_5$ and $\mathscr{P}_6$ are $y=1$ and $z=1$.

(b) Let $\mathbf{n} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a normal vector for the desired plane $\mathscr{P}$. Since $\mathscr{P}$ is perpendicular to the xy-plane, their normal vectors must be orthogonal. Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x \cdot 0 + y \cdot 0 + z \cdot 1 = z = 0.$$

Thus $z = 0$, so the normal vector is of the form $\mathbf{n} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. But the normal vector is also perpendicular to the plane in question, by definition. Since that plane contains both the origin and $(1, 1, 1)$, the normal vector is orthogonal to $(1, 1, 1) - (0, 0, 0)$:

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = x \cdot 1 + y \cdot 1 + z \cdot 0 = x + y = 0.$$

Thus $x + y = 0$, so that $y = -x$. So finally, a normal vector to $\mathscr{P}$ is given by $\mathbf{n} = \begin{bmatrix} x \\ -x \\ 0 \end{bmatrix}$ for any nonzero x. We may as well choose $x = 1$, giving $\mathbf{n} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Since the plane passes through $(0, 0, 0)$, we let $\mathbf{p} = \mathbf{0}$. Then substituting in $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ gives

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad x - y = 0.$$

Thus the general equation for the plane perpendicular to the xy-plane and containing the diagonal from the origin to $(1, 1, 1)$ is $x - y = 0$.

(c) As in Example 1.22 (Figure 1.34) in Section 1.2, use $\mathbf{u} = [0, 1, 1]$ and $\mathbf{v} = [1, 0, 1]$ as two vectors in the required plane. If $\mathbf{n} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a normal vector to the plane, then $\mathbf{n} \cdot \mathbf{u} = 0 = \mathbf{n} \cdot \mathbf{v}$:

$$\mathbf{n} \cdot \mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = y + z = 0 \Rightarrow y = -z, \qquad \mathbf{n} \cdot \mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = x + z = 0 \Rightarrow x = -z.$$

Thus the normal vector is of the form $\mathbf{n} = \begin{bmatrix} -z \\ -z \\ z \end{bmatrix}$ for any z. Taking $z = -1$ gives $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$. Now, the side diagonals pass through $(0, 0, 0)$, so set $\mathbf{p} = \mathbf{0}$. Then $\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot \mathbf{p}$ yields

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{or} \quad x + y - z = 0.$$

The general equation for the plane containing the side diagonals is $x + y - z = 0$.

26. Finding the distance between points A and B is equivalent to finding $\mathrm{d}(\mathbf{a}, \mathbf{b})$, where $\mathbf{a}$ is the vector from the origin to A, and similarly for $\mathbf{b}$. Given $\mathbf{x} = [x, y, z]$, $\mathbf{p} = [1, 0, -2]$, and $\mathbf{q} = [5, 2, 4]$, we want to solve $\mathrm{d}(\mathbf{x}, \mathbf{p}) = \mathrm{d}(\mathbf{x}, \mathbf{q})$; that is,

$$\mathrm{d}(\mathbf{x}, \mathbf{p}) = \sqrt{(x-1)^2 + (y-0)^2 + (z+2)^2} = \sqrt{(x-5)^2 + (y-2)^2 + (z-4)^2} = \mathrm{d}(\mathbf{x}, \mathbf{q}).$$

Squaring both sides gives

$$\begin{gathered}(x-1)^2+(y-0)^2+(z+2)^2=(x-5)^2+(y-2)^2+(z-4)^2 \quad\Rightarrow\\ x^2-2x+1+y^2+z^2+4z+4=x^2-10x+25+y^2-4y+4+z^2-8z+16 \quad\Rightarrow\\ 8x+4y+12z=40 \quad\Rightarrow\\ 2x+y+3z=10.\end{gathered}$$

Thus all such points (x,y,z) lie on the plane $2x+y+3z=10$.

27. To calculate $\mathrm{d}(Q,\ell)=\frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$, we first put ℓ into general form. With $\mathbf{d}=\begin{bmatrix}1\\-1\end{bmatrix}$, we get $\mathbf{n}=\begin{bmatrix}1\\1\end{bmatrix}$ since then $\mathbf{n}\cdot\mathbf{d}=0$. Then we have

$$\mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{p} \;\Rightarrow\; \begin{bmatrix}1\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}=1.$$

Thus $x+y=1$ and thus $a=b=c=1$. Since $Q=(2,2)=(x_0,y_0)$, we have

$$\mathrm{d}(Q,\ell)=\frac{|1\cdot 2+1\cdot 2-1|}{\sqrt{1^2+1^2}}=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}.$$

