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An evolution algebra in population genetics

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A B S T R A C T

We consider an evolution algebra which corresponds to a bisexual population with a set of females partitioned into finitely many different types and the males having only one type. We study basic properties of the algebra. This algebra is commutative (and hence flexible), not associative and not necessarily power-associative, in general. We prove that being alternative is equivalent to being associative. We find conditions to be an associative, a fourth power-associative, or a nilpotent algebra. We also prove that if the algebra is not alternative then to be power-associative is equivalent to be Jordan. Moreover it is not unital. In a general case, we describe the full set of idempotent elements and the full set of absolute nilpotent elements. The set of all operators of left (right) multiplications is described. Under some conditions it is proved that the corresponding algebra is centroidal. Moreover the classification of 2-dimensional and some 3-dimensional algebras are obtained.

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1. Introduction

Description of a sex linked inheritance with algebras involves overcoming the obstacle of asymmetry in the genetic inheritance rules. Inheritance which is not sex linked is symmetrical with respect to the sexes of the organisms [5], while sex linked inheritance is not (see [4,6]). The main problem for a given algebra of a sex linked population is to carefully examine how the basic algebraic model must be altered in order to compensate for this lack of symmetry in the genetic inheritance system. In [2], Etherington began the study of this kind of algebras with the simplest possible case.

Now the methods of mathematical genetics have become probability theory, stochastic processes, nonlinear differential and difference equations and non-associative algebras. The book [5] describes some mathematical methods of studying algebras of genetics. This book mainly considers a *free population*, which means random mating in the population. Evolution of a free population can be given by a dynamical system generated by a quadratic stochastic operator (QSO) and by an evolution algebra of a free population. In [5] an evolution algebra associated to the free population was introduced and, using this non-associative algebra, many results are obtained in explicit form, e.g., the explicit description of stationary quadratic operators, and the explicit solutions of a nonlinear evolutionary equation in the absence of selection, as well as general theorems on convergence to equilibrium in the presence of selection. In [3] some recently obtained results and also several open problems related to the theory of QSOs are discussed. See also [5] for more detailed theory of QSOs.

Recently in [4] an evolution algebra $\mathcal{B}$ was introduced identifying the coefficients of inheritance of a bisexual population as the structure constants of the algebra. The basic properties of the algebra are studied. Moreover a detailed analysis of a special case of the evolution algebra (of bisexual population in which type "1" of females and males have preference) was given. Since the structural constants of the algebra $\mathcal{B}$ are given by two cubic matrices, the study of this algebra is difficult. To avoid such difficulties we have to consider an algebra of bisexual population with a simplified form of matrices of structural constants. In this paper we consider a such simplified model of bisexual population and study corresponding evolution algebra.

The paper is organized as follows. In Section 2 we define our algebra as an evolution algebra which corresponds to a bisexual population with a set of females partitioned into finitely many different types and the males having only one type. Then we study basic properties (associativity, non-associativity, commutativity, power-associativity, nilpotency, unitality, etc.) of the algebra. Section 3 is devoted to subalgebras, absolute nilpotent elements and idempotent elements of the algebra. In Section 4 the set of all operators of left (right) multiplications is described. In Section 5, under some conditions, it is proved that the corresponding algebra is centroidal. The last section gives a classification of 2-dimensional and some 3-dimensional algebras.

2. Definition and basic properties of the EACP

We consider the set H (the set of "hen") and r (a "rooster").

Definition 2.1. Let $(\mathcal{C}, \cdot)$ be an algebra over a field K of characteristic $\neq 2$. If $\mathcal{C} = H \oplus Kr$ admits a multiplication given by

$$\begin{aligned} &xr = rx = \varphi(x) + \mu(x)r, \quad \forall x \in H, \\ &xy = 0, \qquad rr = 0, \quad \forall x, y \in H, \end{aligned} \tag{2.1}$$

where $\varphi \in \mathrm{End}_K(H)$ and $\mu \in H^*$, is a linear map then this algebra is called an evolution algebra of a "chicken" population (EACP).

Remark 2.2. If H is finite-dimensional and $\{h_1, \dots, h_n\}$ is a basis of H and $\{h_1, \dots, h_n, r\}$ the basis of $\mathcal{C}$, then taking $\varphi_1 = \frac{1}{2}\varphi, \mu_1 = \frac{1}{2}\mu$, we have $h_i r = r h_i = \frac{1}{2}(\sum_{j=1}^n a_{ij} h_j + b_i r)$. We call the basis $\{h_1, \dots, h_n, r\}$ a natural basis.

Moreover, if

$$\sum_{j=1}^n a_{ij} = 1; \qquad b_i = 1 \quad \text{for all } i = 1, 2, \dots, n \tag{2.2}$$

then the corresponding $\mathcal{C}$ is a particular case of an evolution algebra of a bisexual population, $\mathcal{B}$, introduced in [4]. The study of the algebra $\mathcal{B}$ is difficult, since it is determined by two cubic matrices. While, in the case finite-dimensional, the algebra $\mathcal{C}$ is simpler, since it is defined by a rectangular $n \times (n+1)$-matrix

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} & b_n \end{pmatrix}.$$

We recall the following definitions: If a, b and c denote arbitrary elements of an algebra then

Associative: $(ab)c = a(bc)$.

Commutative: $ab = ba$.

Anticommutative: $ab = -ba$.

Jacobi identity: $(ab)c + (bc)a + (ca)b = 0$.

Jordan identity: $(ab)a^2 = a(ba^2)$.

Alternative: $(aa)b = a(ab)$ and $(ba)a = b(aa)$.

