CHAPTER I

# LINEAR ALGEBRA, BASIC NOTIONS

## 1. Introduction

In your previous calculus courses, you studied differentiation and integration for functions of more than one variable. Usually, you couldn't do much if the number of variables was greater than two or three. However, in many important applications, the number of relevant variables can be quite large. In such situations, even very basic algebra can get quite complicated. Linear algebra is a tool invented in the nineteenth century and further extended in the twentieth century to enable people to handle such algebra in a systematic and understandable manner.

We start off with a couple of simple examples where it is clear that we may have to deal with a lot of variables.

**Example 1.** Professor Marie Curie has ten students in a chemistry class and gives five exams which are weighted differently in order to obtain a total score for the course. The data as presented in her grade book is as follows.

| student/exam | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 78 | 70 | 74 | 82 | 74 |
| 2 | 81 | 75 | 72 | 85 | 80 |
| 3 | 92 | 90 | 94 | 88 | 94 |
| 4 | 53 | 72 | 65 | 72 | 59 |
| 5 | 81 | 79 | 79 | 82 | 78 |
| 6 | 21 | 92 | 90 | 88 | 95 |
| 7 | 83 | 84 | 76 | 79 | 84 |
| 8 | 62 | 65 | 67 | 73 | 65 |
| 9 | 70 | 72 | 76 | 82 | 73 |
| 10 | 69 | 75 | 70 | 78 | 79 |

The numbers across the top label the exams and the numbers in the left hand column number the students. There are a variety of statistics the teacher might want to calculate from this data. First, she might want to know the average score for each test. For a given test, label the scores $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}$ so that $x_i$ is the score for the $i$th student. Then the average score is

$$\frac{x_1 + x_2 + x_2 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}}{10} = \frac{1}{10}\sum_{i=1}^{10} x_i.$$

For example, for the second test the average is

$$\frac{1}{10}(70 + 75 + 90 + 72 + 79 + 92 + 84 + 65 + 72 + 75) = 77.4.$$

Suppose she decides to weight the five scores as follows: the first, third, and fifth scores are weighted equally at 20 percent or 0.2, the second score is weighted 10 percent or 0.1, and the fourth score is weighted 30 percent or 0.3. Then if the scores for a typical student are denoted $y_1, y_2, y_3, y_4, y_5$, the total weighted score would be

$$0.2\, y_1 + 0.1\, y_2 + 0.2\, y_3 + 0.3\, y_4 + 0.2\, y_5.$$

If we denote the weightings $a_1 = a_3 = a_5 = 0.2, a_2 = 0.1, a_4 = 0.3$, then this could also be written

$$a_1y_1 + a_2y_2 + a_3y_3 + a_4y_4 + a_5y_5 = \sum_{i=1}^{5} a_iy_i.$$

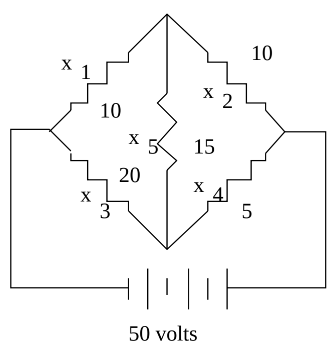
For example, for the third student, the total score would be

$$0.2 \cdot 92 + 0.1 \cdot 90 + 0.2 \cdot 94 + 0.3 \cdot 88 + 0.2 \cdot 94 = 91.4.$$

As you see, in both cases we have a number of variables and we are forming what is called a linear function of those variables, that is, an expression in which each variable appears simply to the first power (with no complicated functions). When we only have two or three variables, the algebra for dealing with such functions is quite simple, but as the number of variables grows, the algebra becomes much more complex.

Such data sets and calculations should be familiar to anyone who has played with a spreadsheet.

**Example 2.** In studying complicated electrical circuits, one uses a collection of rules called *Kirchhoff's laws.* One of these rules says that the currents converging at a node in the circuit add up algebraically to zero. (Currents can be positive or negative.) Other rules put other restrictions on the currents. For example, in the circuit below




Numerical resistances in ohms

Kirchhoff's laws yield the following equations for the currents $x_1, x_2, x_3, x_4, x_5$ in the different branches of the circuit.

$$\begin{aligned} 10x_1 + 10x_2 &= 50 \\ 20x_3 + 5x_4 &= 50 \\ x_1 - x_2 - x_5 &= 0 \\ -x_3 + x_4 - x_5 &= 0 \\ 10x_1 + 5x_4 + 15x_5 &= 50 \end{aligned}$$

Don't worry if you don't know anything about electricity. The point is that the circuit is governed by a system of *linear equations*. In order to understand the circuit, we must have methods to solve such systems. In your high school algebra course, you learned how to solve two equations in two unknowns and perhaps three equations in three unknowns. In this course we shall study how to solve any number of equations in any number of unknowns. Linear algebra was invented in large part to discuss the solutions of such systems in an organized manner. The above example yielded a fairly small system, but electrical engineers must often deal with very large circuits involving many, many currents. Similarly, many other applications in other fields require the solution of systems of very many equations in very many unknowns.

Nowadays, one uses electronic computers to solve such systems. Consider for example the system of 5 equations in 5 unknowns

$$\begin{aligned}
2x_1 + 3x_2 - 5x_3 + 6x_4 - x_5 &= 10\\
3x_1 - 3x_2 + 6x_3 + x_4 - x_5 &= 2\\
x_1 + x_2 - 4x_3 + 2x_4 + x_5 &= 5\\
4x_1 - 3x_2 + x_3 + 6x_4 + x_5 &= 4\\
2x_1 + 3x_2 - 5x_3 + 6x_4 - x_5 &= 3
\end{aligned}$$

How might you present the data needed to solve this system to a computer? Clearly, the computer won't care about the names of the unknowns since it doesn't need such aids to do what we tell it to do. It would need to be given the table of coefficients

$$\begin{matrix}
2 & 3 & -5 & 6 & -1\\
3 & -3 & 6 & 1 & -1\\
1 & 1 & -4 & 2 & 1\\
4 & -3 & 1 & 6 & 1\\
2 & 3 & -5 & 6 & -1
\end{matrix}$$

and the quantities on the right—also called the 'givens'

$$\begin{matrix}
10\\
2\\
5\\
4\\
3
\end{matrix}$$

Each such table is an example of a *matrix*, and in the next section, we shall discuss the algebra of such matrices.

**Exercises for Section 1.**

1. A professor taught a class with three students who took two exams each. The results were

| student/test | 1 | 2 |
|---|---|---|
| 1 | 100 | 95 |
| 2 | 60 | 75 |
| 3 | 100 | 95 |

(a) What were the average scores on each test?

(b) Are there weightings $a_1, a_2$ which result in either of the following weighted scores?

| student | score | | student | score |
|---|---|---|---|---|
| 1 | 98 | or | 1 | 98 |
| 2 | 66 | | 2 | 66 |
| 3 | 98 | | 3 | 97 |

2. Solve each of the following linear systems by any method you know.

(a)

$$\begin{aligned} 2x + 3y &= 3 \\ x + 3y &= 1 \end{aligned}$$

(b)

$$\begin{aligned} x + y &= 3 \\ y + z &= 4 \\ x + y + z &= 5 \end{aligned}$$

(c)

$$\begin{aligned} x + y &= 3 \\ y + z &= 4 \\ x + 2y + z &= 5 \end{aligned}$$

(d)

$$\begin{aligned} x + y + z &= 1 \\ z &= 1 \end{aligned}$$

## 2. Matrix Algebra

In the previous section, we saw examples of rectangular arrays or matrices such as the table of grades

$$\begin{bmatrix} 78 & 70 & 74 & 82 & 74 \\ 81 & 75 & 72 & 85 & 80 \\ 92 & 90 & 94 & 88 & 94 \\ 53 & 72 & 65 & 72 & 59 \\ 81 & 79 & 79 & 82 & 78 \\ 21 & 92 & 90 & 88 & 95 \\ 83 & 84 & 76 & 79 & 84 \\ 62 & 65 & 67 & 73 & 65 \\ 70 & 72 & 76 & 82 & 73 \\ 69 & 75 & 70 & 78 & 79 \end{bmatrix}$$

This is called a $10 \times 5$ matrix. It has 10 rows and 5 columns. More generally, an $m \times n$ matrix is a table or rectangular array of the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

It has $m$ *rows* and $n$ *columns*. The quantities $a_{ij}$ are called the *entries* of the matrix. They are numbered with subscripts so that the first index $i$ tells you which *row* the entry is in, and the second index $j$ tells you which *column* it is in.

**Examples.**

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \quad \text{is a } 2 \times 2 \text{ matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \quad \text{is a } 2 \times 4 \text{ matrix}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \quad \text{is a } 1 \times 4 \text{ matrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \text{is a } 4 \times 1 \text{ matrix}$$

Matrices of various sizes and shapes arise in applications. For example, every financial spreadsheet involves a matrix, at least implicitly. Similarly, every system of linear equations has a *coefficient matrix.*

In computer programming, a matrix is called a *2-dimensional array* and the entry in row $i$ and column $j$ is usually denoted $a[i, j]$ instead of $a_{ij}$. As in programming, it is useful to think of the entire array as a single entity, so we use a single letter to denote it

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}.$$

There are various different special arrangements which play important roles. A matrix with the same number of rows as columns is called a *square* matrix. Matrices of coefficients for systems of linear equations are often square. A $1 \times 1$ matrix

$$\begin{bmatrix} a \end{bmatrix}$$

is not logically distinguishable from a number or *scalar*, so we make no distinction between the two concepts. A matrix with one row

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$$

is called a *row vector* and a matrix with one column

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

is called a *column vector.*

This terminology requires a bit of explanation. In three dimensional calculus, a vector is completely determined by its set of components $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$. Much of the analysis you encountered in that subject was simplified by using vector notation $\mathbf{v}$ to stand for the vector rather than emphasizing its components. When we wish to generalize to larger numbers of variables, it is also useful to think of a set of components

$$\begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

as constituting a higher dimensional vector $\mathbf{v}$. In this way we can use geometric insights which apply in two or three dimensions to help guide us—by analogy—when discussing these more complicated situations. In so doing, there is a formal difference between specifying the components horizontally, as above, or vertically as in

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

but logically speaking the same data is specified. In either case, the entity under consideration should be viewed as a higher dimensional analogue of a vector. For technical reasons which will be clear shortly, we shall usually specify such objects as column vectors.

Matrices are denoted in different ways by different authors. Most people use ordinary (non-boldface) capital letters, e.g., $A, B, X, Q$. However, one sometimes wants to use boldface for row or column vectors, as above, since boldface is commonly used for vectors in two and three dimensions and we want to emphasize that analogy. Since there are no consistent rules about notation, you should make sure you know when a symbol represents a matrix which is not a scalar.

Matrices may be combined in various useful ways. Two matrices *of the same size and shape* are added by adding corresponding entries. You are not allowed to add matrices with different shapes.

**Examples.**

$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 2 & 4 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} x+y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -y \\ -y \\ x \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ x \end{bmatrix}.$$

The $m \times n$ matrix with zero entries is called a *zero matrix* and is usually just denoted 0. Since zero matrices with different shapes are not the same, it is sometimes necessary to indicate the shape by using subscripts, as in '$0_{mn}$', but usually the context makes it clear which zero matrix is needed. The zero matrix of a given shape has the property that if you add it to any matrix $A$ of the same shape, you get the same $A$ again.

**Example.**

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \end{bmatrix}$$

A matrix may also be multiplied by a scalar by multiplying each entry of the matrix by that scalar. More generally, we may multiply several matrices with the same shape by different scalars and add up the result:

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k$$

where $c_1, c_2, \dots, c_k$ are scalars and $A_1, A_2, \dots, A_k$ are $m \times n$ matrices with the same $m$ and $n$. This process is called *linear combination.*

**Example.**

$$2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 5 \\ 2 \end{bmatrix}.$$

Sometimes it is convenient to put the scalar on the other side of the matrix, but the meaning is the same: each entry of the matrix is multiplied by the scalar.

$$cA = Ac.$$

We shall also have occasion to consider matrix valued *functions* $A(t)$ of a scalar variable $t$. That means that each entry $a_{ij}(t)$ is a function of $t$. Such functions are differentiated or integrated entry by entry.