28. Comparing the given equation to $\mathbf{x}=\mathbf{p}+t\mathbf{d}$, we get $P=(1,1,1)$ and $\mathbf{d}=\begin{bmatrix}-2\\0\\3\end{bmatrix}$. As suggested by Figure 1.63, we need to calculate the length of $\overrightarrow{RQ}$, where R is the point on the line at the foot of the perpendicular from Q. So if $\mathbf{v}=\overrightarrow{PQ}$, then

$$\overrightarrow{PR}=\operatorname{proj}_{\mathbf{d}}\mathbf{v}, \qquad \overrightarrow{RQ}=\mathbf{v}-\operatorname{proj}_{\mathbf{d}}\mathbf{v}.$$

Now, $\mathbf{v}=\overrightarrow{PQ}=\mathbf{q}-\mathbf{p}=\begin{bmatrix}0\\1\\0\end{bmatrix}-\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}-1\\0\\-1\end{bmatrix}$, so that

$$\operatorname{proj}_{\mathbf{d}}\mathbf{v}=\left(\frac{\mathbf{d}\cdot\mathbf{v}}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d}=\left(\frac{-2\cdot(-1)+3\cdot(-1)}{-2\cdot(-2)+3\cdot 3}\right)\begin{bmatrix}-2\\0\\3\end{bmatrix}=\begin{bmatrix}\frac{2}{13}\\0\\-\frac{3}{13}\end{bmatrix}.$$

Thus

$$\mathbf{v}-\operatorname{proj}_{\mathbf{d}}\mathbf{v}=\begin{bmatrix}-1\\0\\-1\end{bmatrix}-\begin{bmatrix}\frac{2}{13}\\0\\-\frac{3}{13}\end{bmatrix}=\begin{bmatrix}-\frac{15}{13}\\0\\-\frac{10}{13}\end{bmatrix}.$$

Then the distance $\mathrm{d}(Q,\ell)$ from Q to ℓ is

$$\|\mathbf{v}-\operatorname{proj}_{\mathbf{d}}\mathbf{v}\|=\frac{5}{13}\left\|\begin{bmatrix}3\\0\\2\end{bmatrix}\right\|=\frac{5}{13}\sqrt{3^2+2^2}=\frac{5\sqrt{13}}{13}.$$

29. To calculate $\mathrm{d}(Q,\mathscr{P})=\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$, we first note that the plane has equation $x+y-z=0$, so that $a=b=1$, $c=-1$, and $d=0$. Also, $Q=(2,2,2)$, so that $x_0=y_0=z_0=2$. Hence

$$\mathrm{d}(Q,\mathscr{P})=\frac{|1\cdot 2+1\cdot 2-1\cdot 2-0|}{\sqrt{1^2+1^2+(-1)^2}}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}.$$

30. To calculate $\mathrm{d}(Q,\mathscr{P}) = \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$, we first note that the plane has equation $x-2y+2z=1$, so that $a=1$, $b=-2$, $c=2$, and $d=1$. Also, $Q=(0,0,0)$, so that $x_0=y_0=z_0=0$. Hence

$$\mathrm{d}(Q,\mathscr{P}) = \frac{|1\cdot 0 - 2\cdot 0 + 2\cdot 0 - 1|}{\sqrt{1^2+(-2)^2+2^2}} = \frac{1}{3}.$$

31. Figure 1.66 suggests that we let $\mathbf{v} = \overrightarrow{PQ}$; then $\mathbf{w} = \overrightarrow{PR} = \operatorname{proj}_{\mathbf{d}} \mathbf{v}$. Comparing the given line ℓ to $\mathbf{x} = \mathbf{p} + t\mathbf{d}$, we get $P=(-1,2)$ and $\mathbf{d} = \begin{bmatrix}1\\-1\end{bmatrix}$. Then $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q}-\mathbf{p} = \begin{bmatrix}2\\2\end{bmatrix} - \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}3\\0\end{bmatrix}$. Next,

$$\mathbf{w} = \operatorname{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{d}\cdot\mathbf{v}}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} = \left(\frac{1\cdot 3 + (-1)\cdot 0}{1\cdot 1 + (-1)\cdot(-1)}\right)\begin{bmatrix}1\\-1\end{bmatrix} = \frac{3}{2}\begin{bmatrix}1\\-1\end{bmatrix}.$$

So

$$\mathbf{r} = \mathbf{p} + \overrightarrow{PR} = \mathbf{p} + \operatorname{proj}_{\mathbf{d}} \mathbf{v} = \mathbf{p} + \mathbf{w} = \begin{bmatrix}-1\\2\end{bmatrix} + \begin{bmatrix}\frac{3}{2}\\-\frac{3}{2}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix}.$$

So the point R on ℓ that is closest to Q is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