Flexible: $a(ba) = (ab)a$.

Power-associative: For each element a the subalgebra generated by a is associative, that is $a^n a^m = a^{m+n}$, for all nonnegative integers n, m.

Fourth power-associative: Each element a satisfies the identity $a^2a^2 = a^4$. It is known that if the characteristic of K satisfies char $K \neq 2, 3, 5$ and the algebra is flexible (particularly commutative), then being fourth power-associative implies being power-associative (see [1]).

It is known that these properties are related by

- associative implies alternative implies power-associative;
- associative implies Jordan identity implies power-associative;
- each of the associative, commutative, anticommutative properties, Jordan identity, and Jacobi identity individually imply flexible.

For a field with characteristic not two, being both commutative and anticommutative implies the algebra is just $\{0\}$.

By [4, Theorem 4.1] we have

(1) Algebra $\mathcal{C}$ is not associative, in general.
(2) Algebra $\mathcal{C}$ is commutative, flexible.
(3) $\mathcal{C}$ is not power-associative, in general.

Now we shall give conditions under which $\mathcal{C}$ will be associative and fourth power-associative.

Theorem 2.3. *$\mathcal{C}$ alternative implies $\mathcal{C}^3 = \{0\}$.*

Proof. $\mathcal{C}$ alternative implies that for all $x \in H$, we have $(xr)r = xr^2 = 0$ and $x(xr) = x^2r = 0$. Thus,

$$0 = \big(\varphi(x) + \mu(x)r\big)r = \varphi(x)r = \varphi^2(x) + \mu\big(\varphi(x)\big)r,$$

and

$$0 = x\big(\varphi(x) + \mu(x)r\big) = \mu(x)xr = \mu(x)\big(\varphi(x) + \mu(x)r\big) = \mu(x)\varphi(x) + \mu(x)^2r.$$

Therefore, $\varphi^2(x) = 0$ and $\mu(x) = 0$, $\forall x \in H$.

Let $a = x + \alpha r$, $b = y + \beta r$, $c = z + \gamma r$ elements in $\mathcal{C}$. Since $\mu = 0$ we have that $ab = (x + \alpha r)(y + \beta r) = \beta xr + \alpha yr = \beta\varphi(x) + \alpha\varphi(y)$.

Therefore, using that $\mu = 0$ and $\varphi^2 = 0$, we get

$$(ab)c = \big(\beta\varphi(x) + \alpha\varphi(y)\big)(z + \gamma r) = \beta\gamma\varphi(x)r + \alpha\gamma\varphi(y)r = \beta\gamma\varphi^2(x) + \alpha\gamma\varphi^2(y) = 0,$$

and so $\mathcal{C}^3 = \{0\}$. □

Corollary 2.4. *$\mathcal{C}$ alternative if and only if $\mathcal{C}$ is associative if and only if $\mu = 0$ and $\varphi^2 = 0$. Moreover, $\mathcal{C}$ satisfies Jacobi and Jordan identities.*

We note that the conditions (2.2) and of Corollary 2.4 cannot be satisfied simultaneously, so the corresponding algebra $\mathcal{B}$ of a bisexual population is not associative.

Example 2.5. For $n = 2$, the following matrix

$$\begin{pmatrix} a_{11} & a_{12} & 0 \\ a_{21} & -a_{11} & 0 \end{pmatrix},$$

satisfies the conditions of Corollary 2.4, for any a_{11}, a_{12}, a_{21} with $a_{11}^2 = -a_{12}a_{21}$.

Theorem 2.6. *$\mathcal{C}$ is fourth power-associative if and only if $\mu = 0$ and $\varphi^3 = 0$, that is, φ is a nilpotent operator.*

Proof. Let $a = x + \alpha r$ be an element in $\mathcal{C}$. Then

$$\begin{aligned} a^2 &= 2\alpha xr = 2\alpha\big(\varphi(x) + \mu(x)r\big), \\ a^2a^2 &= \big(2\alpha\big(\varphi(x) + \mu(x)r\big)\big)^2 = 4\alpha^2\big(2\mu(x)\big)\varphi(x)r = 8\alpha^2\mu(x)\big(\varphi^2(x) + \mu\big(\varphi(x)\big)r\big). \end{aligned}$$

On the other hand,

$$\begin{aligned} a^3 = a^2a &= 2\alpha\big(\varphi(x) + \mu(x)r\big)(x + \alpha r) = 2\alpha\big(\alpha\varphi(x)r + \mu(x)xr\big) \\ &= 2\alpha\big[\alpha\varphi^2(x) + \alpha\mu\big(\varphi(x)\big)r + \mu(x)\varphi(x) + \mu(x)^2r\big] \\ &= 2\alpha\big[\alpha\varphi^2(x) + \mu(x)\varphi(x) + \big(\alpha\mu\big(\varphi(x)\big) + \mu(x)^2\big)r\big], \end{aligned}$$

and

$$\begin{aligned} a^4 = a^3a &= 2\alpha\big[\alpha\varphi^2(x) + \mu(x)\varphi(x) + \big(\alpha\mu\big(\varphi(x)\big) + \mu(x)^2\big)r\big](x + \alpha r) \\ &= 2\alpha\big[\alpha^2\varphi^2(x)r + \alpha\mu(x)\varphi(x)r + \big(\alpha\mu\big(\varphi(x)\big) + \mu(x)^2\big)xr\big] \\ &= 2\alpha\big[\alpha^2\big(\varphi^3(x) + \mu\big(\varphi^2(x)\big)r\big) + \alpha\mu(x)\big(\varphi^2(x) + \mu\big(\varphi(x)\big)r\big) \\ &\quad + \alpha\mu\big(\varphi(x)\big)\big(\varphi(x) + \mu(x)r\big) + \mu(x)^2\big(\varphi(x) + \mu(x)r\big)\big] \\ &= 2\alpha\big[\alpha^2\varphi^3(x) + \alpha\mu(x)\varphi^2(x) + \alpha\mu\big(\varphi(x)\big)\varphi(x) + \mu(x)^2\varphi(x) \\ &\quad + \big(\alpha^2\mu\big(\varphi^2(x)\big) + 2\alpha\mu(x)\mu\big(\varphi(x)\big) + \mu(x)^3\big)r\big]. \end{aligned}$$