**Examples.**

$$\frac{d}{dt}\begin{bmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} & -e^{-t} \\ 4e^{2t} & e^{-t} \end{bmatrix}$$

$$\int_0^1 \begin{bmatrix} t \\ t^2 \end{bmatrix} dt = \begin{bmatrix} 1/2 \\ 1/3 \end{bmatrix}$$

There are various ways to *multiply* matrices. For example, one sometimes multiplies matrices of the same shape by multiplying corresponding entries. This is useful only in very special circumstances. Another kind of multiplication generalizes the *dot product* of vectors. In three dimensions, if $\mathbf{a}$ has components $[\,a_1 \quad a_2 \quad a_3\,]$ and $\mathbf{b}$ has components $[\,b_1 \quad b_2 \quad b_3\,]$, then the dot product $\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$; that is, corresponding components are multiplied and the results are added. If

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}$$

is a row vector of size $n$, and

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

is a column vector of the same size $n$, the row by column product is defined to be the sum of the products of corresponding entries

$$\begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = a_1b_1 + a_2b_2 + \dots + a_nb_n = \sum_{i=1}^{n} a_ib_i.$$

This product is of course a *scalar*, and except for the distinction between row and column vectors, it is an obvious generalization of notion of dot product in two or three dimensions. You should be familiar with its properties.

More generally, let $A$ be an $m \times n$ matrix and $B$ an $n \times p$ matrix. Then each row of $A$ has the same size as each column of $B$. The *matrix product* $AB$ is defined to be the $m \times p$ matrix with $i, j$ entry the row by column product of the $i$th row of $A$ with the $j$th column of $B$. Thus, if $C = AB$, then $C$ has the same number of rows as $A$, the same number of columns as $B$, and

$$c_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj}.$$

**Examples.**

$$\underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 1 \end{bmatrix}}_{2\times 3} = \underbrace{\begin{bmatrix} 2-1 & 0+2 & 2+1 \\ 1-0 & 0+0 & 1+0 \end{bmatrix}}_{2\times 3}$$
$$= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}}_{3\times 2}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} x-y \\ x \\ 2x+y \end{bmatrix}}_{3\times 1}$$

The most immediate use for matrix multiplication is a simplification of the notation used to describe a system of linear equations.

Consider the system in the previous section

$$\begin{aligned} 2x_1 + 3x_2 - 5x_3 + 6x_4 - x_5 &= 10 \\ 3x_1 - 3x_2 + 6x_3 + x_4 - x_5 &= 2 \\ x_1 + x_2 - 4x_3 + 2x_4 + x_5 &= 5 \\ 4x_1 - 3x_2 + x_3 + 6x_4 + x_5 &= 4 \\ 2x_1 + 3x_2 - 5x_3 + 6x_4 - x_5 &= 3 \end{aligned}$$

If you look closely, you will notice that the expressions on the left are the entries of a matrix product:

$$\begin{bmatrix} 2 & 3 & -5 & 6 & -1 \\ 3 & -3 & 6 & 1 & -1 \\ 1 & 1 & -4 & 2 & 1 \\ 4 & -3 & 1 & 6 & 1 \\ 2 & 3 & -5 & 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 - 5x_3 + 6x_4 - x_5 \\ 3x_1 - 3x_2 + 6x_3 + x_4 - x_5 \\ x_1 + x_2 - 4x_3 + 2x_4 + x_5 \\ 4x_1 - 3x_2 + x_3 + 6x_4 + x_5 \\ 2x_1 + 3x_2 - 5x_3 + 6x_4 - x_5 \end{bmatrix}$$

Note that what appears on the right—although it looks rather complicated—is just a $5 \times 1$ column vector. Thus, the system of equations can be written as a single matrix equation

$$\begin{bmatrix} 2 & 3 & -5 & 6 & -1 \\ 3 & -3 & 6 & 1 & -1 \\ 1 & 1 & -4 & 2 & 1 \\ 4 & -3 & 1 & 6 & 1 \\ 2 & 3 & -5 & 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 5 \\ 4 \\ 3 \end{bmatrix}.$$

If we use the notation

$$A = \begin{bmatrix} 2 & 3 & -5 & 6 & -1 \\ 3 & -3 & 6 & 1 & -1 \\ 1 & 1 & -4 & 2 & 1 \\ 4 & -3 & 1 & 6 & 1 \\ 2 & 3 & -5 & 6 & -1 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 10 \\ 2 \\ 5 \\ 4 \\ 3 \end{bmatrix},$$

then the system can be written even more compactly

$$A\mathbf{x} = \mathbf{b}.$$

Of course, this notational simplicity hides a lot of real complexity, but it does help us to think about the essentials of the problem.

More generally, an arbitrary system of $m$ equations in $n$ unknowns has the form

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

is the coefficient matrix and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

are column vectors of unknowns and 'givens' respectively.

Later in this chapter, we shall investigate systematic methods for solving systems of linear equations.

**Special Operations for Row or Column Vectors.** We have already remarked that a column vector

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

may be viewed as a generalization of a vector in two or three dimensions. We also used a generalization of the *dot product* of two such vectors in defining the matrix product. In a similar fashion, we may define the length of a row vector or column vector to be the square root of the sum of the squares of its components. For example. for

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -3 \\ 4 \end{bmatrix}, \qquad \text{we have } |\mathbf{v}| = \sqrt{1^2 + 2^2 + (-3)^2 + 4^2} = \sqrt{30}.$$

**Exercises for Section 2.**

1. Let

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \qquad \mathbf{z} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Calculate $\mathbf{x} + \mathbf{y}$ and $3\mathbf{x} - 5\mathbf{y} + \mathbf{z}$.

2. Let

$$A = \begin{bmatrix} 2 & 7 & 4 & -3 \\ -3 & 0 & 1 & -2 \\ 1 & 3 & -2 & 3 \\ 0 & 0 & 5 & -5 \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 5 \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} -2 \\ 2 \\ 0 \\ 4 \end{bmatrix}.$$

Compute $A\mathbf{x}$, $A\mathbf{y}$, $A\mathbf{x} + A\mathbf{y}$, and $A(\mathbf{x} + \mathbf{y})$.

3. Let

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & -2 & 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ -3 & 2 \end{bmatrix}, C = \begin{bmatrix} -1 & 1 & -3 \\ 0 & 2 & -2 \end{bmatrix}, D = \begin{bmatrix} -1 & -2 & 0 \\ 1 & -2 & 1 \\ 2 & 1 & -4 \end{bmatrix}.$$

Calculate each of the following quantities *if it is defined*: $A + 3B$, $A + C$, $C + 2D$, $AB, BA, CD, DC$.

4. Suppose $A$ is a $2 \times 2$ matrix such that

$$A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \qquad A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

Find $A$.

5. Let $\mathbf{e}_i$ denote the $n \times 1$ column vector, with all entries zero except the $i$th which is 1, e.g., for $n = 3$,

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let $A$ be an arbitrary $m \times n$ matrix. Show that $A\mathbf{e}_i$ is the $i$th column of $A$. You may verify this just in the case $n = 3$ and $A$ is $3 \times 3$. That is sufficiently general to understand the general argument.

6. Write each of the following systems in matrix form.

(a)

$$\begin{aligned} 2x_1 - 3x_2 &= 2 \\ -4x_1 + 2x_2 &= 3 \end{aligned}$$

(b)

$$\begin{aligned} 2x_1 - 3x_2 &= 4 \\ -4x_1 + 2x_2 &= 1 \end{aligned}$$

(c)

$$\begin{aligned} x_1 + x_2 \qquad &= 1 \\ x_2 + x_3 &= 1 \\ 2x_1 + 3x_2 - x_3 &= 0 \end{aligned}$$

7. (a) Determine the lengths of the following column vectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}.$$

(b) Are any of these vectors mutually perpendicular?
(c) Find unit vectors proportional to each of these vectors.

8. One kind of magic square is a square array of numbers such that the sum of every row and the sum of every column is the same number.

(a) Which of the following matrices present magic squares?

$$\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

(b) Use matrix multiplication to describe the condition that an $n \times n$ matrix $A$ presents a magic square.

9. Population is often described by a first order differential equation of the form $\dfrac{dp}{dt} = rp$ where $p$ represents the population and $r$ is a parameter called the growth rate. However, real populations are more complicated. For example, human populations come in different ages with different fertility. Matrices are used to create more realistic population models. Here is an example of how that might be done

Assume a human population is divided into 10 age groups between 0 and 99. Let $x_i$, $i = 1, 2, \dots, 10$ be the number of women in the $i$th age group, and consider the vector $\mathbf{x}$ with those components. (For the sake of this exercise, we ignore men.) Suppose the following table gives the birth and death rates for each age group in each ten year period.

| $i$ | Age | BR | DR |
|---|---|---|---|
| 1 | $0 \dots 9$ | 0 | .01 |
| 2 | $10 \dots 19$ | .01 | .01 |
| 3 | $20 \dots 29$ | .04 | .01 |
| 4 | $30 \dots 39$ | .03 | .01 |
| 5 | $40 \dots 49$ | .01 | .02 |
| 6 | $50 \dots 59$ | .001 | .03 |
| 7 | $60 \dots 69$ | 0 | .04 |
| 8 | $70 \dots 79$ | 0 | .10 |
| 9 | $80 \dots 89$ | 0 | .30 |
| 10 | $90 \dots 99$ | 0 | 1.00 |

For example, the fourth age group is women age 30 to 39. In a ten year period, we expect this group to give birth to $.03x_4$ girls, all of whom will be in the first age group at the beginning of the next ten year period. We also expect $.01x_4$ of them to die, which tells us something about the value of $x_5$ at the beginning of the next ten year period.

Construct a $10 \times 10$ matrix $A$ which incorporates this information about birth and death rates so that $A\mathbf{x}$ gives the population vector after one ten year period has elapsed.

Note that $A^n\mathbf{x}$ keeps track of the population structure after $n$ ten year periods have elapsed.

## 3. Formal Rules

The *usual rules of algebra* apply to matrices with a few exceptions. Here are *some* of these rules and warnings about when they apply.

The *associative law*

$$A(BC) = (AB)C$$

works as long as the shapes of the matrices match. That means that the length of each row of $A$ must be the same as the length of each column of $B$ and the length of each row of $B$ must be the same as the length of each column of $C$. Otherwise, none of the products in the formula will be defined.

**Example 1.** Let

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -2 & -2 \end{bmatrix}, \qquad (AB)C = \begin{bmatrix} 7 \\ -6 \end{bmatrix},$$

while

$$BC = \begin{bmatrix} 7 \\ 1 \end{bmatrix}, \qquad A(BC) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}.$$

Note that this is bit more complicated than the associative law for ordinary numbers (scalars).

For those who are interested, the proof of the general associative law is outlined in the exercises.

For each positive integer $n$, the $n \times n$ matrix

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

is called the *identity matrix* of degree $n$. As in the case of the zero matrices, we get a different identity matrix for each $n$, and if we need to note the dependence on $n$, we shall use the notation $I_n$. The identity matrix of degree $n$ has the property $IA = A$ for any matrix $A$ with $n$ rows and the property $BI = B$ for any matrix $B$ with $n$ columns.

**Example 2.** Let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

be the $3 \times 3$ identity matrix. Then, for example,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 4 & 2 \\ -1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \\ -1 & 6 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

The entries of the identity matrix are usually denoted $\delta_{ij}$. $\delta_{ij} = 1$ if $i = j$ (the *diagonal entries*) and $\delta_{ij} = 0$ if $i \neq j$. The indexed expression $\delta_{ij}$ is often called the *Kronecker* $\delta$.

The *commutative law* $AB = BA$ is *not generally true* for matrix multiplication. First of all, the products won't be defined unless the shapes match. Even if the shapes match on both sides, the resulting products may have different sizes. Thus, if $A$ is $m \times n$ and $B$ is $n \times m$, then $AB$ is $m \times m$ and $BA$ is $n \times n$. Finally, even if the shapes match and the products have the same sizes (if both $A$ and $B$ are $n \times n$), it may still be true that the products are different.

**Example 3.** Suppose

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \qquad BA = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \neq 0$$

so $AB \neq BA$. Lest you think that this is a specially concocted example, let me assure you that it is the exception rather than the rule for the commutative law to hold for a randomly chosen pair of square matrices.

Another rule of algebra which holds for scalars but *does not generally hold* for matrices is the *cancellation law.*

**Example 4.** Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$AB = 0 \qquad \text{and} \qquad AC = 0$$

so we cannot necessarily conclude from $AB = AC$ that $B = C$.

The *distributive laws*

$$\begin{aligned} A(B + C) &= AB + AC \\ (A + B)C &= AC + BC \end{aligned}$$

do hold as long as the operations are defined. Note however that since the commutative law does not hold in general, the distributive law must be stated for both possible orders of multiplication.