32. Figure 1.66 suggests that we let $\mathbf{v} = \overrightarrow{PQ}$; then $\overrightarrow{PR} = \operatorname{proj}_{\mathbf{d}} \mathbf{v}$. Comparing the given line ℓ to $\mathbf{x} = \mathbf{p}+t\mathbf{d}$, we get $P=(1,1,1)$ and $\mathbf{d} = \begin{bmatrix}-2\\0\\3\end{bmatrix}$. Then $\mathbf{v} = \overrightarrow{PQ} = \mathbf{q}-\mathbf{p} = \begin{bmatrix}0\\1\\0\end{bmatrix} - \begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\-1\end{bmatrix}$. Next,

$$\operatorname{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{d}\cdot\mathbf{v}}{\mathbf{d}\cdot\mathbf{d}}\right)\mathbf{d} = \left(\frac{-2\cdot(-1) + 3\cdot(-1)}{(-2)^2+3^2}\right)\begin{bmatrix}-2\\0\\3\end{bmatrix} = \begin{bmatrix}\frac{2}{13}\\0\\-\frac{3}{13}\end{bmatrix}.$$

So

$$\mathbf{r} = \mathbf{p} + \overrightarrow{PR} = \mathbf{p} + \operatorname{proj}_{\mathbf{d}} \mathbf{v} = \begin{bmatrix}1\\1\\1\end{bmatrix} + \begin{bmatrix}\frac{2}{13}\\0\\-\frac{3}{13}\end{bmatrix} = \begin{bmatrix}\frac{15}{13}\\1\\\frac{10}{13}\end{bmatrix}.$$

So the point R on ℓ that is closest to Q is $\left(\frac{15}{13}, 1, \frac{10}{13}\right)$.

33. Figure 1.67 suggests we let $\mathbf{v} = \overrightarrow{PQ}$, where P is some point on the plane; then $\overrightarrow{QR} = \operatorname{proj}_{\mathbf{n}} \mathbf{v}$. The equation of the plane is $x+y-z=0$, so $\mathbf{n} = \begin{bmatrix}1\\1\\-1\end{bmatrix}$. Setting $y=0$ shows that $P=(1,0,1)$ is a point on the plane. Then

$$\mathbf{v} = \overrightarrow{PQ} = \mathbf{q}-\mathbf{p} = \begin{bmatrix}2\\2\\2\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix},$$

so that

$$\operatorname{proj}_{\mathbf{n}} \mathbf{v} = \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \left(\frac{1\cdot 1 + 1\cdot 1 - 1\cdot 1}{1^2+1^2+(-1)^2}\right)\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{bmatrix}.$$

Finally,

$$\mathbf{r} = \mathbf{p} + \overrightarrow{PQ} + \overrightarrow{PR} = \mathbf{p} + \mathbf{v} - \operatorname{proj}_{\mathbf{n}} \mathbf{v} = \begin{bmatrix}1\\0\\1\end{bmatrix} + \begin{bmatrix}1\\2\\1\end{bmatrix} - \begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{bmatrix} = \begin{bmatrix}\frac{4}{3}\\\frac{4}{3}\\\frac{8}{3}\end{bmatrix}.$$

Therefore, the point R in $\mathscr{P}$ that is closest to Q is $\left(\frac{4}{3}, \frac{4}{3}, \frac{8}{3}\right)$.

34. Figure 1.67 suggests we let $\mathbf{v} = \overrightarrow{PQ}$, where P is some point on the plane; then $\overrightarrow{QR} = \operatorname{proj}_{\mathbf{n}} \mathbf{v}$. The equation of the plane is $x - 2y + 2z = 1$, so $\mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$. Setting $y = z = 0$ shows that $P = (1, 0, 0)$ is a point on the plane. Then

$$\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix},$$

so that

$$\operatorname{proj}_{\mathbf{n}} \mathbf{v} = \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right) \mathbf{n} = \left(\frac{1 \cdot (-1)}{1^2 + (-2)^2 + 2^2}\right) \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ -\frac{2}{9} \end{bmatrix}.$$

Finally,

$$\mathbf{r} = \mathbf{p} + \overrightarrow{PQ} + \overrightarrow{PR} = \mathbf{p} + \mathbf{v} - \operatorname{proj}_{\mathbf{n}} \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ -\frac{2}{9} \end{bmatrix} = \begin{bmatrix} -\frac{1}{9} \\ \frac{2}{9} \\ -\frac{2}{9} \end{bmatrix}.$$

Therefore, the point R in $\mathscr{P}$ that is closest to Q is $\left(-\frac{1}{9}, \frac{2}{9}, -\frac{2}{9}\right)$.