Therefore, since $\operatorname{char}(K) \neq 2$, we get $a^2a^2 = a^4$ for all $a \in \mathcal{C}$ if and only if

$$\begin{aligned} 4\alpha\mu(x)\big(\varphi^2(x) + \mu\big(\varphi(x)\big)r\big) = {} & \alpha^2\varphi^3(x) + \alpha\mu(x)\varphi^2(x) + \alpha\mu\big(\varphi(x)\big)\varphi(x) + \mu(x)^2\varphi(x) \\ & + \big(\alpha^2\mu\big(\varphi^2(x)\big) + 2\alpha\mu(x)\mu\big(\varphi(x)\big) + \mu(x)^3\big)r, \end{aligned}$$

for all $x \in H, \alpha \in K$. Thus, $\mu(x)^3 = 0$ and $\mu(x) = 0$. So, $\varphi^3(x) = 0$, that is $\varphi^3 = 0$ and Theorem 2.6 follows. □

Example 2.7. The EACP of dimension 4 given by the matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

is fourth power-associative ($\mu = 0$ and $\varphi^3 = 0$) but not associative, since it does not satisfy $\varphi^2 = 0$. Moreover, if char $K \neq 2, 3, 5$, then the algebra is power-associative.

Definition 2.8. An element x of an algebra $\mathcal{A}$ is called nil if there exists $n(x) \in \mathbb{N}$ such that $(\cdots \underbrace{((x \cdot x) \cdot x) \cdots x}_{n(x)}) = 0$. The algebra $\mathcal{A}$ is called nil if every element of the algebra is nil.

For $k \geq 1$, we introduce the following sequences:

$$\mathcal{A}^{(1)} = \mathcal{A}, \qquad \mathcal{A}^{(k+1)} = \mathcal{A}^{(k)}\mathcal{A}^{(k)};$$
$$\mathcal{A}^{\langle 1\rangle} = \mathcal{A}, \qquad \mathcal{A}^{\langle k+1\rangle} = \mathcal{A}^{\langle k\rangle}\mathcal{A};$$
$$\mathcal{A}^{1} = \mathcal{A}, \qquad \mathcal{A}^{k} = \sum_{i=1}^{k-1} \mathcal{A}^{i}\mathcal{A}^{k-i}.$$

Definition 2.9. An algebra $\mathcal{A}$ is called

(i) solvable if there exists $n \in \mathbb{N}$ such that $\mathcal{A}^{(n)} = 0$ and the minimal such number is called index of solvability;
(ii) right nilpotent if there exists $n \in \mathbb{N}$ such that $\mathcal{A}^{\langle n\rangle} = 0$ and the minimal such number is called index of right nilpotency;
(iii) nilpotent if there exists $n \in \mathbb{N}$ such that $\mathcal{A}^{n} = 0$ and the minimal such number is called index of nilpotency.

We note that for an EACP, notions as nil, nilpotent and right nilpotent algebras are equivalent. However, the indexes of nility, right nilpotency and nilpotency do not coincide in general.

From Theorem 2.3 it follows

Corollary 2.10. *$\mathcal{C}$ alternative implies that $\mathcal{C}$ is nilpotent with nilindex* 3.

Remark 2.11. $\mu = 0$ implies that $\mathcal{C}$ is solvable. In fact, if $\mu = 0$ and $a = x + \alpha r \in \mathcal{C}$ then $a^2 = 2\alpha\varphi(x)$. Therefore $\mathcal{C}^{(2)} = \varphi(H) \subset H$ and $\mathcal{C}^{(2)}\mathcal{C}^{(2)} \subset H^2 = 0$. So, $\mathcal{C}$ is solvable.

Proposition 2.12. *Let $\mathcal{C}$ be a non alternative algebra. If $\mathcal{C}$ is fourth power-associative then $\mathcal{C}$ is nilpotent with nilindex* 4.

Proof. It is enough to prove that $\mathcal{C}$ is right nilpotent. From Theorem 2.6, $\mathcal{C}$ is fourth power-associative if and only if $\mu = 0$ and $\varphi^3 = 0$. Since $\mu = 0$ we have

$$\begin{aligned} \mathcal{C}^{\langle 2\rangle} &= \varphi(H), \\ \mathcal{C}^{\langle 3\rangle} &= \mathcal{C}^{\langle 2\rangle}\mathcal{C} = \varphi(H)\mathcal{C} = \varphi(H)r = \varphi^2(H), \\ \mathcal{C}^{\langle 4\rangle} &= \mathcal{C}^{\langle 3\rangle}\mathcal{C} = \varphi^2(H)\mathcal{C} = \varphi^2(H)r = \varphi^3(H) = 0, \end{aligned}$$

since $\varphi^3 = 0$. □

Corollary 2.13. *Let $\mathcal{C}$ be a non alternative algebra. Then $\mathcal{C}$ is a Jordan algebra if and only if $\mathcal{C}$ power-associative algebra.*

Proof. It is known that a Jordan algebra is power-associative. Conversely, let $\mathcal{C}$ power-associative then $\mathcal{C}$ is fourth power-associative and Proposition 2.12 implies $\mathcal{C}^4 = 0$ and $\mathcal{C}$ satisfies the Jordan identity. □

Recall that an algebra is *unital* or *unitary* if it has an element a with $ab = b = ba$ for all b in the algebra.