Another useful rule is

$$c(AB) = (cA)B = A(cB)$$

where $c$ is a scalar and $A$ and $B$ are matrices whose shapes match so the products are defined.

The rules of calculus apply in general to matrix valued functions except that you have to be careful about orders whenever products are involved. For example, we have

$$\frac{d}{dt}(A(t)B(t)) = \frac{dA(t)}{dt}B(t) + A(t)\frac{dB(t)}{dt}$$

for matrix valued functions $A(t)$ and $B(t)$ with matching shapes.

We have just listed *some* of the rules of algebra and calculus, and we haven't discussed any of the proofs. Generally, you can be confident that matrices can be manipulated like scalars if you are careful about matters like commutativity discussed above. However, in any given case, if things don't seem to be working properly, you should look carefully to see if some operation you are using is valid for matrices.

**Exercises for Section 3.**

1. (a) Let $I$ be the $3 \times 3$ identity matrix. What is $I^2$? How about $I^3, I^4, I^5$, etc.?

(b) Let $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. What is $J^2$?

2. Find two $2 \times 2$ matrices $A$ and $B$ such that neither has *any* zero entries but such that $AB = 0$.

3. Let $A$ be an $m \times n$ matrix, let $\mathbf{x}$ and $\mathbf{y}$ be $n \times 1$ column vectors, and let $a$ and $b$ be scalars. Using the rules of algebra discussed in Section 3, prove

$$A(a\mathbf{x} + b\mathbf{y}) = a(A\mathbf{x}) + b(A\mathbf{y}).$$

4. (Optional) Prove the associative law $(AB)C = A(BC)$. Hint: If $D = AB$, then $d_{ik} = \sum_{j=1}^{n} a_{ij}b_{jk}$, and if $E = BC$ then $e_{jr} = \sum_{k=1}^{p} b_{jk}c_{kr}$, where $A$ is $m \times n$, $B$ is $n \times p$, and $C$ is $p \times q$.

5. Verify the following relation

$$\frac{d}{dt}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}.$$

## 4. Linear Systems of Algebraic Equations

We start with a problem you ought to be able to solve from what you learned in high school

**Example 1.** Consider the algebraic system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ x_1 - x_2 + x_3 &= 0 \\ x_1 + x_2 + 2x_3 &= 1 \end{aligned} \tag{1}$$

which is a system of 3 equations in 3 unknowns $x_1, x_2, x_3$. This system may also be written more compactly as a matrix equation

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The method we shall use to solve (1) is the method of *elimination* of unknowns. Subtract the first equation from each of the other equations to eliminate $x_1$ from those equations.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ -3x_2 + 2x_3 &= -1 \\ -x_2 + 3x_3 &= 0 \end{aligned}$$

Now subtract 3 times the third equation from the second equation.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ -7x_3 &= -1 \\ -x_2 + 3x_3 &= 0 \end{aligned}$$

which may be reordered to obtain

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ -x_2 + 3x_3 &= 0 \\ 7x_3 &= 1. \end{aligned}$$

We may now solve as follows. According to the last equation $x_3 = 1/7$. Putting this in the second equation yields

$$-x_2 + 3/7 = 0 \qquad \text{or} \qquad x_2 = 3/7.$$

Putting $x_3 = 1/7$ and $x_2 = 3/7$ in the first equation yields

$$x_1 + 2(3/7) - 1/7 = 1 \qquad \text{or} \qquad x_1 = 1 - 5/7 = 2/7.$$

Hence, we get

$$\begin{aligned} x_1 &= 2/7 \\ x_2 &= 3/7 \\ x_3 &= 1/7 \end{aligned}$$

To check, we calculate

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2/7 \\ 3/7 \\ 1/7 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The above example illustrates the general procedure which may be applied to any system of $m$ equations in $n$ unknowns

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

or, using matrix notation,

$$A\mathbf{x} = \mathbf{b}$$

with

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

As in Example 1, a sequence of elimination steps yields a set of equations each involving at least one fewer unknowns than the one above it. This process is called *Gaussian reduction* after the famous 19th century German mathematician C. F. Gauss. To complete the solution, we start with the last equation and substitute back recursively in each of the previous equations. This process is called appropriately *back-substitution*. The combined process will generally lead to a complete solution, but, as we shall see later, there can be some difficulties.

**Row Operations and Gauss-Jordan reduction.** Generally a system of $m$ equations in $n$ unknowns can be written in matrix form

$$A\mathbf{x} = \mathbf{b}$$

where $A$ is an $m \times n$ matrix of coefficients, $\mathbf{x}$ is a $n \times 1$ column vector of unknowns and $\mathbf{b}$ is a $m \times 1$ column vector of givens. It turns out to be just about as easy to study more general systems of the form

$$AX = B$$

where $A$ is an $m \times n$ matrix, $X$ is an $n \times p$ matrix of *unknowns*, and $B$ is a *known* $m \times p$ matrix. Usually, $p$ will be 1, so $X$ and $B$ will be column vectors, but the procedure is basically the same for any $p$. For the moment we emphasize the case in which the coefficient matrix $A$ is *square*, i.e., $m = n$, but we shall return later to the general case ($m$ and $n$ possibly different).

If you look carefully at Example 1, you will see that we employed three basic types of operations:

(1) adding or subtracting a multiple of one equation from another,
(2) multiplying or dividing an equation by a non-zero scalar,
(3) interchanging two equations.

Translated into matrix notation, these operations correspond to applying the following operations to the matrices on both sides of the equation $AX = B$:

(1) adding or subtracting one row of a matrix to another,
(2) multiplying or dividing one row of a matrix by a non-zero scalar,
(3) interchanging two rows of a matrix.

(The rows of the matrices correspond to the equations.)

These operations are called *elementary row operations*.

An important principle about row operations that we shall use over and over again is the following: *To apply a row operation to a product $AX$, it suffices to apply the row operation to $A$ and then to multiply the result by $X$.* It is easy to convince yourself that this rule is valid by looking at examples.

**Example.** Suppose

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \qquad X = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

Apply the operation of adding $-2$ times the first row of $A$ to the second row of $A$.

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix}$$

and multiply by $X$ to get

$$\begin{bmatrix} 1 & 3 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 8 & 6 \\ -6 & -4 & -2 \end{bmatrix}.$$

On the other hand, first compute

$$AX = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 8 & 6 \\ 14 & 12 & 10 \end{bmatrix}$$

and then add $-2$ times its first row to its second row to obtain

$$\begin{bmatrix} 10 & 8 & 6 \\ 14 & 10 & 12 \end{bmatrix} \to \begin{bmatrix} 10 & 8 & 6 \\ -6 & -4 & -2 \end{bmatrix}.$$

Note that the result is the same by either route.

If you want to see a general explanation of why this works, see the appendix to this section.

This suggests a procedure for solving a system of the form

$$AX = B.$$

Apply row operations to both sides until we obtain a system which is easy to solve (or for which it is clear there is no solution.) Because of the principle just enunciated, we may apply the row operations on the left just to the matrix $A$ and omit reference to $X$ since that is not changed. For this reason, it is usual to collect $A$ on the left and $B$ on the right in a so-called *augmented matrix*

$$[A \,|\, B]$$

where the '|' (or other appropriate divider) separates the two matrices. We illustrate this by redoing Example 1, but this time using matrix notation.

**Example 1, redone.** The system was

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ x_1 - x_2 + x_3 &= 0 \\ x_1 + x_2 + 2x_3 &= 1 \end{aligned}$$

so the augmented matrix is

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{array}\right]$$

We first do the Gaussian part of the reduction using row operations. The row operations are indicated to the right with the *rows that are changed* in bold face.

$$\begin{aligned}
\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{array}\right] &\to \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -3 & 2 & -1 \\ 1 & 1 & 2 & 1 \end{array}\right] & & -1[\text{row}1] + \textbf{row2} \\
&\to \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -3 & 2 & -1 \\ 0 & -1 & 3 & 0 \end{array}\right] & & -1[\text{row}1] + \textbf{row3} \\
&\to \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & 0 & -7 & -1 \\ 0 & -1 & 3 & 0 \end{array}\right] & & -3[\text{row}3] + \textbf{row2} \\
&\to \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -7 & -1 \end{array}\right] & & \textbf{row3} \leftrightarrow \textbf{row2}
\end{aligned}$$

Compare this with the previous reduction using equations. We can reconstruct the corresponding system from the augmented matrix, and, as before, we get

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ -x_2 + 3x_3 &= 0 \\ -7x_3 &= -1 \end{aligned}$$

Earlier, we applied back-substitution to find the solution. However, it is better for matrix computation to use an essentially equivalent process. Starting with the last row, use the leading non-zero entry to eliminate the entries above it. (That corresponds to substituting the value of the corresponding unknown in the previous equations.) This process is called *Jordan reduction*. The combined process is called *Gauss–Jordan reduction*. or, sometimes, *reduction to row reduced echelon form*.

$$\begin{aligned}
\left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & -7 & -1 \end{array}\right] &\to \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1/7 \end{array}\right] & & (1/7)[\mathbf{row}3] \\
&\to \left[\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -1 & 0 & -3/7 \\ 0 & 0 & 1 & 1/7 \end{array}\right] & & -3[\mathrm{row}3] + \mathbf{row}2 \\
&\to \left[\begin{array}{rrr|r} 1 & 2 & 0 & 8/7 \\ 0 & -1 & 0 & -3/7 \\ 0 & 0 & 1 & 1/7 \end{array}\right] & & [\mathrm{row}3] + \mathbf{row}1 \\
&\to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2/7 \\ 0 & -1 & 0 & -3/7 \\ 0 & 0 & 1 & 1/7 \end{array}\right] & & 2[\mathrm{row}2] + \mathbf{row}1 \\
&\to \left[\begin{array}{rrr|r} 1 & 0 & 0 & 2/7 \\ 0 & 1 & 0 & 3/7 \\ 0 & 0 & 1 & 1/7 \end{array}\right] & & -1[\mathbf{row}2]
\end{aligned}$$

This corresponds to the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} 2/7 \\ -3/7 \\ 1/7 \end{bmatrix} \qquad \text{or} \qquad X = IX = \begin{bmatrix} 2/7 \\ -3/7 \\ 1/7 \end{bmatrix}$$

which is the desired solution: $x_1 = 2/7, x_2 = -3/7, x_3 = 1/7$.

Here is another similar example.

**Example 2.**

$$\begin{aligned}
x_1 + x_2 - x_3 &= 0 \\
2x_1 \qquad + x_3 &= 2 \\
x_1 - x_2 + 3x_3 &= 1
\end{aligned}$$

or

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Reduce the augmented matrix as follows.

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 2 & 0 & 1 & | & 2 \\ 1 & -1 & 3 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 3 & | & 2 \\ 1 & -1 & 3 & | & 1 \end{bmatrix} \qquad -2[r1] + [\mathbf{r2}]$$

$$\to \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 3 & | & 2 \\ 0 & -2 & 4 & | & 1 \end{bmatrix} \qquad -[r1] + [\mathbf{r3}]$$

$$\to \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 3 & | & 2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \qquad -[r2] + [\mathbf{r3}]$$

This completes the Gaussian reduction. Now continue with the Jordan reduction.

$$\begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 3 & | & 2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 & | & 0 \\ 0 & -2 & 0 & | & 5 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \qquad -3[r3] + [\mathbf{r2}]$$

$$\to \begin{bmatrix} 1 & 1 & 0 & | & -1 \\ 0 & -2 & 0 & | & 5 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \qquad [r3] + [\mathbf{r1}]$$

$$\to \begin{bmatrix} 1 & 1 & 0 & | & -1 \\ 0 & 1 & 0 & | & -5/2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \qquad -(1/2)[\mathbf{r2}]$$

$$\to \begin{bmatrix} 1 & 0 & 0 & | & -3/2 \\ 0 & 1 & 0 & | & -5/2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \qquad -[r2] + [\mathbf{r1}]$$

This corresponds to the system

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X = \begin{bmatrix} -3/2 \\ -5/2 \\ -1 \end{bmatrix} \qquad \text{or} \qquad X = IX = \begin{bmatrix} -3/2 \\ -5/2 \\ -1 \end{bmatrix}$$

which is the desired solution: $x_1 = -3/2, x_2 = -5/2, x_3 = -1$. (Check it by plugging back into the original matrix equation.)