35. Since the given lines ℓ_1 and ℓ_2 are parallel, choose arbitrary points Q on ℓ_1 and P on ℓ_2, say $Q = (1, 1)$ and $P = (5, 4)$. The direction vector of ℓ_2 is $\mathbf{d} = [-2, 3]$. Then

$$\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix},$$

so that

$$\operatorname{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}}\right) \mathbf{d} = \left(\frac{-2 \cdot (-4) + 3 \cdot (-3)}{(-2)^2 + 3^2}\right) \begin{bmatrix} -2 \\ 3 \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} -2 \\ 3 \end{bmatrix}.$$

Then the distance between the lines is given by

$$\|\mathbf{v} - \operatorname{proj}_{\mathbf{d}} \mathbf{v}\| = \left\| \begin{bmatrix} -4 \\ -3 \end{bmatrix} + \frac{1}{13} \begin{bmatrix} -2 \\ 3 \end{bmatrix} \right\| = \left\| \frac{1}{13} \begin{bmatrix} -54 \\ -36 \end{bmatrix} \right\| = \frac{18}{13} \left\| \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\| = \frac{18}{13}\sqrt{13}.$$

36. Since the given lines ℓ_1 and ℓ_2 are parallel, choose arbitrary points Q on ℓ_1 and P on ℓ_2, say $Q = (1, 0, -1)$ and $P = (0, 1, 1)$. The direction vector of ℓ_2 is $\mathbf{d} = [1, 1, 1]$. Then

$$\mathbf{v} = \overrightarrow{PQ} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix},$$

so that

$$\operatorname{proj}_{\mathbf{d}} \mathbf{v} = \left(\frac{\mathbf{d} \cdot \mathbf{v}}{\mathbf{d} \cdot \mathbf{d}}\right) \mathbf{d} = \left(\frac{1 \cdot 1 + 1 \cdot (-1) + 1 \cdot (-2)}{1^2 + 1^2 + 1^2}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -\frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Then the distance between the lines is given by

$$\|\mathbf{v} - \operatorname{proj}_{\mathbf{d}} \mathbf{v}\| = \left\| \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + \frac{2}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{5}{3} \\ -\frac{1}{3} \\ -\frac{4}{3} \end{bmatrix} \right\| = \frac{1}{3}\sqrt{5^2 + (-1)^2 + (-4)^2} = \frac{\sqrt{42}}{3}.$$

37. Since $\mathscr{P}_1$ and $\mathscr{P}_2$ are parallel, we choose an arbitrary point on $\mathscr{P}_1$, say $Q = (0, 0, 0)$, and compute $d(Q, \mathscr{P}_2)$. Since the equation of $\mathscr{P}_2$ is $2x + y - 2z = 5$, we have $a = 2$, $b = 1$, $c = -2$, and $d = 5$; since $Q = (0, 0, 0)$, we have $x_0 = y_0 = z_0 = 0$. Thus the distance is

$$d(\mathscr{P}_1, \mathscr{P}_2) = d(Q, \mathscr{P}_2) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|2 \cdot 0 + 1 \cdot 0 - 2 \cdot 0 - 5|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{5}{3}.$$

38. Since $\mathscr{P}_1$ and $\mathscr{P}_2$ are parallel, we choose an arbitrary point on $\mathscr{P}_1$, say $Q = (1, 0, 0)$, and compute $\mathrm{d}(Q, \mathscr{P}_2)$. Since the equation of $\mathscr{P}_2$ is $x + y + z = 3$, we have $a = b = c = 1$ and $d = 3$; since $Q = (1, 0, 0)$, we have $x_0 = 1$, $y_0 = 0$, and $z_0 = 0$. Thus the distance is

$$\mathrm{d}(\mathscr{P}_1, \mathscr{P}_2) = \mathrm{d}(Q, \mathscr{P}_2) = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0 - 3|}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}.$$

39. We wish to show that $\mathrm{d}(B, \ell) = \frac{|ax_0+by_0-c|}{\sqrt{a^2+b^2}}$, where $\mathbf{n} = \begin{bmatrix} a \\ b \end{bmatrix}$, $\mathbf{n} \cdot \mathbf{a} = c$, and $B = (x_0, y_0)$. If $\mathbf{v} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, then

$$\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} - c = ax_0 + by_0 - c.$$

Then from Figure 1.65, we see that

$$\mathrm{d}(B, \ell) = \|\mathrm{proj}_{\mathbf{n}}\, \mathbf{v}\| = \left\| \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} \right\| = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|ax_0 + by_0 - c|}{\sqrt{a^2 + b^2}}.$$

40. We wish to show that $\mathrm{d}(B, \ell) = \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$, where $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $\mathbf{n} \cdot \mathbf{a} = d$, and $B = (x_0, y_0, z_0)$. If $\mathbf{v} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, then

$$\mathbf{n} \cdot \mathbf{v} = \mathbf{n} \cdot (\mathbf{b} - \mathbf{a}) = \mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} - d = ax_0 + by_0 + cz_0 - d.$$