Proposition 2.14. *The algebra $\mathcal{C}$ is not unital.*

Proof. Assume $a = x + \alpha r$ is a unity element. We then have $ar = r$ which gives $\varphi(x) + \mu(x)r = r$, so $\varphi(x) = 0$. From $ax = x$ we get $\alpha xr = x$. Since $\varphi(x) = 0$, we obtain $\alpha\mu(x)r = x$, which is a contradiction. This completes the proof. □

An algebra $\mathcal{A}$ is a *division algebra* if for every $a, b \in \mathcal{A}$ with $a \neq 0$ the equations $ax = b$ and $xa = b$ are solvable in $\mathcal{A}$.

Proposition 2.15. *If $\mathcal{C}$ is finite-dimensional, then the algebra $\mathcal{C}$ is not a division algebra.*

Proof. Since $\mathcal{C}$ is finite-dimensional we will use Remark 2.2 to have the multiplication $h_i r = r h_i = \frac{1}{2}(\sum_{j=1}^n a_{ij}h_j + b_i r)$. Since $\mathcal{C}$ is a commutative algebra we shall check only $ax = b$. For coordinates of any $a = \sum_{i=1}^n \alpha_i h_i + \alpha r$, $b = \sum_{i=1}^n \beta_i h_i + \beta r$, $x = \sum_{i=1}^n \chi_i h_i + \chi r$, the equation $ax = b$ has the following form

$$\begin{aligned} \left(\sum_{i=1}^n a_{ij}\alpha_i\right)\chi + \alpha\sum_{i=1}^n a_{ij}\chi_i &= 2\beta_j, \\ \left(\sum_{i=1}^n b_i\alpha_i\right)\chi + \alpha\sum_{i=1}^n b_i\chi_i &= 2\beta, \quad j = 1, \dots, n. \end{aligned}$$

So this is a linear system with $n+1$ unknowns $\chi_1, \dots, \chi_n, \chi$. This system can be written as $\mathbf{M}y = B$ where $y^T = (\chi_1, \dots, \chi_n, \chi)$, $B = 2(\beta_1, \dots, \beta_n, \beta)$ and

$$\mathbf{M} = \begin{pmatrix} \alpha a_{11} & \alpha a_{21} & \dots & \alpha a_{n1} & \sum_{i=1}^n a_{i1}\alpha_i \\ \alpha a_{12} & \alpha a_{22} & \dots & \alpha a_{n2} & \sum_{i=1}^n a_{i2}\alpha_i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha a_{1n} & \alpha a_{2n} & \dots & \alpha a_{nn} & \sum_{i=1}^n a_{in}\alpha_i \\ \alpha b_1 & \alpha b_2 & \dots & \alpha b_n & \sum_{i=1}^n b_i\alpha_i \end{pmatrix}.$$

By the very known Kronecker–Capelli theorem the system of linear equations $\mathbf{M}y = B$ has a solution if and only if the rank of the matrix $\mathbf{M}$ is equal to the rank of its augmented matrix $(\mathbf{M}|B)$. Since the last column of the matrix $\mathbf{M}$ is a linear combination of the other columns of the matrix, we have $\operatorname{rank} \mathbf{M} \leq n$. Moreover, since the dimension of the algebra $\mathcal{C}$ is $n+1$ we can choose b, i.e., the vector B such that $\operatorname{rank}(\mathbf{M}|B) = 1 + \operatorname{rank} \mathbf{M}$. Then for such b the equation $ax = b$ is not solvable. This completes the proof. □

3. Evolution subalgebras, absolute nilpotent elements and idempotents elements of $\mathcal{C}$

By analogues of [7, Definition 4, p. 23] we give the following

Definition 3.1. Let $\mathcal{C}$ be an EACP, and $\mathcal{C}_1$ be a subspace of $\mathcal{C}$.

If $\mathcal{C}_1 = H_1 \oplus Kr, H_1 \subseteq H, \varphi_1 = \varphi|_{H_1}$ and $\mu_1 = \mu|_{H_1}$, with multiplication table like (2.1), then we call $\mathcal{C}_1$ an evolution subalgebra of the chicken population (CP) $\mathcal{C}$.

The following proposition gives some evolution subalgebras of a CP.

Proposition 3.2. *Let $\mathcal{C}$ be a finite-dimensional EACP with basis $\{h_1, \dots, h_n, r\}$ and matrix*

$$M = \begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 & b_1 \\ a_{21} & a_{22} & 0 & \dots & 0 & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} & b_n \end{pmatrix}.$$

Then for each m, $1 \leq m \leq n$, the algebra $\mathcal{C}_m = \langle h_1, \dots, h_m, r\rangle \subset \mathcal{C}$ is an evolution subalgebra of a CP.

Proof. Recall that since $\mathcal{C}$ is finite-dimensional we will use Remark 2.2 to have the multiplication $h_i r = r h_i = \frac{1}{2}(\sum_{j=1}^n a_{ij} h_j + b_i r)$. For a given M it is easy to see that $\mathcal{C}_m$ is closed under multiplication. The chosen subset of the natural basis of $\mathcal{C}$ satisfies (2.1). □

The following is an example of a subalgebra of $\mathcal{C}$, which is not an evolution subalgebra of a CP.