The strategy is clear. Use the sequence of row operations as indicated above to reduce the coefficient matrix $A$ to the identity matrix $I$. *If this is possible*, the same sequence of row operations will transform the matrix $B$ to a new matrix $B'$, and the corresponding matrix equation will be

$$IX = B' \qquad \text{or} \qquad X = B'.$$

It is natural at this point to conclude that $X = B'$ is the solution of the original system, but there is a subtle problem with that. As you may have learned in high school, the process of solving an equation or a system of equations may introduce *extraneous* solutions. These are not actually solutions of the original equations, but are introduced by some algebraic manipulation along the way. In particular, it might be true that the original system $AX = B$ has *no* solutions, and the conclusion $X = B'$ is an artifact of the process. The best we conclude from the above logic is the following: *if there is a solution*, and if it is possible to reduce $A$ to $I$ by a sequence of row operations, then the solution is $X = B'$. That is, *if a solution exists*, it is unique, i.e., there is only one solution. To see that the solution must be unique, argue as follows. Suppose there were two solutions $X$ and $Y$. Then we would have

$$AX = B$$
$$AY = B.$$

Subtraction would then yield

$$A(X - Y) = 0 \qquad \text{or} \qquad AZ = 0 \qquad \text{where } Z = X - Y.$$

However, we could apply our sequence of row operations to the equation $AZ = 0$ to obtain $IZ = 0$ since row operations have no effect on the zero matrix. Thus, we would conclude that $Z = X - Y = 0$ or $X = Y$.

How about the question of whether or not there is a solution in the first place? Of course, in any given case, we can simply check that $X = B'$ is a solution by substituting $B'$ for $X$ in $AX = B$ and seeing that it works. However, this relies on knowing $A$ and $B$ explicitly. So, it would be helpful if we had a general argument which assured us that $X = B'$ *is* a solution when the reduction is possible. (Among other things, we could skip the checking process if we were sure we did all the arithmetic correctly.)

To understand why $X = B'$ definitely is a solution, we need another argument. First note that *every possible row operation is reversible.* Thus, to reverse the effect of adding a multiple of one row to another, just subtract the same multiple of the first row from the (modified) second row. To reverse the effect of multiplying a row by a non-zero scalar, just multiply the (modified) row by the reciprocal of that scalar. Finally, to reverse the effect of interchanging two rows, just interchange them back. Hence, the effect of any sequence of row operations on a system of equations is to produce an *equivalent* system of equations. Anything which is a solution of the initial system is necessarily a solution of the transformed system and vice-versa. Thus, the system $AX = B$ is *equivalent* to the system $X = IX = B'$, which is to say $X = B'$ is a solution of $AX = B$.

**Appendix. Elementary Matrices and the Effect of Row Operations on Products.** Each of the elementary row operations may be accomplished by multiplying by an appropriate square matrix on the left. Such matrices of course should have the proper size for the matrix being multiplied.

To add $c$ times the $j$th row of a matrix to the $i$th row (with $i \neq j$), multiply that matrix on the left by the matrix $E_{ij}(c)$ which has diagonal entries 1, the $i,j$-entry $c$, and all other entries 0. This matrix may also be obtained by applying the specified row operation to the identity matrix. You should try out a few examples to convince yourself that it works.

**Example.** For $n = 3$,

$$E_{13}(-4) = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To multiply the $i$th row of a matrix by $c \neq 0$, multiply that matrix on the left by the matrix $E_i(c)$ which has diagonal entries 1 except for the $i, i$-entry which is $c$ and which has all other entries zero. $E_i(c)$ may also be obtained by multiplying the $i$th row of the identity matrix by $c$.

**Example.** For $n = 3$,

$$E_2(6) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

To interchange the $i$th and $j$th rows of a matrix, with $i \neq j$, multiply by the matrix on the left by the matrix $E_{ij}$ which is obtained from the identity matrix by interchanging its $i$th and $j$th rows. The diagonal entries of $E_{ij}$ are 1 except for its $i, i$, and $j, j$-entries which are zero. Its $i, j$ and $j, i$-entries are both 1, and all other entries are zero.

**Examples.** For $n = 3$,

$$E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

Matrices of the above type are called *elementary matrices*.

The fact that row operations may be accomplished by matrix multiplication by elementary matrices has many important consequences. Thus, let $E$ be an elementary matrix corresponding to a certain elementary row operation. The associative law tells us

$$E(AX) = (EA)X$$

as long as the shapes match. However, $E(AX)$ is the result of applying the row operation to the product $AX$ and $(EA)X$ is the result of applying the row operation to $A$ and then multiplying by $X$. This establishes the important principle enunciated earlier in this section and upon which Gauss-Jordan reduction is based. *A row operation on a product $AX$ may be accomplished by first applying that row operation to $A$ and then multiplying the result by $X$.*

**Exercises for Section 4.**

1. Solve each of the following systems by Gauss-Jordan elimination *if there is a solution.*

(a)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + 2x_3 &= -1 \\ x_1 \qquad\quad + x_3 &= 0 \end{aligned}$$

(b)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 4 \\ 2x_1 + 3x_2 + 2x_3 &= -1 \\ x_1 + x_2 - x_3 &= 10 \end{aligned}$$

(c)

$$\begin{bmatrix} 1 & 1 & -2 & 3 \\ 2 & 1 & 0 & 1 \\ 1 & -1 & 1 & 0 \\ 3 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ -18 \\ -9 \\ 9 \end{bmatrix}.$$

2. Use Gaussian elimination to solve

$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

where $X$ is an unknown $2 \times 2$ matrix.

3. Solve each of the following matrix equations

$$\text{(a)} \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{(b)} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$$

4. What is the effect of multiplying a $2 \times 2$ matrix $A$ on the right by the matrix

$$E = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}?$$

What general rule is this a special case of? ($E$ is a special case of an elementary matrix, as discussed in the Appendix.)

5. (Optional) Review the material in the Appendix on elementary matrices. Then calculate

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

Hint: Use the row operations suggested by the first four matrices.

## 5. Singularity, Pivots, and Invertible Matrices

Let $A$ be a square coefficient matrix. Gauss-Jordan reduction will work as indicated in the previous section if $A$ can be reduced by a sequence of elementary row operations to the identity matrix $I$. A square matrix with this property is called *non-singular* or *invertible*. (The reason for the latter terminology will be clear shortly.) If it cannot be so reduced, it is called *singular*. Clearly, there are singular matrices. For example, the matrix equation

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

is equivalent to the system of 2 equations in 2 unknowns

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + x_2 &= 0 \end{aligned}$$

which is *inconsistent* and has no solution. Thus Gauss-Jordan reduction certainly can't work on its coefficient matrix.

To understand how to tell if a square matrix $A$ is non-singular or not, we look more closely at the Gauss-Jordan reduction process. The basic strategy is the following. Start with the first row, and use type (1) row operations to eliminate all entries in the first column below the $1,1$-position. A leading non-zero entry of a row, when used in this way, is called a *pivot*. There is one problem with this course of action: the leading non-zero entry in the first row may not be in the $1,1$-position. In that case, *first* interchange the first row with a succeeding row which does have a non-zero entry in the first column. (If you think about it, you may still see a problem. We shall come back to this and related issues later.)

After the first reduction, the coefficient matrix will have been transformed to a matrix of the form

$$\begin{bmatrix} p_1 & * & \dots & * \\ 0 & * & \dots & * \\ \vdots & \vdots & \dots & \vdots \\ 0 & * & \dots & * \end{bmatrix}$$

where $p_1$ is the (first) pivot. We now do something mathematicians (and computer scientists) love: repeat the same process for the submatrix consisting of the second and subsequent rows. If we are fortunate, we will be able to transform $A$ ultimately by a sequence of elementary row operations into matrix of the form

$$\begin{bmatrix} p_1 & * & * & \dots & * \\ 0 & p_2 & * & \dots & * \\ 0 & 0 & p_3 & \dots & * \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & p_n \end{bmatrix}$$

with pivots on the diagonal and nonzero-entries in those pivot positions. (Such a matrix is also called an *upper triangular matrix* because it has zeroes below the diagonal.) If we get this far, we are bound to succeed. Start in the lower right hand corner and apply the Jordan reduction process. In this way each entry above the pivots on the diagonal may be eliminated. We obtain this way a diagonal matrix

$$\begin{bmatrix} p_1 & 0 & 0 & \dots & 0 \\ 0 & p_2 & 0 & \dots & 0 \\ 0 & 0 & p_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & \dots & p_n \end{bmatrix}$$

with non-zero entries on the diagonal. We may now finish off the process by applying type (2) operations to the rows as needed and finally obtain the identity matrix $I$ as required.

The above analysis makes clear that the placement of the pivots is what is essential to non-singularity. What can go wrong? Let's look at an example.

**Example 1.**

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \qquad \text{clear 1st column}$$

$$\to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \text{no pivot in } 2,2 \text{ position}$$

Note that the last row consists of zeroes.

The general case is similar. It may happen for a given row that the leading non-zero entry *is not in the diagonal position*, and there is no way to remedy this by interchanging with a subsequent row. In that case, we just do the best we can. *We use a pivot as far to the left as possible (after suitable row interchange with a subsequent row where necessary).* In the extreme case, it may turn out that the submatrix we are working with consists only of zeroes, and there are no possible pivots to choose, so we stop. For a singular square matrix, this extreme case must occur, since we will run out of pivot positions before we run out of rows. Thus, the Gaussian reduction will still transform $A$ to an upper triangular matrix $A'$, but some of the diagonal entries will be zero and some of the last rows (perhaps only the last row) will consist of zeroes. That is the singular case.

We showed in the previous section that if the $n \times n$ matrix $A$ is non-singular, then every equation of the form $AX = B$ (where both $X$ and $B$ are $n \times p$ matrices) does have a solution and also that the solution $X = B'$ is *unique*. On the other hand, if $A$ is *singular*, an equation of the form $AX = B$ *may* have a solution, but *there will certainly be matrices $B$ for which $AX = B$ has no solutions.* This is best illustrated by an example.

**Example 2.** Consider the system

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & 0 \\ 1 & 2 & -2 \end{bmatrix} \mathbf{x} = \mathbf{b}$$

where $\mathbf{x}$ and $\mathbf{b}$ are $3 \times 1$ column vectors. Without specifying $\mathbf{b}$, the reduction of the augmented matrix for this system would follow the scheme

$$\begin{bmatrix} 1 & 2 & -1 & | & b_1 \\ 1 & 2 & 0 & | & b_2 \\ 1 & 2 & -2 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & | & * \\ 0 & 0 & 1 & | & * \\ 0 & 0 & -1 & | & * \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & | & b'_1 \\ 0 & 0 & 1 & | & b'_2 \\ 0 & 0 & 0 & | & b'_3 \end{bmatrix}.$$

Now simply choose $b'_3 = 1$ (or any other non-zero value), so the reduced system is inconsistent. (Its last equation would be $0 = b'_3 \neq 0$.) Since, the two row operations may be reversed, we can now work back to a system with the original coefficient matrix which is also inconsistent. (Check in this case that if you choose $b'_1 = 0, b'_2 = 1, b'_3 = 1$, then reversing the operations yields $b_1 = -1, b_2 = 0, b_3 = -1$.)

The general case is completely analogous. Suppose

$$A \to \cdots \to A'$$

is a sequence of elementary row operations which transforms $A$ to a matrix $A'$ for which the last row consists of zeroes. Choose any $n \times p$ matrix $B'$ for which the last row *does not* consist of zeroes. Then the equation

$$A'X = B'$$

cannot be valid since the last row on the left will necessarily consist of zeroes. Now reverse the row operations in the sequence which transformed $A$ to $A'$. Let $B$ be the effect of this reverse sequence on $B'$.

$$A \leftarrow \cdots \leftarrow A'$$
$$B \leftarrow \cdots \leftarrow B'$$

Then the equation

$$AX = B$$

cannot be consistent because the equivalent system $A'X = B'$ is not consistent.

We shall see later that when $A$ is a singular $n \times n$ matrix, if $AX = B$ has a solution $X$ for a particular $B$, then it has infinitely many solutions.

There is one unpleasant possibility we never mentioned. It is *conceivable* that the standard sequence of elementary row operations transforms $A$ to the identity matrix, so we decide it is non-singular, but some other bizarre sequence of elementary row operations transforms it to a matrix with some rows consisting of zeroes, in which case we should decide it is singular. Fortunately this can never happen because singular matrices and non-singular matrices have diametrically opposed properties. For example, if $A$ is non-singular then $AX = B$ has a solution for every $B$, while if $A$ is singular, there are many $B$ for which $AX = B$ has no solution. This fact does not depend on the method we use to find solutions.