Then from Figure 1.65, we see that

$$\mathrm{d}(B, \ell) = \|\mathrm{proj}_{\mathbf{n}}\, \mathbf{v}\| = \left\| \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}} \right) \mathbf{n} \right\| = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|ax_0 + by_0 + cz_0 - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

41. Choose $B = (x_0, y_0)$ on ℓ_1; since ℓ_1 and ℓ_2 are parallel, the distance between them is $\mathrm{d}(B, \ell_2)$. Then since B lies on ℓ_1, we have $\mathbf{n} \cdot \mathbf{b} = \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = ax_0 + by_0 = c_1$. Choose A on ℓ_2, so that $\mathbf{n} \cdot \mathbf{a} = c_2$. Set $\mathbf{v} = \mathbf{b} - \mathbf{a}$. Then using the formula in Exercise 39, the distance is

$$\mathrm{d}(\ell_1, \ell_2) = \mathrm{d}(B, \ell_2) = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a}|}{\|\mathbf{n}\|} = \frac{|c_1 - c_2|}{\|\mathbf{n}\|}.$$

42. Choose $B = (x_0, y_0, z_0)$ on $\mathscr{P}_1$; since $\mathscr{P}_1$ and $\mathscr{P}_2$ are parallel, the distance between them is $\mathrm{d}(B, \mathscr{P}_2)$. Then since B lies on $\mathscr{P}_1$, we have $\mathbf{n} \cdot \mathbf{b} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = ax_0 + by_0 + cz_0 = d_1$. Choose A on $\mathscr{P}_2$, so that $\mathbf{n} \cdot \mathbf{a} = d_2$. Set $\mathbf{v} = \mathbf{b} - \mathbf{a}$. Then using the formula in Exercise 40, the distance is

$$\mathrm{d}(\mathscr{P}_1, \mathscr{P}_2) = \mathrm{d}(B, \mathscr{P}_2) = \frac{|\mathbf{n} \cdot \mathbf{v}|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot (\mathbf{b} - \mathbf{a})|}{\|\mathbf{n}\|} = \frac{|\mathbf{n} \cdot \mathbf{b} - \mathbf{n} \cdot \mathbf{a}|}{\|\mathbf{n}\|} = \frac{|d_1 - d_2|}{\|\mathbf{n}\|}.$$

43. Since $\mathscr{P}_1$ has normal vector $\mathbf{n}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathscr{P}_2$ has normal vector $\mathbf{n}_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$, the angle θ between the normal vectors satisfies

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{1 \cdot 2 + 1 \cdot 1 + 1 \cdot (-2)}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{1}{3\sqrt{3}}.$$

Thus

$$\theta = \cos^{-1}\left(\frac{1}{3\sqrt{3}}\right) \approx 78.9^\circ.$$

44. Since $\mathscr{P}_1$ has normal vector $\mathbf{n}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ and $\mathscr{P}_2$ has normal vector $\mathbf{n}_2 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, the angle θ between the normal vectors satisfies

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{3 \cdot 1 - 1 \cdot 4 + 2 \cdot (-1)}{\sqrt{3^2 + (-1)^2 + 2^2}\sqrt{1^2 + 4^2 + (-1)^2}} = -\frac{3}{\sqrt{14}\sqrt{18}} = -\frac{1}{\sqrt{28}}.$$

This is an obtuse angle, so the acute angle is

$$\pi - \theta = \pi - \cos^{-1}\left(-\frac{1}{\sqrt{28}}\right) \approx 79.1^\circ.$$

45. First, to see that $\mathscr{P}$ and ℓ intersect, substitute the parametric equations for ℓ into the equation for $\mathscr{P}$, giving

$$x + y + 2z = (2 + t) + (1 - 2t) + 2(3 + t) = 9 + t = 0,$$

so that $t = -9$ represents the point of intersection, which is thus $(2 + (-9), 1 - 2(-9), 3 + (-9)) = (-7, 19, -6)$. Now, the normal to $\mathscr{P}$ is $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, and a direction vector for ℓ is $\mathbf{d} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. So if θ is the angle between $\mathbf{n}$ and $\mathbf{d}$, then θ satisfies

$$\cos\theta = \frac{\mathbf{n} \cdot \mathbf{d}}{\|\mathbf{n}\|\,\|\mathbf{d}\|} = \frac{1 \cdot 1 + 1 \cdot (-2) + 2 \cdot 1}{\sqrt{1^2 + 1^2 + 1^2}\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{1}{6},$$

so that

$$\theta = \cos^{-1}\left(\frac{1}{6}\right) \approx 80.4^\circ.$$

Thus the angle between the line and the plane is $90^\circ - 80.4^\circ \approx 9.6^\circ$.