Example 3.3. Let $\mathcal{C}$ be a finite-dimensional EACP with basis $\{h_1, h_2, h_3, r\}$ and multiplication defined by $h_i r = h_i + r$, $i = 1, 2, 3$. Take $u_1 = h_1 + r$, $u_2 = h_2 + r$. Then

$$(au_1 + bu_2)(cu_1 + du_2) = acu_1^2 + (ad + bc)u_1u_2 + bdu_2^2$$
$$= (2ac + ad + bc)u_1 + (2bd + ad + bc)u_2.$$

Hence, $\mathcal{F} = \langle u_1, u_2 \rangle$ is a subalgebra of $\mathcal{C}$, but it is not an evolution subalgebra of the CP $\mathcal{C}$. Indeed, assume $\{v_1, v_2\}$ is a basis of $\mathcal{F}$. Then $v_1 = au_1 + bu_2$ and $v_2 = cu_1 + du_2$ for some $a, b, c, d \in K$ such that $D = ad - bc \neq 0$. We have $v_1^2 = (2a^2 + 2ab)u_1 + (2b^2 + 2ab)u_2$ and $v_2^2 = (2c^2 + 2cd)u_1 + (2d^2 + 2cd)u_2$. We must have $v_1^2 = v_2^2 = 0$, i.e.,

$$a^2 + ab = 0, \qquad b^2 + ab = 0, \qquad c^2 + cd = 0, \qquad d^2 + cd = 0.$$

From this we get $a = -b$ and $c = -d$. Then $D = 0$, which is a contradiction. If $a = 0$ then $b = 0$ (resp. $c = 0$ then $d = 0$), we reach the same contradiction. Hence $v_1^2 \neq 0$ and $v_2^2 \neq 0$, and consequently $\mathcal{F}$ is not an evolution subalgebra of the CP $\mathcal{C}$.

If we write $x^{[k]}$ for the power $(\cdots(x^2)^2\cdots)$ (k times) with $x^{[0]} = x$ then the trajectory with initial x is given by k times iteration of the operator V, i.e., $V^k(x) = x^{[k]}$. This algebraic interpretation of the trajectory is useful to connect powers of an element of the algebra and with the dynamical system generated by the evolution operator V. For example, zeros of V, i.e., $V(x) = 0$, correspond to absolute nilpotent elements of $\mathcal{C}$ and fixed points of V, i.e., $V(x) = x$, correspond to idempotent elements of $\mathcal{C}$.

The following proposition describes the nonzero absolute nilpotent elements ($a^2 = 0$) of $\mathcal{C}$.

Proposition 3.4. *$a = x + \alpha r$ is a nonzero absolute nilpotent element of $\mathcal{C}$ if and only if $\alpha = 0$ or ($\varphi(x) = 0$ and $\mu(x) = 0$).*

Proof. Let $a = x + \alpha r$ be a nonzero absolute nilpotent element of $\mathcal{C}$. Then $\alpha \neq 0$ or $x \neq 0$. Therefore, $a^2 = (x + \alpha r)^2 = 2\alpha xr = 2\alpha(\varphi(x) + \mu(x)r)$. Then $a^2 = 0 \iff 2\alpha(\varphi(x) + \mu(x)r) = 0 \iff \alpha = 0$ or ($\varphi(x) = 0$ and $\mu(x) = 0$). □

Now we shall describe idempotent elements of $\mathcal{C}$, these are solutions to $a^2 = a$.

Proposition 3.5. *$e = x + \alpha r$ is a nonzero idempotent of $\mathcal{C}$ if and only if $\alpha \neq 0$, $\mu(x) = \frac{1}{2}$ and x is an eigenvector of φ with eigenvalue $\frac{1}{2\alpha}$.*

Proof. Let $e = x + \alpha r$ be a nonzero idempotent of $\mathcal{C}$. Then $\alpha \neq 0$ or $x \neq 0$. Therefore, $e^2 = e \iff 2\alpha\varphi(x) + 2\alpha\mu(x)r = x + \alpha r \iff 2\alpha\varphi(x) = x$ and $2\alpha\mu(x) = \alpha \iff \varphi(x) = \frac{1}{2\alpha}x$ and $\mu(x) = \frac{1}{2}$ and proposition follows. □

Remark 3.6. Let x be an element of H such that $\varphi(x) = \lambda x$ (for some $\lambda \neq 0$) and $\mu(x) \neq 0$, then $e = \frac{1}{2\mu(x)}x + \frac{1}{2\lambda}r$ is a nonzero idempotent of $\mathcal{C}$.

4. The enveloping algebra of an EACP

For a given algebra $\mathcal{A}$ with ground field K, we recall that multiplication by elements of $\mathcal{A}$ on the left or on the right give rise to left and right K-linear transformations of $\mathcal{A}$ given by $L_a(x) = ax$ and $R_a(x) = xa$. The *enveloping algebra*, denoted by $\mathcal{E}(\mathcal{A})$, of a non-associative algebra $\mathcal{A}$ is the subalgebra of the full algebra of K-endomorphisms of $\mathcal{A}$ which is generated by the left and right multiplication maps of $\mathcal{A}$. This enveloping algebra is necessarily associative, even though $\mathcal{A}$ may be non-associative. In a sense this makes the enveloping algebra "the smallest associative algebra containing $\mathcal{A}$".

Since an EACP, $\mathcal{C}$, is a commutative algebra the right and left operators coincide, so we use only L_a.