**Inverses of Non-singular Matrices.** Let $A$ be a non-singular $n \times n$ matrix. According to the above analysis, the equation

$$AX = I$$

(where we take $B$ to be the $n \times n$ identity matrix $I$) has a unique $n \times n$ solution matrix $X = B'$. This $B'$ is called the *inverse of* $A$, and it is usually denoted $A^{-1}$. That explains why non-singular matrices are also called *invertible*.

**Example 3.** Consider

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}$$

To solve $AX = I$, we reduce the augmented matrix $[A\,|\,I]$.

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] &\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 1 & -1 & 0 & 1 \end{array}\right] \\ &\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & -2 & 1 \end{array}\right] \\ &\to \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right] \\ &\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 2 & -1 \end{array}\right]. \end{aligned}$$

(You should make sure you see which row operations were used in each step.) Thus, the solution is

$$X = A^{-1} = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix}.$$

Check the answer by calculating

$$A^{-1}A = \begin{bmatrix} 0 & 2 & -1 \\ 0 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

There is a subtle point about the above calculations. The matrix inverse $X = A^{-1}$ was derived as the *unique* solution of the equation $AX = I$, but we checked it by calculating $A^{-1}A = I$. The definition of $A^{-1}$ told us only that $AA^{-1} = I$. Since matrix multiplication is not generally commutative, how could we be sure that the product *in the other order* would also be the identity $I$? The answer is provided by the following tricky argument. Let $Y = A^{-1}A$. Then

$$AY = A(A^{-1}A) = (AA^{-1})A = IA = A$$

so that $Y$ is the *unique* solution of the equation $AY = A$. However, $Y = I$ is *also* a solution of that equation, so we may conclude that $A^{-1}A = Y = I$. The upshot is that for a non-singular square matrix $A$, we have *both* $AA^{-1} = I$ and $A^{-1}A = I$.

The existence of matrix inverses for non-singular square matrices suggests the following scheme for solving matrix equations of the form

$$AX = B.$$

First, find the matrix inverse $A^{-1}$, and then take $X = A^{-1}B$. This is indeed the solution since

$$AX = A(A^{-1}B) = (AA^{-1})B = IB = B.$$

However, as easy as this looks, one should not be misled by the formal algebra. The only method we have for finding the matrix inverse is to apply Gauss-Jordan reduction to the augmented matrix $[A\,|\,I]$. If $B$ has fewer than $n$ columns, then applying Gauss-Jordan reduction directly to $[A\,|\,B]$ would ordinarily involve less computation that finding $A^{-1}$. Hence, it is usually the case that applying Gauss-Jordan reduction to the original system of equations is the best strategy. An exception to this rule is where we have one common coefficient matrix $A$ and many different matrices $B$, or perhaps a $B$ with a very large number of columns. In that case, it seems as though it would make sense to find $A^{-1}$ first. However, there is a variation of Gauss-Jordan reduction called the *LU decomposition*, that is more efficient and avoids the necessity for calculating the inverse and multiplying by it. See the appendix to this section for a brief discussion of the LU decomposition.

**A Note on Strategy.** The methods outlined in this and the previous section call for us first to reduce the coefficient matrix to one with zeroes below the diagonal and pivots on the diagonal. Then, *starting in the lower right hand corner*, we use each pivot to eliminate the non-zero entries in the column above the pivot. Why is it important to start at the lower right and work backwards? For that matter, why not just clear each column above and below the pivot as we go? There is a very good reason for that. *We want to do as little arithmetic as possible.* If we clear the column above the rightmost pivot first, then nothing we do subsequently will affect the entries in that column. Doing it in some other order would require lots of unnecessary arithmetic in that column. For a system with two or three unknowns, this makes little difference. However, for large systems, the number of operations saved can be considerable. Issues like this are specially important in designing computer algorithms for solving systems of equations.

**Numerical Considerations in Computation.** The examples we have chosen to illustrate the principles employ small matrices for which one may do exact arithmetic. The worst that will happen is that some of the fractions may get a bit messy. In real applications, the matrices are often quite large, and it is not practical to do exact arithmetic. The introduction of rounding and similar numerical approximations complicates the situation, and computer programs for solving systems of equations have to deal with problems which arise from this. If one is not careful in designing such a program, one can easily generate answers which are very far off, and even deciding when an answer is sufficiently accurate sometimes involves rather subtle considerations. Typically, one encounters problems for matrices in which the entries differ radically in size. Also, because of rounding, few matrices are ever *exactly* singular since one can never be sure that a very small numerical value at a potential pivot would have been zero if the calculations had been done exactly. On the other hand, it is not surprising that matrices which are close to being singular can give computer programs indigestion.

In practical problems on a computer, the organization and storage of data can also be quite important. For example, it is usually not necessary to keep the old entries as the reduction proceeds. It is important, however, to keep track of the row operations. The memory locations which become zero in the reduction process are ideally suited for storing the relevant information to keep track of the row operations. (The LU factorization method is well suited to this type of programming.)

If you are interested in such questions, there are many introductory texts which discuss numerical linear algebra. Two such are *Introduction to Linear Algebra* by Johnson, Riess, and Arnold and *Applied Linear Algebra* by Noble and Daniel.

**Appendix. The LU Decomposition.** If one needs to solve many equations of the form $A\mathbf{x} = \mathbf{b}$ with the same $A$ but different $\mathbf{b}$s, we noted that one could first calculate $A^{-1}$ by Gauss-Jordan reduction and then calculate $A^{-1}\mathbf{b}$. However, it is more efficient to store the row operations which were performed in order to do the Gaussian reduction and then apply these to the given $\mathbf{b}$ by another method which does not require a time consuming matrix multiplication. This is made precise by a formal decomposition of $A$ as a product in a special form.

First assume that $A$ is non-singular and that the Gaussian reduction of $A$ can be done in the usual systematic manner starting in the upper left hand corner, but *without using any row interchanges*. We will illustrate the method by an example, and save an explanation for why it works for later. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ -2 & 3 & 4 \end{bmatrix}.$$

Proceed with the Gaussian reduction while at the same time storing the *inverses* of the row operations which were performed. In practice in a computer program, the operation (or actually its inverse) is stored in the memory location containing the entry which is no longer needed, but we shall indicate it more schematically. We start with $A$ on the right and the identity matrix on the left.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ -2 & 3 & 4 \end{bmatrix}.$$

Now apply the first row operation to $A$ on the right. Add $-1$ times the first row to the second row. At the same time put $+1$ in the $2,1$ entry in the matrix on the left. (Note that this is *not a row operation*, we are just storing the important part of the inverse of the operation just performed, i.e., the multiplier.)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 3 & 4 \end{bmatrix}.$$

Next add 2 times the first row to the third row of the matrix on the right and store a $-2$ in the $3,1$ position of the matrix on the left.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 3 & 6 \end{bmatrix}.$$

Now multiply the second row of the matrix on the right by $\frac{1}{2}$, and store a 2 in the $2,2$ position in the matrix on the left.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 3 & 6 \end{bmatrix}.$$

Next add $-3$ times the second row to the third row of the matrix on the right and store a 3 in the $3,2$ position of the matrix on the left.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -2 & 3 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Finally, multiply the third row of the matrix on the right by $\frac{1}{6}$ and store 6 in the $3,3$ position of the matrix on the left.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -2 & 3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The net result is that we have stored the row operations (or rather their inverses) in the matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -2 & 3 & 6 \end{bmatrix}$$

on the left and we have by Gaussian reduction reduced $A$ to the matrix

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

on the right. Note that $L$ is a lower triangular matrix and $U$ is an upper triangular matrix with *ones on the diagonal*. Also,

$$LU = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -2 & 3 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 1 \\ -2 & 3 & 4 \end{bmatrix} = A.$$

$A = LU$ is called the LU decomposition of $A$. We shall see below why this worked, but let's see how we can use it to solve a system of the form $A\mathbf{x} = \mathbf{b}$. Using the decomposition, we may rewrite this $LU\mathbf{x} = \mathbf{b}$. Put $\mathbf{y} = L\mathbf{x}$, and consider the system $L\mathbf{y} = \mathbf{b}$. To be explicit take

$$\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

so the system we need to solve is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -2 & 3 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

But this system is very easy to solve. We may simply use Gaussian reduction (Jordan reduction being unnecessary) or equivalently we can use what is called *forward substitution.* as below:

$$\begin{aligned} y_1 &= 1 \\ y_2 &= \frac{1}{2}(1 - y_1) = 0 \\ y_3 &= \frac{1}{6}(2 + 2y_1 - 3y_2) = \frac{2}{3}. \end{aligned}$$

So the intermediate solution is

$$\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 2/3 \end{bmatrix}.$$

Now we need only solve $U\mathbf{x} = \mathbf{y}$ or

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 2/3 \end{bmatrix}.$$

To do this, either we may use Jordan reduction or equivalently, what is usually done, back substitution.

$$\begin{aligned} x_3 &= \frac{2}{3} \\ x_2 &= 0 - 0\,x_3 = 0 \\ x_1 &= 1 - 0\,x_2 - 1\,x_3 = \frac{1}{3}. \end{aligned}$$

So the solution we obtain finally is

$$\mathbf{x} = \begin{bmatrix} 1/3 \\ 0 \\ 2/3 \end{bmatrix}.$$

You should check that this is actually a solution of the original system.

Note that all this would have been silly had we been interested just in solving the single system $A\mathbf{x} = \mathbf{b}$. In that case, Gauss-Jordan reduction would have sufficed, and it would not have been necessary to store the row operations in the matrix $L$. However, if we had many such equations to solve with the same coefficient matrix $A$, we would save considerable time by having saved the important parts of the row operations in $L$. And unlike the inverse method, forward and back substitution eliminate the need to multiply any matrices.

**Why the LU decomposition method works.** Assume as above, that $A$ is non-singular and can be reduced in the standard order without any row interchanges. Recall that each row operation may be accomplished by pre-multiplying by an appropriate elementary matrix. Let $E_{ij}(c)$ be the elementary matrix which adds $c$ times the $i$th row to the $j$th row, and let $E_i(c)$ be the elementary matrix which multiplies the $i$th row by $c$. Then in the above example, the Gaussian part of the reduction could be described schematically by

$$\begin{aligned} A \to E_{12}(-1)A \to E_{13}(2)E_{12}(-1)A \to E_2(1/3)E_{13}(2)E_{12}(-1)A \\ \to E_{23}(-3)E_2(1/3)E_{13}(2)E_{12}(-1)A \\ \to E_3(1/6)E_{23}(-3)E_2(1/3)E_{13}(2)E_(12(-1)A = U \end{aligned}$$

where

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is the end result of the Gaussian reduction and is upper triangular with ones on the diagonal. To get the LU decomposition of $A$, simply multiply the left hand side of the last equation by the inverses of the elementary matrices, and remember that the inverse of an elementary matrix is a similar elementary matrix with the scalar replaced by its negative for type one operations or its reciprocal for type two operations. So

$$A = E_{12}(1)E_{13}(-2)E_2(3)E_{23}(3)E_3(6)U = LU$$

where $L$ is just the product of the elementary matrices to the left of $A$. Because we have been careful of the order in which the operations were performed, all that is necessary to compute this matrix, is to place the indicated scalar in the indicated position. Nothing that is done later can effect the placement of the scalars done earlier, So $L$ ends up being the matrix we derived above.

**The case in which switching rows is required.** In many cases, Gaussian reduction cannot be done without some row interchanges, To see how this affects the procedure, imagine that the row interchanges are not actually done as needed, but the pivots are left in the rows they happen to appear in. This will result in a matrix which is a permuted version of a matrix in Gauss reduced form. We may then straighten it out by applying the row interchanges at the end.

Here is how to do this in actual practice. We illustrate it with an example. Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 4 \end{bmatrix}.$$

We apply Gaussian reduction, writing over each step the appropriate elementary matrix which accomplishes the desired row operation.