46. First, to see that $\mathscr{P}$ and ℓ intersect, substitute the parametric equations for ℓ into the equation for $\mathscr{P}$, giving

$$4x - y - z = 4 \cdot t - (1 + 2t) - (2 + 3t) = -t - 3 = 6,$$

so that $t = -9$ represents the point of intersection, which is thus $(-9, 1 + 2 \cdot (-9), 2 + 3 \cdot (-9)) = (-9, -17, -25)$. Now, the normal to $\mathscr{P}$ is $\mathbf{n} = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$, and a direction vector for ℓ is $\mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. So if θ is the angle between $\mathbf{n}$ and $\mathbf{d}$, then θ satisfies

$$\cos\theta = \frac{\mathbf{n} \cdot \mathbf{d}}{\|\mathbf{n}\|\,\|\mathbf{d}\|} = \frac{4 \cdot 1 - 1 \cdot 2 - 1 \cdot 3}{\sqrt{4^2 + 1^2 + 1^2}\sqrt{1^2 + 2^2 + 3^2}} = -\frac{1}{\sqrt{18}\sqrt{14}}.$$

This corresponds to an obtuse angle, so the acute angle between the two is

$$\theta = \pi - \cos^{-1}\left(-\frac{1}{\sqrt{18}\sqrt{14}}\right) \approx 86.4^\circ.$$

Thus the angle between the line and the plane is $90^\circ - 86.4^\circ \approx 3.6^\circ$.

47. We have $\mathbf{p} = \mathbf{v} - c\,\mathbf{n}$, so that $c\,\mathbf{n} = \mathbf{v} - \mathbf{p}$. Take the dot product of both sides with $\mathbf{n}$, giving

$$\begin{aligned}
(c\,\mathbf{n}) \cdot \mathbf{n} &= (\mathbf{v} - \mathbf{p}) \cdot \mathbf{n} \quad \Rightarrow \\
c(\mathbf{n} \cdot \mathbf{n}) &= \mathbf{v} \cdot \mathbf{n} - \mathbf{p} \cdot \mathbf{n} \quad \Rightarrow \\
c(\mathbf{n} \cdot \mathbf{n}) &= \mathbf{v} \cdot \mathbf{n} \ \text{ (since } \mathbf{p} \text{ and } \mathbf{n} \text{ are orthogonal)} \quad \Rightarrow \\
c &= \frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}.
\end{aligned}$$

Note that another interpretation of the figure is that $c\mathbf{n} = \operatorname{proj}_{\mathbf{n}} \mathbf{v} = \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}$, which also implies that $c = \frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}$.

Now substitute this value of c into the original equation, giving

$$\mathbf{p} = \mathbf{v} - c\mathbf{n} = \mathbf{v} - \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n}.$$

48. (a) A normal vector to the plane $x + y + z = 0$ is $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then

$$\begin{aligned}\mathbf{n}\cdot\mathbf{v} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 1\cdot 1 + 1\cdot 0 + 1\cdot(-2) = -1 \\ \mathbf{n}\cdot\mathbf{n} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1\cdot 1 + 1\cdot 1 + 1\cdot 1 = 3,\end{aligned}$$

so that $c = -\frac{1}{3}$. Then

$$\mathbf{p} = \mathbf{v} - \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \\ -\frac{5}{3} \end{bmatrix}.$$

(b) A normal vector to the plane $3x - y + z = 0$ is $\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$. Then

$$\begin{aligned}\mathbf{n}\cdot\mathbf{v} &= \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 3\cdot 1 - 1\cdot 0 + 1\cdot(-2) = 1 \\ \mathbf{n}\cdot\mathbf{n} &= \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = 3\cdot 3 - 1\cdot(-1) + 1\cdot 1 = 11,\end{aligned}$$

so that $c = \frac{1}{11}$. Then

$$\mathbf{p} = \mathbf{v} - \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{11}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{8}{11} \\ \frac{1}{11} \\ -\frac{23}{11} \end{bmatrix}.$$

(c) A normal vector to the plane $x - 2z = 0$ is $\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$. Then

$$\begin{aligned}\mathbf{n}\cdot\mathbf{v} &= \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 1\cdot 1 + 0\cdot 0 - 2\cdot(-2) = 5 \\ \mathbf{n}\cdot\mathbf{n} &= \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 1\cdot 1 + 0\cdot 0 - 2\cdot(-2) = 5,\end{aligned}$$

so that $c = 1$. Then

$$\mathbf{p} = \mathbf{v} - \left(\frac{\mathbf{n}\cdot\mathbf{v}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \mathbf{0}.$$

Note that the projection is **0** because the vector is normal to the plane, so its projection onto the plane is a single point.