Theorem 4.1. *Let $\mathcal{C}$ be a finite-dimensional EACP with a natural basis $\{h_1, \dots, h_n, r\}$. If $\operatorname{Ker}(\varphi) \cap \operatorname{Ker}(\mu) = 0$ and $H \nsubseteq \operatorname{Ker}(\varphi)$ (or $H \nsubseteq \operatorname{Ker}(\mu)$) then $\{L_1, \dots, L_n, L_r\}$ (where $L_i = L_{h_i}$) spans a linear space, denoted by $\operatorname{span}(L, \mathcal{C})$, which is the set of all operators of left multiplication. The vector space $\operatorname{span}(L, \mathcal{C})$ and $\mathcal{C}$ have the same dimension.*

Proof. If we prove that L_a is an injection for every $a \in \mathcal{C}$, the linear space that is spanned by all operators of left multiplication can be spanned by the set $\{L_i, i = 1, \dots, n, r\}$. This set is a basis for $\operatorname{span}(L, \mathcal{C})$.

Therefore, we will prove that L_a is an injection for every $a \in \mathcal{C}$. Let $L_a = L_c$ then $L_a(b) = L_c(b)$ for all $b \in \mathcal{C}$. Since a, b, c are elements in $\mathcal{C}$ we have $a = x + \alpha r$, $b = y + \beta r$, $c = z + \gamma r$. We have for all $b \in \mathcal{C}$,

$$L_a(b) = L_c(b) \iff ab = cb \iff \beta xr + \alpha yr = \beta zr + \gamma yr.$$

If $\beta = 0$, then $\alpha yr = \gamma yr$ or $\alpha(\varphi(y) + \mu(y)r) = \gamma(\varphi(y) + \mu(y)r)$. Since$H \nsubseteq \operatorname{Ker}(\varphi)$ (or $H \nsubseteq \operatorname{Ker}(\mu)$) we have that $\alpha = \gamma$.

If $\beta \neq 0$, $y = 0$, then $\beta xr = \beta zr$ implies that $xr = zr$, that is $\varphi(x) + \mu(x)r = \varphi(z) + \mu(z)r$. Therefore $\varphi(x) = \varphi(z)$ and $\mu(x) = \mu(z)$ and $x - z \in \operatorname{Ker}(\varphi) \cap \operatorname{Ker}(\mu) = 0$. Thus, $x = z$ and $a = c$. So L_a is an injection. $\square$

Proposition 4.2. *For any $b = y + \beta r \in \mathcal{C}$ and any $h, h_1, h_2, \dots, h_m \in H$ the following hold*

$$L_{h_m} \circ L_{h_{m-1}} \circ \dots \circ L_{h_1}(b) = \left(\prod_{j=1}^{m-1} \mu(h_j)\right) L_{h_m}(b), \tag{4.1}$$

$$L_r \circ L_h(b) = L_{\varphi(h)}(b), \tag{4.2}$$

$$L_h \circ L_r(b) = \mu(y) L_h(r). \tag{4.3}$$

Proof. Recall that for any $h \in H$, $rh = hr = \varphi(h) + \mu(h)r$.

(1) To prove (4.1) we use mathematical induction over m. For $m = 2$ we have

$$(L_{h_2}\circ L_{h_1})(b) = L_{h_2}\big(h_1(y+\beta r)\big) = L_{h_2}(\beta h_1 r) = L_{h_2}\big(\beta\varphi(h_1)+\beta\mu(h_1)r\big)$$
$$= h_2\big(\beta\varphi(h_1)+\beta\mu(h_1)r\big) = \beta\mu(h_1)h_2 r = \mu(h_1)L_{h_2}(b).$$

Assume now that the formula (4.1) is true for m, we shall prove it for $m+1$:

$$L_{h_{m+1}}\circ L_{h_m}\circ\ldots\circ L_{h_1}(b) = L_{h_{m+1}}\circ\left(\prod_{j=1}^{m-1}\mu(h_j)L_{h_m}(b)\right)$$
$$= \prod_{j=1}^{m-1}\mu(h_j)L_{h_{m+1}}\circ L_{h_m}(b) = \left(\prod_{j=1}^{m}\mu(h_j)\right)L_{h_{m+1}}(b).$$

(2) Proof of (4.2):

$$L_r\circ L_h(b) = L_r\big(h(y+\beta r)\big) = L_r(\beta hr) = r\big(\beta\varphi(h)+\beta\mu(h)r\big) = \beta\varphi(h)r = L_{\varphi(h)}(b).$$

(3) Proof of (4.3):

$$L_h\circ L_r(b) = h\big(r(y+\beta r)\big) = h(ry) = h\big(\varphi(y)+\mu(y)r\big)$$
$$= h\varphi(y) + h\big(\mu(y)r\big) = \mu(y)hr = \mu(y)L_h(r). \qquad \square$$

5. The centroid of an EACP

We recall (see [7]) that the *centroid* of an algebra $\mathcal{A}$, $\Gamma(\mathcal{A})$, is the set of all linear transformations $T\in \operatorname{Hom}(\mathcal{A},\mathcal{A})$ that commute with all left and right multiplication operators

$$TL_x = L_xT, \qquad TR_y = R_yT, \quad \text{for all } x,y\in\mathcal{A}.$$

An algebra $\mathcal{A}$ over a field K is centroidal if $\Gamma(\mathcal{A})\cong K$.