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 0 & 4 \end{bmatrix} \xrightarrow{E_{12}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 2 & 0 & 4 \end{bmatrix} \xrightarrow{E_{13}(-2)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & -2 & 2 \end{bmatrix}$$
$$\xrightarrow{E_{23}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\xrightarrow{E_2(-1/2)} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that two of the steps involved row interchanges: Pre-multiplication by $E_{12}$ switches rows one and two and $E_{23}$ switches rows two and three. Do these row interchanges to the original matrix

$$E_{23}E_{12}A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $Q = E_{23}E_{12}$, and now apply the LU decomposition procedure to $QA$ as described above. No row interchanges will be necessary, and we get

$$QA = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = LU$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

are respectively lower triangular and upper triangular with ones on the diagonal. Now multiply by

$$P = Q^{-1} = (E_{12})^{-1}(E_{23})^{-1} = E_{12}E_{23} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

We obtain

$$A = PLU$$

Here is a brief description of the process. First do the Gaussian reduction noting the row interchanges required. Then apply those to the original matrix and find its LU decomposition. Finally apply the same row interchanges *in the opposite order* to the identity matrix to obtain $P$. Then $A = PLU$. The matrix $P$ has the property that each row and each column has precisely one nonzero entry which is one. It is obtained by an appropriate permutation of the rows of the identity matrix. Such matrices are called *permutation matrices*.

Once one has the decomposition $A = PLU$, one may solve systems of the form $A\mathbf{x} = \mathbf{b}$ by methods similar to that described above, except that there is also a permutation of the unknowns required.

Note. If you are careful, you can recover the constituents of $L$ and $U$ from the original Gaussian elimination, if you apply permutations of indices at the intermediate stages.

**Exercises for Section 5.**

1. In each of the following cases, find the matrix inverse if one exists. Check your answer by multiplication.

(a) $\begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 4 & 1 \\ 1 & 1 & 2 \\ 1 & 3 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & 3 \\ 4 & 7 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 & 2 & 1 & 1 \\ -1 & 1 & -1 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & 2 & 1 & 2 \end{bmatrix}$

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and suppose $\det A = ad - bc \neq 0$. Show that

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Hint: It is not necessary to 'find' the solution by applying Gauss–Jordan reduction. You were told what it is. All you have to do is show that it works, i.e., that it satisfies the defining condition for an inverse.

Just compute $AA^{-1}$ and see that you get $I$.

Note that this formula is probably the fastest way to find the inverse of a $2 \times 2$ matrix. In words, you do the following: interchange the diagonal entries, change the signs of the off diagonal entries, and divide by the determinant $ad - bc$. Unfortunately, there no rule for $n \times n$ matrices, even for $n = 3$, which is quite so simple.

3. Let $A$ and $B$ be invertible $n \times n$ matrices. Show that $(AB)^{-1} = B^{-1}A^{-1}$. Note the reversal of order! Hint: As above, if you are given a candidate for an inverse, you needn't 'find' it; you need only check that it works.

4. In the general discussion of Gauss-Jordan reduction, we assumed for simplicity that there was at least one non-zero entry in the first column of the coefficient matrix $A$. That was done so that we could be sure there would be a non-zero entry in the $1,1$-position (after a suitable row interchange) to use as a pivot. What if the first column consists entirely of zeroes? Does the basic argument (for the singular case) still work?

5. (a) Solve each of the following systems by any method you find convenient.

$$\begin{aligned} x_1 + x_2 &= 2.0000 \\ 1.0001x_1 + x_2 &= 2.0001 \end{aligned} \qquad\qquad \begin{aligned} x_1 + x_2 &= 2.0000 \\ 1.0001x_1 + x_2 &= 2.0002 \end{aligned}$$

(b) You should notice that although these systems are very close together, the solutions are quite different. Can you see some characteristic of the coefficient matrix which might suggest a reason for expecting trouble?

6. Below, do all your arithmetic as though you were a calculator which can only handle four significant digits. Thus, for example, a number like 1.0001 would have to be rounded to 1.000. (a) Solve

$$\begin{aligned} .0001x_1 + x_2 &= 1 \\ x_1 - x_2 &= 0. \end{aligned}$$

by the standard Gauss-Jordan approach using the given $1,1$ position as pivot. Check your answer by substituting back in the original equations. You should be surprised by the result.

(b) Solve the same system but first interchange the two rows, i.e., choose the original $2,1$ position as pivot. Check your answer by substituting back in the original equations.

7. Find the LU decomposition of the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$. Use forward and back substitution to solve the system

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Also solve the system directly by Gauss Jordan reduction and compare the results in terms of time and effort.

## 6. Gauss-Jordan Reduction in the General Case

Gauss-Jordan reduction works just as well if the coefficient matrix $A$ is singular or even if it is not a square matrix. Consider the system

$$A\mathbf{x} = \mathbf{b}$$

where the coefficient matrix $A$ is an $m \times n$ matrix. The method is to apply elementary row operations to the augmented matrix

$$[A\,|\,\mathbf{b}] \to \cdots \to [A'\,|\,\mathbf{b}']$$

making the best of it with the coefficient matrix $A$. We may not be able to transform $A$ to the identity matrix, but we can always pick out a set of pivots, one in each non-zero row, and otherwise mimic what we did in the case of a square non-singular $A$. If we are fortunate, the resulting system $A'\mathbf{x} = \mathbf{b}'$ will have solutions.

**Example 1.** Consider

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}.$$

Reduce the augmented matrix as follows

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ -1 & -1 & 1 & 5 \\ 1 & 1 & 3 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 1 & 2 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This completes the 'Gaussian' part of the reduction with pivots in the $1,1$ and $2,3$ positions, and the last row of the transformed coefficient matrix consists of zeroes. Let's now proceed with the 'Jordan' part of the reduction. Use the last pivot to clear the column above it.

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 3 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and the resulting augmented matrix corresponds to the system

$$\begin{aligned} x_1 + x_2 \qquad &= -3 \\ x_3 &= 2 \\ 0 &= 0 \end{aligned}$$

Note that the last equation could just as well have read $0 = 6$ (or some other non-zero quantity) in which case the system would be inconsistent and not have a solution. Fortunately, that is not the case in this example. The second equation tells us $x_3 = 2$, but the first equation only gives a relation $x_1 = -3 - x_2$ between $x_1$ and $x_2$. That means that the solution has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 - x_2 \\ x_2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

where $x_2$ can have *any value whatsoever*. We say that $x_2$ is a free variable, and the fact that it is arbitrary means that there are *infinitely many solutions*. $x_1$ and $x_3$ are called *bound* variables. Note that *the bound variables are in the pivot positions.*

It is instructive to reinterpret this geometrically in terms of vectors in space. The original system of equations may be written

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ -x_1 - x_2 + x_3 &= 5 \\ x_1 + x_2 + 3x_3 &= 3 \end{aligned}$$

which are equations for 3 planes in space. Here we are using $x_1, x_2, x_3$ to denote the coordinates instead of the more familiar $x, y, z$. Solutions

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

correspond to points lying in the common intersection of those planes. Normally, we would expect three planes to intersect in a single point. That would have been the case had the coefficient matrix been non-singular. However, in this case the planes intersect in a line, and the solution obtained above may be interpreted as the vector equation of that line. If we put $x_2 = s$ and rewrite the equation using the vector notation you are familiar with from your course in vector calculus, we obtain

$$\mathbf{x} = \langle -3, 0, 2 \rangle + s\langle -1, 1, 0 \rangle.$$

You should recognize this as the line passing through the endpoint of the vector $\langle -3, 0, 3 \rangle$ and parallel to the vector $\langle -1, 1, 0 \rangle$.

Example (1) illustrates many features of the general procedure. Gauss–Jordan reduction of the coefficient matrix is always possible, but *the pivots don't always end up on the diagonal*. In any case, the Jordan part of the reduction will yield a 1 in each pivot position with zeroes elsewhere in the column containing the pivot. The position of a pivot in a row will be on the diagonal or to its right, and all entries in that row *to the left of the pivot* will be zero. Some of the entries to the right of the pivot may be non-zero.

If the number of pivots is smaller than the number of rows (which will always be the case for a singular square matrix), then some rows of the reduced coefficient matrix will consist entirely of zeroes. If there are non-zero entries in those rows to the right of the divider *in the augmented matrix*, the system is inconsistent and has no solutions.

Otherwise, the system does have solutions. Such solutions are obtained by writing out the corresponding system, and transposing all terms *not associated with the pivot position* to the right side of the equation. Each unknown in a pivot position is then expressed in terms of the non-pivot unknowns (if any). The pivot unknowns are said to be *bound*. The non-pivot unknowns may be assigned any value and are said to be *free*.

**The vector space $\mathbf{R}^n$.** As we saw in Example 1, it is helpful to visualize solutions geometrically. Thus although there were infinitely many solutions, we saw we could capture all the solutions by means of a single parameter $s$. Thus, it makes sense to describe the set of all solutions as being 'one dimensional', in the same sense that we think of a line as being one dimensional. We would like to be able to use such geometric visualization for general systems. To this end, we have to generalize our notion of 'space' and 'geometry'.

Let $\mathbf{R}^n$ denote the set of all $n \times 1$ column vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

Here the $\mathbf{R}$ indicates that the entries are supposed to be real numbers. (As mentioned earlier, we could just as well have considered the set of all $1 \times n$ row vectors.) Thus, for $n = 1$, $\mathbf{R}^1$ consists of all $1 \times 1$ matrices or scalars and as such can be identified with the number line. Similarly, $\mathbf{R}^2$ may be identified with the usual coordinate plane, and $\mathbf{R}^3$ with space. In making this definition, we hope to encourage you to think of $\mathbf{R}^4, \mathbf{R}^5$, etc. as higher dimensional analogues of these familiar geometric objects. Of course, we can't really visualize such things geometrically, but we can use the same algebra that works for $n = 1, 2$, or 3, and we can proceed by analogy.

For example, as we noted in Section 2, we can define the length $|\mathbf{v}|$ of a column vector as the square root of the sum of the squares of its components, and we may define the dot product $\mathbf{u} \cdot \mathbf{v}$ of two such vectors as the sum of products of corresponding components. The vectors are said to be perpendicular if they are not zero and their dot product is zero. These are straight forward generalizations of the corresponding notions in $\mathbf{R}^2$ and $\mathbf{R}^3$.

As another example, we can generalize the notion of plane as follows. In $\mathbf{R}^3$, the graph of a single linear equation

$$a_1x_1 + a_2x_2 + a_3x_3 = b$$

is a plane. Hence, by analogy, we call the 'graph' in $\mathbf{R}^4$ of

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b$$

a *hyperplane*.

**Example 2.** Consider the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ x_1 + 2x_2 + x_3 + 3x_4 &= 0 \\ 2x_1 + 4x_2 + 3x_4 &= 0 \end{aligned}$$

which can be rewritten in matrix form

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{1}$$

Reducing the augmented matrix yields

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 1 & 2 & 1 & 3 & 0 \\ 2 & 4 & 0 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 2 & 3 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3/2 & 0 \\ 0 & 0 & 1 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

(Note that since there are zeroes to the right of the divider, we don't have to worry about possible inconsistency in this case.) The system corresponding to the reduced augmented matrix is

$$\begin{aligned} x_1 + 2x_2 \qquad + (3/2)x_4 &= 0 \\ x_3 + (3/2)x_4 &= 0 \\ 0 &= 0 \end{aligned}$$

Thus,

$$\begin{aligned} x_1 &= -2x_2 - (3/2)x_4 \\ x_3 &= \qquad\;\; - 3(/2)x_4 \end{aligned}$$

with $x_1$ and $x_3$ *bound* and $x_2$ and $x_4$ *free*. A general solution has the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - (3/2)x_4 \\ x_2 \\ - (3/2)x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -(3/2)x_4 \\ 0 \\ -(3/2)x_4 \\ x_4 \end{bmatrix}$$
$$\mathbf{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3/2 \\ 0 \\ -3/2 \\ 1 \end{bmatrix}$$

where the free variables $x_2$ and $x_4$ can assume any value. The bound variables $x_1$ and $x_3$ are then determined.

This solution may also be interpreted geometrically in $\mathbf{R}^4$. The original set of equations may be thought of as determining a 'graph' which is the intersection of three hyperplanes (each defined by one of the equations.) Note also that each of these hyperplanes passes through the origin since the zero vector is certainly a solution. Introduce two vectors (using vector calculus notation)

$$\begin{aligned} \mathbf{v}_1 &= \langle -2, 1, 0, 0\rangle \\ \mathbf{v}_2 &= \langle -3/2, 0, -3/2, 1\rangle \end{aligned}$$

in $\mathbf{R}^4$. Note that neither of these vectors is a multiple of the other. Hence, we may think of them as spanning a (2-dimensional) plane in $\mathbf{R}^4$. Putting $s_1 = x_2$ and $s_2 = x_4$, we may express the general solution vector as

$$\mathbf{x} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2,$$

so the solution set of the system (1) may be identified with the plane spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$. Of course, we can't hope to actually draw a picture of this.