(d) A normal vector to the plane $2x - 3y + z = 0$ is $\mathbf{n} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$. Then

$$\mathbf{n}\cdot\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = 2\cdot 1 - 3\cdot 0 + 1\cdot(-2) = 0$$

$$\mathbf{n}\cdot\mathbf{n} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = 2\cdot 2 - 3\cdot(-3) + 1\cdot 1 = 14,$$

so that $c = 0$. Thus $\mathbf{p} = \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$. Note that the projection is the vector itself because the vector is parallel to the plane, so it is orthogonal to the normal vector.

Exploration: The Cross Product

1. **(a)** $\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 - 0\cdot(-1) \\ 1\cdot 3 - 0\cdot 2 \\ 0\cdot(-1) - 1\cdot 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -3 \end{bmatrix}.$

(b) $\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} -1\cdot 1 - 2\cdot 1 \\ 2\cdot 0 - 3\cdot 1 \\ 3\cdot 1 - (-1)\cdot 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}.$

(c) $\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} 2\cdot(-6) - 3\cdot(-4) \\ 3\cdot 2 - (-1)\cdot(-6) \\ -1\cdot(-4) - 2\cdot 2 \end{bmatrix} = \mathbf{0}.$

(d) $\mathbf{u}\times\mathbf{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} = \begin{bmatrix} 1\cdot 3 - 1\cdot 2 \\ 1\cdot 1 - 1\cdot 3 \\ 1\cdot 2 - 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$

2. We have

$$\mathbf{e}_1\times\mathbf{e}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0\cdot 0 - 0\cdot 1 \\ 0\cdot 0 - 1\cdot 0 \\ 1\cdot 1 - 0\cdot 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{e}_3$$

$$\mathbf{e}_2\times\mathbf{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1 - 0\cdot 0 \\ 0\cdot 0 - 0\cdot 1 \\ 0\cdot 0 - 1\cdot 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{e}_1$$

$$\mathbf{e}_3\times\mathbf{e}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0\cdot 0 - 1\cdot 0 \\ 1\cdot 1 - 0\cdot 0 \\ 0\cdot 0 - 0\cdot 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \mathbf{e}_2.$$

3. Two vectors are orthogonal if their dot product equals zero. But

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot\mathbf{u} &= \begin{bmatrix} u_2v_3-u_3v_2\\ u_3v_1-u_1v_3\\ u_1v_2-u_2v_1\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\\ &= (u_2v_3-u_3v_2)u_1+(u_3v_1-u_1v_3)u_2+(u_1v_2-u_2v_1)u_3\\ &= (u_2v_3u_1-u_1v_3u_2)+(u_3v_1u_2-u_2v_1u_3)+(u_1v_2u_3-u_3v_2u_1)=0\\ (\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} &= \begin{bmatrix} u_2v_3-u_3v_2\\ u_3v_1-u_1v_3\\ u_1v_2-u_2v_1\end{bmatrix}\cdot\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}\\ &= (u_2v_3-u_3v_2)v_1+(u_3v_1-u_1v_3)v_2+(u_1v_2-u_2v_1)v_3\\ &= (u_2v_3v_1-u_2v_1v_3)+(u_3v_1v_2-u_3v_2v_1)+(u_1v_2v_3-u_1v_3v_2)=0.\end{aligned}$$

4. **(a)** By Exercise 1, a vector normal to the plane is

$$\mathbf{n}=\mathbf{u}\times\mathbf{v}=\begin{bmatrix}0\\1\\1\end{bmatrix}\times\begin{bmatrix}3\\-1\\2\end{bmatrix}=\begin{bmatrix}1\cdot 2-1\cdot(-1)\\ 1\cdot 3-0\cdot 2\\ 0\cdot(-1)-1\cdot 3\end{bmatrix}=\begin{bmatrix}3\\3\\-3\end{bmatrix}.$$

So the normal form for the equation of this plane is $\mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{p}$, or

$$\begin{bmatrix}3\\3\\-3\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\3\\-3\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-2\end{bmatrix}=9.$$

This simplifies to $3x+3y-3z=9$, or $x+y-z=3$.

(b) Two vectors in the plane are $\mathbf{u}=\overrightarrow{PQ}=\begin{bmatrix}2\\1\\1\end{bmatrix}$ and $\mathbf{v}=\overrightarrow{PR}=\begin{bmatrix}1\\3\\-2\end{bmatrix}$. So by Exercise 1, a vector normal to the plane is

$$\mathbf{n}=\mathbf{u}\times\mathbf{v}=\begin{bmatrix}2\\1\\1\end{bmatrix}\times\begin{bmatrix}1\\3\\-2\end{bmatrix}=\begin{bmatrix}1\cdot(-2)-1\cdot 3\\ 1\cdot 1-2\cdot(-2)\\ 2\cdot 3-1\cdot 1\end{bmatrix}=\begin{bmatrix}-5\\5\\5\end{bmatrix}.$$

So the normal form for the equation of this plane is $\mathbf{n}\cdot\mathbf{x}=\mathbf{n}\cdot\mathbf{p}$, or

$$\begin{bmatrix}-5\\5\\5\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-5\\5\\5\end{bmatrix}\cdot\begin{bmatrix}0\\-1\\1\end{bmatrix}=0.$$

This simplifies to $-5x+5y+5z=0$, or $x-y-z=0$.