Theorem 5.1. *Let $\mathcal{C}$ be a finite-dimensional EACP and $\{h_1,\ldots,h_n,r\}$ be a natural basis of $\mathcal{C}$. If μ is an injection then $\mathcal{C}$ is centroidal.*

Proof. Let $T\in\Gamma(\mathcal{C})$, then $TL_a(c) = L_aT(c)$ for all $a,c\in\mathcal{C}$. We will prove that there exists $\lambda\in K$, such that $T(r)=\lambda r$ and $T(h_i)=\lambda h_i$ for all h_i, $i=1,\ldots,n$.

Assume that $T(r) = y+\beta r$, $T(h_i) = z_i+\gamma_i r$ with $y, z_i\in H$, $\beta,\gamma_i\in K$.

We have that

$$0 = TL_r(r) = L_r\big(T(r)\big) \iff 0 = r\big(T(r)\big) \iff 0 = ry = \varphi(y)+\mu(y)r.$$

This implies that $\varphi(y)=0$ and $\mu(y)=0$. Since $\operatorname{Ker}(\mu)=0$, we get $y=0$ and

$$T(r) = \beta r. \tag{5.1}$$

On the other hand we have

$$
\begin{aligned}
& L_r T(h_i) = T L_r(h_i) \\
& \quad \Longleftrightarrow \quad rT(h_i) = T(rh_i) \\
& \quad \Longleftrightarrow \quad rz_i = T\big(\varphi(h_i) + \mu(h_i)r\big) \\
& \quad \Longleftrightarrow \quad \varphi(z_i) + \mu(z_i)r = T\big(\varphi(h_i)\big) + T\big(\mu(h_i)r\big) \\
& \quad \Longleftrightarrow \quad \varphi(z_i) + \mu(z_i)r = T\big(\varphi(h_i)\big) + \mu(h_i)T(r) \\
& \quad \Longleftrightarrow \quad \varphi(z_i) + \mu(z_i)r = T\big(\varphi(h_i)\big) + \mu(h_i)\beta r \\
& \quad \Longleftrightarrow \quad \varphi(z_i) = T\big(\varphi(h_i)\big) \quad \text{and} \quad \mu(z_i) = \mu(h_i)\beta \\
& \quad \Longleftrightarrow \quad \varphi(z_i) = T\big(\varphi(h_i)\big) \quad \text{and} \quad z_i - \beta h_i \in \operatorname{Ker}(\mu) = \{0\}. \\
& z_i = \beta h_i. \qquad (5.2)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
& L_{h_i} T(h_i) = T L_{h_i}(h_i) \\
& \quad \Longleftrightarrow \quad h_i T(h_i) = 0 \quad \Longleftrightarrow \quad h_i(\beta h_i + \gamma_i r) = 0 \\
& \quad \Longleftrightarrow \quad \gamma_i h_i r = 0 \quad \Longleftrightarrow \quad \gamma_i \varphi(h_i) + \gamma_i \mu(h_i) r = 0 \\
& \quad \Longleftrightarrow \quad \gamma_i \varphi(h_i) = 0 \quad \text{and} \quad \gamma_i \mu(h_i) = 0 \quad \Longleftrightarrow \quad \gamma_i \varphi(h_i) = 0 \quad \text{and} \quad \mu(\gamma_i h_i) = 0.
\end{aligned}
$$

This implies that $\gamma_i h_i \in \operatorname{Ker}(\mu) = \{0\}$. Then

$$\gamma_i = 0. \qquad (5.3)$$

Using Eqs. (5.1), (5.2) and (5.3), we get

$$T(h_i) = \beta h_i, \qquad T(r) = \beta r,$$

where β is a scalar in the ground field K. That is, T is a scalar multiplication. Consequently, $\Gamma(\mathcal{C}) \cong K$ and $\mathcal{C}$ is centroidal. □

6. Classification of 2 and 3-dimensional EACP

Let $\mathcal{C}$ be a 2-dimensional EACP and $\{h, r\}$ be a basis of this algebra.

It is evident that if $\dim \mathcal{C}^2 = 0$ then $\mathcal{C}$ is an abelian algebra, i.e., an algebra with all products equal to zero.

Proposition 6.1. *Any 2-dimensional, non-trivial EACP $\mathcal{C}$ is isomorphic to one of the following pairwise non isomorphic algebras:*

$\mathcal{C}_1$: $rh = hr = h$, $h^2 = r^2 = 0$,
$\mathcal{C}_2$: $rh = hr = \frac{1}{2}(h + r)$, $h^2 = r^2 = 0$.

Proof. For an EACP $\mathcal{C}$ we have

$$rh = hr = \frac{1}{2}(ah + br), \quad h^2 = r^2 = 0.$$

Case: $a \neq 0$, $b = 0$. By change of basis $h' = h$ and $r' = \frac{2}{a}r$ we get the algebra $\mathcal{C}_1$.
Case: $a = 0$, $b \neq 0$. Take $h' = r$ and $r' = \frac{2}{b}h$ then we get the algebra $\mathcal{C}_1$.
Case: $a \neq 0$, $b \neq 0$. The change $h' = \frac{1}{a}r$, and $r' = \frac{1}{b}h$ implies the algebra $\mathcal{C}_2$.
Since $\mathcal{C}_1^2\mathcal{C}_1^2 = 0$ and $\mathcal{C}_2^2\mathcal{C}_2^2 \neq 0$, the algebras $\mathcal{C}_1$ and $\mathcal{C}_2$ are not isomorphic. □

We note that the algebra $\mathcal{C}_2$ is known as the sex differentiation algebra [6].
Let now $\mathcal{C}$ be a 3-dimensional EACP and $\{h_1, h_2, r\}$ be a basis of this algebra.