Make sure you understand the procedure used in the above examples to express the general solution vector $\mathbf{x}$ entirely in terms of the free variables. We shall use it quite generally.

Any system of equations with real coefficients may be interpreted as defining a locus in $\mathbf{R}^n$, and studying the structure—in particular, the dimensionality—of such a locus is something which will be of paramount concern.

**Example 3.** Consider

$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ -1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ -7 \\ 10 \end{bmatrix}.$$

Reducing the augmented matrix yields

$$\begin{aligned} &\left[\begin{array}{rr|r} 1 & 2 & 1 \\ 1 & 0 & 5 \\ -1 & 1 & -7 \\ 2 & 0 & 10 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & -2 & 4 \\ 0 & 3 & -6 \\ 0 & -4 & 8 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \\ &\to \left[\begin{array}{rr|r} 1 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

which is equivalent to

$$\begin{aligned} x_1 &= 5 \\ x_2 &= -2. \end{aligned}$$

Thus the unique solution vector is

$$\mathbf{x} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}.$$

Geometrically, what we have here is four lines in the plane which happen to intersect in the common point with coordinates $(5, -2)$.

**Rank and Nullity.** These examples and the preceding discussion lead us to certain conclusions about a system of the form

$$A\mathbf{x} = \mathbf{b}$$

where $A$ is an $m \times n$ matrix, $\mathbf{x}$ is an $n \times 1$ column vector of unknowns, and $\mathbf{b}$ is an $m \times 1$ column vector that is given.

The number $r$ of pivots of $A$ is called the *rank* of $A$, and clearly it plays an crucial role. It is the same as the number of non-zero rows at the end of the Gauss-Jordan reduction since there is exactly one pivot in each non-zero row. The rank is certainly not greater than either the number of rows $m$ or the number of columns $n$ of $A$.

If $m = n$, i.e., $A$ is a square matrix, then $A$ is non-singular when its rank is $n$ and it is singular when its rank is smaller than $n$.

More generally, suppose $A$ is not square, i.e., $m \neq n$. In this case, if the rank $r$ is smaller than the number of rows $m$, then there are column vectors $\mathbf{b}$ in $\mathbf{R}^m$ for which the system $A\mathbf{x} = \mathbf{b}$ *does not have any solutions.* The argument is basically the same as for the case of a singular square matrix. Transform $A$ by a sequence of elementary row operations to a matrix $A'$ with its last row consisting of zeroes, choose $\mathbf{b}'$ so that $A'\mathbf{x} = \mathbf{b}'$ is inconsistent, and reverse the operations to find an inconsistent $A\mathbf{x} = \mathbf{b}$.

If for a given $\mathbf{b}$ in $\mathbf{R}^m$, the system $A\mathbf{x} = \mathbf{b}$ does have solutions, then the unknowns $x_1, x_2, \dots, x_n$ may be partitioned into two sets: $r$ bound unknowns and $n - r$ free unknowns. The bound unknowns are expressed in terms of the free unknowns. The number $n - r$ of free unknowns is sometimes called the *nullity* of the matrix $A$. If the nullity $n - r > 0$, i.e., $n > r$, then (if there are any solutions at all) there are infinitely many solutions.

Systems of the form

$$A\mathbf{x} = \mathbf{0}$$

are called *homogeneous.* Example 2 is a homogeneous system. Gauss-Jordan reduction of a homogeneous system always succeeds since the matrix $\mathbf{b}'$ obtained from $\mathbf{b} = 0$ is also zero. If $m = n$, i.e., the matrix is square, and $A$ is non-singular, the *only solution* is 0, but if $A$ is singular, i.e., $r < n$, then there are definitely non-zero solutions since there are some free unknowns which can be assigned non-zero values. This rank argument works for any $m$ and $n$: if $r < n$, then there are definitely non-zero solutions for the homogeneous system $A\mathbf{x} = 0$. One special case of interest is $m < n$. Since $r \leq m$, we must have $r < n$ in that case. That leads to the following important principle: *a homogeneous system of linear algebraic equations for which there are more unknowns than equations always has some non-trivial solutions.*

Note that the nullity $n - r$ of $A$ measures the 'number' of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$ in the sense that it tells us the number of free variables in a general solution. (Of course, it plays a similar role for a general system, but only if it is consistent, i.e., it has solutions.) This explains the etymology of the term 'nullity'. It measures the ease with which multiplication by $A$ can transform a vector $\mathbf{x}$ in $\mathbf{R}^n$ to the zero vector in $\mathbf{R}^m$.

**Pseudo-inverses.** (This section is not essential for what follows. ) It some applications, one needs to try to find 'inverses' of non-square matrices. Thus, if $A$ is a $m \times n$ matrix, one might need to find an $n \times m$ matrix $A'$ such that

$$AA' = I \qquad \text{the } m \times m \text{ identity.}$$

Such an $A'$ would be called a *right pseudo-inverse.* Similarly, an $n \times m$ matrix $A''$ such that

$$A''A = I \qquad \text{the } n \times n \text{ identity}$$

is called a left pseudo-inverse.

**Example.** Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. To find a right pseudo-inverse, we try to solve

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for the unknown $3 \times 2$ matrix $X$. Apply Gauss–Jordan reduction to the augmented matrix

$$\left[\begin{array}{ccc|cc} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|cc} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right].$$

The corresponding system is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

This may be written out explicitly as

$$\begin{aligned} x_{11} \qquad - x_{31} &= 1 \qquad & x_{12} \qquad - x_{32} &= -1 \\ x_{21} + x_{31} &= 0 & x_{22} + x_{32} &= 1 \end{aligned}$$

Here $x_{31}$ and $x_{32}$ play the roles of free variables, and the other variables are bound. If we put both of these equal to zero, we obtain $x_{11} = 1, x_{12} = -1, x_{21} = 0, x_{22} = 1$. Thus, a right pseudo-inverse for $A$ is

$$A' = X = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

You should check that $AA' = I$. Of course, there are infinitely many other solutions obtained by letting $x_{31}$ and $x_{32}$ assume other values.

Note that

$$A'A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which is definitely not the $3 \times 3$ identity matrix. So $A'$ is not a left pseudo-inverse for $A$.

If $m < n$, i.e., $A$ has fewer rows than columns, then *no left pseudo-inverse is possible*. Similarly, if $m > n$, i.e., $A$ has more rows than columns, then *no right pseudo-inverse is possible.*

We shall prove the second statement. Suppose we could find an $n \times m$ matrix $A'$ such that $AA' = I$ (the $m \times m$ identity matrix). Then for any $m \times 1$ column vector $\mathbf{b}$, $\mathbf{x} = A'\mathbf{b}$ is a solution of $A\mathbf{x} = \mathbf{b}$ since

$$A\mathbf{x} = A(A'\mathbf{b}) = (AA')\mathbf{b} = I\mathbf{b} = \mathbf{b}.$$

On the other hand, we know that since $m > n \geq r$, we can always find a $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ does *not* have a solution.

On the other hand, if $m < n$ and the rank of $A$ is $m$ (which is as large as it can get in any case), then it is always possible to find a right pseudo-inverse. To see this, let

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \\ \vdots & \vdots & \dots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nm} \end{bmatrix}$$

and consider the matrix equation

$$AX = I.$$

It may be viewed as $m$ separate equations of the form

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, A\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix},$$

one for each column of $I$. Since $r = m$, each of these equations has a solution. (In fact it will generally have infinitely many solutions.)

**Exercises for Section 6.**

1. In each of the following cases, apply the Gauss-Jordan reduction process to find the complete solution, if one exists. As in the text, the answer should express the solution $\mathbf{x}$ as a 'particular solution' (possibly zero) plus a linear combination of 'basic vectors' with the free unknowns (if any) as coefficients.

(a) $\begin{bmatrix} 1 & -6 & -4 \\ 3 & -8 & -7 \\ -2 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix}.$

(b) $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix}.$

(c) $\begin{bmatrix} 1 & -2 & 2 & 1 \\ 1 & -2 & 1 & 2 \\ 3 & -6 & 4 & 5 \\ 1 & -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 14 \\ 8 \end{bmatrix}.$

2. What is wrong with the following reduction and the ensuing logic?

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right].$$

The equivalent system is

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 + x_3 &= 0 \end{aligned}$$

which yields the general solution $x_1 = 1 - x_2 - x_3, x_2 = -x_3$.

3. Find a general solution vector of the system $A\mathbf{x} = 0$ where

$$\text{(a) } A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 0 \\ -1 & 4 & -1 & -2 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} 1 & 3 & 4 & 0 & 2 \\ 2 & 7 & 6 & 1 & 1 \\ 4 & 13 & 14 & 1 & 3 \end{bmatrix}$$

4. Consider the vectors

$$\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

in $\mathbf{R}^4$. If these were vectors in $\mathbf{R}^3$, we could use the formula

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|}$$

to determine the angle $\theta$ between the two vectors. In $\mathbf{R}^4$, we can't of course talk directly about angles in the geometric sense we are familiar with, but we can still use the above formula to *define* the angle between the two vectors. In this example, find that angle.

5. What is the rank of the coefficient matrix for each of the matrices in the previous problem.

6. What is the rank of each of the following matrices?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

7. Let $A$ be an $m \times n$ matrix with $m < n$, and let $r$ be its rank. Which of the following is always true, sometimes true, never true?
(a) $r \le m < n$. (b) $m < r < n$. (c) $r = m$. (d) $r = n$. (e) $r < m$. (f) $r = 0$.

8. (a) A system $A\mathbf{x} = \mathbf{b}$ with $A$ an $m \times n$ matrix of rank $r$ will always have solutions if $m = r$. Explain.
(b) It will not have solutions for some choices of $\mathbf{b}$ if $r < m$. Explain.

9. How do you think the rank of a product $AB$ compares to the rank of $A$? Is the former rank always $\le$, $\ge$, or $=$ the latter rank? Try some examples, make a conjecture, and see if you can prove it. Hint: Look at the number of rows of zeroes after you reduce $A$ completely to $A'$. Could further reduction transform $A'B$ to a matrix with more rows of zeroes?

10. (Optional) Find a right pseudo-inverse $A'$ for

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix}.$$

Note that there are infinitely many answers to this problem. You need only find one, but if you are ambitious, you can find all of them. Is there a left pseudo-inverse for $A$. If there is find one, if not explain why not.

11. (Optional) If $A$ is an $m \times n$ matrix with $m > n$ (more rows that columns), we showed in the text that there can be no right pseudo-inverse $A'$ for $A$. How can we use this fact to conclude that if $m < n$ (fewer rows than columns), there is no left pseudo-inverse for $A$?

## 7. Homogeneous Systems and Vector Subspaces

As mentioned in the previous section, a system of equations of the form

$$A\mathbf{x} = \mathbf{0}$$

is called *homogeneous*. (The '$\mathbf{b}$' for such a system consists of zeroes.) A system of the form

$$A\mathbf{x} = \mathbf{b}$$

where $\mathbf{b} \neq \mathbf{0}$ is called *inhomogeneous*. Every inhomogeneous system has an associated homogeneous system, and the solutions of the two systems are closely related. To see this, review the example from the previous section

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}.$$

We showed that its general solution has the form

$$\mathbf{x} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \tag{1}$$

where $x_2$ is free and may assume any value. On the other hand, it is easy to check that the homogeneous system

$$\begin{bmatrix} 1 & 1 & 2 \\ -1 & -1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has the general solution

$$\mathbf{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \tag{2}$$

where $x_2$ is also free. (You should go back and verify that for yourself, which should be easy since the Gauss-Jordan reduction is exactly the same; the only difference is that you have zeroes to the right of the vertical bar.) A close reading of (1) and (2) is informative. First note that if we set $x_2 = 0$ in (1), we obtain the specific solution

$$\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$$

and then the remaining part of the solution is a general solution (2) of the homogeneous equation.