5. **(a)** $\mathbf{v}\times\mathbf{u}=\begin{bmatrix}v_2u_3-v_3u_2\\ v_3u_1-u_3v_1\\ v_1u_2-v_2u_1\end{bmatrix}=-\begin{bmatrix}u_2v_3-u_3v_2\\ u_3v_1-u_1v_3\\ u_1v_2-u_2v_1\end{bmatrix}=-(\mathbf{u}\times\mathbf{v}).$

(b) $\mathbf{u}\times\mathbf{0}=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\times\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}u_2\cdot 0-u_3\cdot 0\\ u_3\cdot 0-u_1\cdot 0\\ u_1\cdot 0-u_2\cdot 0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\mathbf{0}.$

(c) $\mathbf{u}\times\mathbf{u}=\begin{bmatrix}u_2u_3-u_3u_2\\ u_3u_1-u_1u_3\\ u_1u_2-u_2u_1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\mathbf{0}.$

(d) $\mathbf{u}\times k\mathbf{v}=\begin{bmatrix}u_2kv_3-u_3kv_2\\ u_3kv_1-u_1kv_3\\ u_1kv_2-u_2kv_1\end{bmatrix}=k\begin{bmatrix}u_2v_3-u_3v_2\\ u_3v_1-u_1v_3\\ u_1v_2-u_2v_1\end{bmatrix}=k(\mathbf{u}\times\mathbf{v}).$

(e) $\mathbf{u} \times k\mathbf{u} = k(\mathbf{u} \times \mathbf{u}) = k(\mathbf{0}) = \mathbf{0}$ by parts (d) and (c).

(f) Compute the cross-product:

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} + \mathbf{w}) &= \begin{bmatrix} u_2(v_3 + w_3) - u_3(v_2 + w_2) \\ u_3(v_1 + w_1) - u_1(v_3 + w_3) \\ u_1(v_2 + w_2) - u_2(v_1 + w_1) \end{bmatrix} \\
&= \begin{bmatrix} (u_2v_3 - u_3v_2) + (u_2w_3 - u_3w_2) \\ (u_3v_1 - u_1v_3) + (u_3w_1 - u_1w_3) \\ (u_1v_2 - u_2v_1) + (u_1w_2 - u_2w_1) \end{bmatrix} \\
&= \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} + \begin{bmatrix} u_2w_3 - u_3w_2 \\ u_3w_1 - u_1w_3 \\ u_1w_2 - u_2w_1 \end{bmatrix} \\
&= \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}.
\end{aligned}$$

6. In each case, simply compute:

(a)

$$\begin{aligned}
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \cdot \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} \\
&= u_1v_2w_3 - u_1v_3w_2 + u_2v_3w_1 - u_2v_1w_3 + u_3v_1w_2 - u_3v_2w_1 \\
&= (u_2v_3 - u_3v_2)w_1 + (u_3v_1 - u_1v_3)w_2 + (u_1v_2 - u_2v_1)w_3 \\
&= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}.
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \times \begin{bmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{bmatrix} \\
&= \begin{bmatrix} u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) \\ u_3(v_2w_3 - v_3w_2) - u_1(v_1w_2 - v_2w_1) \\ u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2) \end{bmatrix} \\
&= \begin{bmatrix} (u_1w_1 + u_2w_2 + u_3w_3)v_1 - (u_1v_1 + u_2v_2 + u_3v_3)w_1 \\ (u_1w_1 + u_2w_2 + u_3w_3)v_2 - (u_1v_1 + u_2v_2 + u_3v_3)w_2 \\ (u_1w_1 + u_2w_2 + u_3w_3)v_3 - (u_1v_1 + u_2v_2 + u_3v_3)w_3 \end{bmatrix} \\
&= (u_1w_1 + u_2w_2 + u_3w_3) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} - (u_1v_1 + u_2v_2 + u_3v_3) \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\
&= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}.
\end{aligned}$$

(c)

$$\begin{aligned}
\|\mathbf{u} \times \mathbf{v}\|^2 &= \left\| \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix} \right\|^2 \\
&= (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2 \\
&= (u_1^2 + u_2^2 + u_3^2)^2(v_1^2 + v_2^2 + v_3^2)^2 - (u_1v_1 + u_2v_2 + u_3v_3)^2 \\
&= \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2.
\end{aligned}$$