Theorem 6.2. *Any 3-dimensional EACP $\mathcal{C}$ with $\dim(\mathcal{C}^2) = 1$ is isomorphic to one of the following pairwise non isomorphic algebras:*

$\mathcal{C}_1$: $h_1 r = r$;
$\mathcal{C}_2$: $h_1 r = h_2$;
$\mathcal{C}_3$: $h_1 r = h_1 + r$.

In each algebra we take $rh_i = h_i r$, $i = 1, 2$ and all omitted products are zero.

Proof. For a 3-dimensional EACP $\mathcal{C}$ we have

$$h_1 r = rh_1 = \frac{1}{2}(ah_1 + bh_2 + Ar), \qquad h_2 r = rh_2 = \frac{1}{2}(ch_1 + dh_2 + Br),$$
$$h_1^2 = h_2^2 = h_1 h_2 = r^2 = 0.$$

First we note that non-zero coefficients of $h_1 r$ can be taken 1. Indeed, if $abA \neq 0$ then the change of basis $h_1' = \frac{2}{A}h_1$, $h_2' = \frac{2b}{aA}h_2$, $r' = \frac{2}{a}r$ makes all coefficients of $h_1 r$ equal 1. In case some a, b, A is equal 0 then one can choose a suitable change of basis to make non-zero coefficients equal to 1. Therefore we have three parametric families: $h_2 r = rh_2 = \frac{1}{2}(ch_1 + dh_2 + Br)$ with one of the following conditions

(i) $h_1 r = rh_1 = r$,
(ii) $h_1 r = rh_1 = h_2$,
(iii) $h_1 r = rh_1 = h_1 + r$,
(iv) $h_1 r = rh_1 = h_2 + r$,
(v) $h_1 r = rh_1 = h_1 + h_2 + r$,
(vi) $h_1 r = rh_1 = h_1$,
(vii) $h_1 r = rh_1 = h_1 + h_2$.

If $\dim(\mathcal{C}^2) = 1$, then h_2r is proportional to h_1r. From above-mentioned (i)–(vii) it follows the following cases for h_1r and h_2r.

Case (i): In this case $h_1r = r$ and $h_2r = cr$ for some $c \in K$. If $c = 0$ we get the algebra $\mathcal{C}_1$. If $c \neq 0$ then by the change

$$h_1' = h_1, \qquad h_2' = -h_1 + \frac{1}{c}h_2, \qquad r' = r,$$

we again obtain the algebra $\mathcal{C}_1$.

Case (ii): In this case $h_1r = h_2$ and $h_2r = ch_2$ for some $c \in K$. If $c = 0$ we get the algebra $\mathcal{C}_2$. If $c \neq 0$ then by the change

$$h_1' = \frac{1}{c}r, \qquad h_2' = ch_1 - h_2, \qquad r' = h_2,$$

we get the algebra $\mathcal{C}_1$.

Case (iii): In this case we have $h_1r = h_1 + r$ and $h_2r = c(h_1 + r)$ for some $c \in K$. If $c = 0$ we get the algebra $\mathcal{C}_3$. If $c \neq 0$ then by the change

$$h_1' = h_1, \qquad h_2' = \frac{1}{c}h_2 - h_1, \qquad r' = r,$$

we get the algebra $\mathcal{C}_3$.

Case (iv): We have $h_1r = h_2 + r$ and $h_2r = c(h_2 + r)$ for some $c \in K$. If $c = 0$ then by the change

$$h_1' = h_1, \qquad h_2' = h_2, \qquad r' = h_2 + r,$$

we get the algebra $\mathcal{C}_1$. If $c \neq 0$ then by the change

$$h_1' = \frac{1}{c}h_2, \qquad h_2' = \frac{1}{c}h_2 - h_1, \qquad r' = \frac{1}{c}r,$$

we get the algebra $\mathcal{C}_3$.

Case (v): We have $h_1r = h_1 + h_2 + r$ and $h_2r = c(h_1 + h_2 + r)$ for some $c \in K$. If $c \neq -1$ then by the change

$$h_1' = \frac{1}{1+c}(h_1 + h_2), \qquad h_2' = \frac{1}{1+c}(-ch_1 + h_2), \qquad r' = \frac{1}{1+c}r,$$

we get the algebra $\mathcal{C}_3$. If $c = -1$ then by the change

$$h_1' = h_1, \qquad h_2' = h_1 + h_2, \qquad r' = h_1 + h_2 + r,$$

we get the algebra $\mathcal{C}_1$.

Case (vi): We have $h_1r = h_1$ and $h_2r = ch_1$ for some $c \in K$. In this case by the change

$$h_1' = r, \qquad h_2' = ch_1 - h_2, \qquad r' = h_1,$$

we get the algebra $\mathcal{C}_1$.

Case (vii): In this case $h_1 r = h_1 + h_2$ and $h_2 r = c(h_1 + h_2)$ for some $c \in K$. In the case $c = -1$, by making the change

$$h_1' = \frac{r}{2}, \qquad h_2' = h_1 + h_2, \qquad r' = h_1 - h_2,$$

we get the algebra $\mathcal{C}_2$. For $c \neq -1$, by taking the change

$$h_1' = \frac{r}{1+c}, \qquad h_2' = h_2 - ch_1, \qquad r' = h_1 + h_2,$$

we get the algebra $\mathcal{C}_1$.

The obtained algebras are pairwise non-isomorphic. This may be checked by comparison of the algebraic properties listed in the following table.

	$\mathcal{C}_i^2\mathcal{C}_i^2 = 0$	Nilpotent
$\mathcal{C}_1$	Yes	No
$\mathcal{C}_2$	Yes	Yes
$\mathcal{C}_3$	No	No

□

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