The phenomenon illustrated above is part of a general principle. *You can always find a general solution of an inhomogeneous linear system by adding one particular solution to a general solution of the corresponding homogeneous system.* The reason for this is fairly clear algebraically. Let $\mathbf{x}_0$ denote the particular solution of the inhomogeneous equation and let $\mathbf{x}$ denote any other solution. Then we have

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A\mathbf{x}_0 &= \mathbf{b} \end{aligned}$$

which simply asserts that both are solutions. Now subtract

$$A\mathbf{x} - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

However, since $A\mathbf{x} - A\mathbf{x}_0 = A(\mathbf{x} - \mathbf{x}_0)$, this yields

$$A(\mathbf{x} - \mathbf{x}_0) = \mathbf{0}$$

from which we conclude that $\mathbf{z} = \mathbf{x} - \mathbf{x}_0$ is a solution of the homogeneous system. Transposition yields

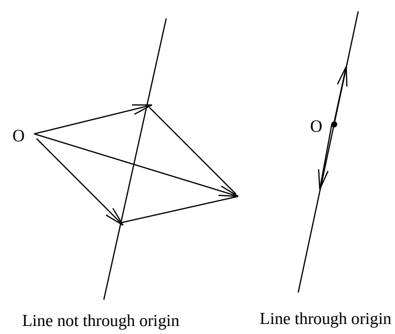
$$\mathbf{x} = \underbrace{\mathbf{x}_0}_{\text{par. sol.}} + \underbrace{\mathbf{z}}_{\text{hom. sol.}}.$$

**Vector Subspaces.** Because of the above remarks, homogeneous systems play a specially important role, so we want to concentrate on the *solution sets* of such systems. Let $A$ be an $m \times n$ matrix. The *set* of all solutions $\mathbf{x}$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the *null space* of $A$. Notice that the null space of an $m \times n$ matrix is a *subset* of $\mathbf{R}^n$.

Null spaces have an important property which we now discuss.

A non-empty subset $V$ of $\mathbf{R}^n$ is called a vector subspace if it has the property that any linear combination of vectors in $V$ is also in $V$. In symbols, if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $V$, and $a$ and $b$ are scalars, then $a\mathbf{u} + b\mathbf{v}$ is also a vector in $V$.

In two and three dimensions, the subsets which are subspaces are pretty much what you would expect. In $\mathbf{R}^2$ any line through the origin is a subspace, but lines not through the origin are not. The diagram below indicates why.



Also, curves are not vector subspaces. (See the exercises at the end of the section). In $\mathbf{R}^3$ any line through the origin is also a subspace and lines not through the origin are not. Similarly, planes through the origin are vector subspaces, but other planes are not, and of course curves or curved surfaces are not.

There is one slightly confusing point about the way we use this terminology. The entire set $\mathbf{R}^n$ is considered a *sub*set of itself, and it certainly has the desired property, so it is considered a vector subspace of itself.

It is not hard to see that the zero vector must be in every vector subspace $W$. Indeed, just pick any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $W$—$\mathbf{v}$ could even be a multiple of $\mathbf{u}$. Then $\mathbf{0} = (0)\mathbf{u} + (0)\mathbf{v}$, the linear combination with both scalars $a = b = 0$, must also be in $W$. The upshot is that *any set which does not contain the zero vector cannot be a vector subspace.*

The set consisting only of the zero vector $\mathbf{0}$ has the desired property—any linear combination of zero with itself is also zero. Hence, that set is also a vector subspace, called the zero subspace.

The term 'subspace' is sometimes used more generally to refer to any subset of $\mathbf{R}^n$. Hence the adjective 'vector' is crucial. Sometimes people use the term 'linear subspace' instead.

There are two ways vector subspaces come about. First of all, as noted above, they arise as null spaces, i.e., as solution sets of homogeneous systems $A\mathbf{x} = \mathbf{0}$. That is the main reason we are interested in them. To see why a null space satisfies the definition, suppose $\mathbf{u}$ and $\mathbf{v}$ are both solutions of $A\mathbf{x} = 0$. That is, $A\mathbf{u} = 0$ and $A\mathbf{v} = 0$. Then

$$A(a\mathbf{u} + b\mathbf{v}) = A(a\mathbf{u}) + A(b\mathbf{v}) = aA\mathbf{u} + bA\mathbf{v} = a\,\mathbf{0} + b\,\mathbf{0} = \mathbf{0}.$$

So any linear combination of solutions is again a solution and is again in the null space of $A$.

There is another related way in which vector subspaces arise, and this will play an important role in analyzing solutions of linear systems. Recall the homogeneous system

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 1 & 2 & 1 & 3 \\ 2 & 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2}$$

discussed in the previous section. We saw that its null space consists of all vectors of the form

$$x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3/2 \\ 0 \\ -3/2 \\ 0 \end{bmatrix}$$

as the free scalars $x_2$ and $x_4$ range over all possible values. Let

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} -3/2 \\ 0 \\ -3/2 \\ 0 \end{bmatrix}.$$

Then, what we have discovered is that the solution set or null space consists of all linear combinations of the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ of vectors. This is a much more useful way of presenting the answer, since we specify it in terms of a small number of objects—in this case just two. Since the null space itself is infinite, this simplifies things considerably.

In general, suppose $W$ is a vector subspace of $\mathbf{R}^n$ and $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a finite subset of $W$. We say that $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a *spanning set* for $W$ (or more simply that it *spans* $W$) if each vector $\mathbf{v}$ in $W$ can be expressed as a linear combination

$$\mathbf{v} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k,$$

for appropriate scalars $s_1, s_2, \dots, s_k$. The simplest case of this is when $k = 1$, i.e., the spanning set consists of a single vector $\mathbf{v}$. Then the subspace spanned by this vector is just the set of all $s\mathbf{v}$ with $s$ an arbitrary scalar. If $\mathbf{v} \neq \mathbf{0}$, this set is just the line through the origin containing $\mathbf{v}$.

**Example.** Consider the set of solutions $\mathbf{x}$ in $\mathbf{R}^4$ of the single homogeneous equation

$$x_1 - x_2 + x_3 - 2x_4 = 0.$$

This is the null space of the $1 \times 4$ matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & -2 \end{bmatrix}.$$

The matrix is already reduced with pivot 1 in the $1,1$-position. The general solution is

$$x_1 = x_2 - x_3 + 2x_4 \qquad x_2, x_3, x_4 \quad \text{free},$$

and the general solution vector is

$$\mathbf{x} = \begin{bmatrix} x_2 - x_3 + 2x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

It follows that the null space is spanned by

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

This is a special case of a more general principle: *Gauss-Jordan reduction for a homogeneous system always results in a description of the null space as the vector subspace spanned by a finite set of basic solution vectors*. We shall elaborate a bit more on this principle in the next section.

**Exercises for Section 7.**

1. What is the general solution of the equation $x_1 - 2x_2 + x_3 = 4$? Express it as the sum of a particular solution plus the general solution of the equation $x_1 - 2x_2 + x_3 = 0$.

2. Determine if each of the following subsets of $\mathbf{R}^3$ is a vector subspace of $\mathbf{R}^3$. If it is not a subspace, explain what fails.

(a) The set of all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $2x_1 - x_2 + 4x_3 = 0$.

(b) The set of all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that $2x_1 - x_2 + 4x_3 = 3$.

(c) The set of all $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ such that ${x_1}^2 + {x_2}^2 - {x_3}^2 = 1$.

(d) The set of all $\mathbf{x}$ of the form $\mathbf{x} = \begin{bmatrix} 1 + 2t \\ -3t \\ 2t \end{bmatrix}$ where $t$ is allowed to assume any real value.

(e) The set of all $\mathbf{x}$ of the form $\mathbf{x} = \begin{bmatrix} s + 2t \\ 2s - 3t \\ s + 2t \end{bmatrix}$ where $s$ and $t$ are allowed to assume any real values.

3. Let $L_1$ and $L_2$ be two distinct lines through the origin in $\mathbf{R}^2$. Is the set $S$ consisting of all vectors pointing along one or the other of these two lines a vector subspace of $\mathbf{R}^2$?

4. Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}.$$

What is the subspace of $\mathbf{R}^3$ spanned by these two vectors? Describe it another way.

5. (a) What is the subspace of $\mathbf{R}^3$ spanned by the set

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\}?$$

(b) What is the subspace of $\mathbf{R}^3$ spanned by

$$\left\{ \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}?$$

6. (a) Find a spanning set for the plane in $\mathbf{R}^3$ through the origin defined by the equation $x_1 - 2x_2 + 5x_3 = 0$. Check that each element of your spanning set is perpendicular to the normal vector with components $\langle 1, -2, 5\rangle$.

(b) Find a spanning set for the line in $\mathbf{R}^2$ through the origin defined by the equation $x_1 + x_2 = 0$.

## 8. Linear Independence, Bases, and Dimension

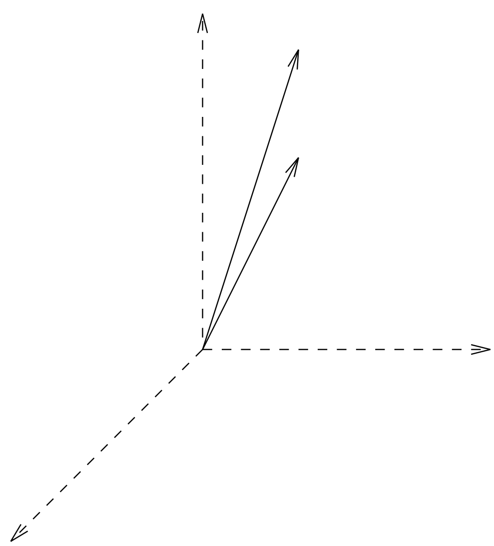
Let $V$ be a vector subspace of $\mathbf{R}^n$. If $V$ is not the zero subspace, it will have infinitely many elements, but it turns out that it is *always* possible to specify $V$ as the subspace spanned by some finite subset $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ of elements of $V$. (If you are curious why, see the appendix to this section where it is proved.)

When doing this, we want to make sure that we don't have any *superfluous* vectors in the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$.

**Example 1.** Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

The subspace of $\mathbf{R}^3$ spanned by these two vectors is a plane through the origin.



Neither of these vectors is superfluous since if you omit either, what you get is the line through the origin containing the other. You don't get the entire plane.

Consider instead the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}.$$

In this case, the second vector is twice the first vector. Hence, for any linear combination, we have

$$s_1\mathbf{v}_1 + s_2\mathbf{v}_2 = s_1\mathbf{v}_1 + s_2(2\mathbf{v}_1) = (s_1 + 2s_2)\mathbf{v}_1.$$

If we put $s = s_1 + 2s_2$, then $s$ also ranges over all possible scalars as $s_1$ and $s_2$ do, so the subspace in fact consists of all

$$s\mathbf{v}_1,$$

that is, it is a *line* through the origin. Thus, the vector $\mathbf{v}_2$ may be dropped. Similarly, we could have kept $\mathbf{v}_2$ and eliminated $\mathbf{v}_1$ since $\mathbf{v}_1 = (1/2)\mathbf{v}_2$. In any case, one of the two vectors is superfluous.

In order to deal with the issue of superfluous vectors in a spanning set, we introduce an important new concept. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a non-empty set of vectors in $\mathbf{R}^n$, not all of which are zero. Such a *set* is called *linearly independent* if no element of the set can be expressed as a *linear combination* of the other elements in the set. For a set $\{\mathbf{v}_1, \mathbf{v}_2\}$ with two vectors, this is the same as saying that neither vector is a scalar multiple of the other. For a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ with three elements it means that *no* relation of any of the following forms is possible:

$$\begin{aligned}
\mathbf{v}_1 &= a_2\mathbf{v}_2 + a_3\mathbf{v}_3 \\
\mathbf{v}_2 &= b_1\mathbf{v}_1 + b_3\mathbf{v}_3 \\
\mathbf{v}_3 &= c_1\mathbf{v}_1 + c_2\mathbf{v}_2.
\end{aligned}$$

The opposite of 'linearly independent' is 'linearly dependent'. Thus, in a linearly dependent set, there is at least one vector which is expressible as a linear combination of the others. It is important to note that linear independence and linear dependence are properties of the entire set, not the individual vectors in the set.

**Example 2.** Consider the set consisting of the following four vectors in $\mathbf{R}^4$.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

This set is not linearly independent since

$$\mathbf{v}_2 = \mathbf{v}_1 - \mathbf{v}_3. \tag{1}$$

Thus, any element in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ can be rewritten

$$\begin{aligned}
c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 &= c_1\mathbf{v}_1 + c_2(\mathbf{v}_1 - \mathbf{v}_3) + c_3\mathbf{v}_3 + c_4\mathbf{v}_4 \\
&= (c_1 + c_2)\mathbf{v}_1 + (c_3 - c_2)\mathbf{v}_3 + c_4\mathbf{v}_4 \\
&= c_1'\mathbf{v}_1 + c_3'\mathbf{v}_3 + c_4\mathbf{v}_4.
\end{aligned}$$

On the other hand, if we delete the element $\mathbf{v}_2$, the set consisting of the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}.$$

is linearly independent. To see this, just look carefully at the pattern of zeroes. For example, $\mathbf{v}_1$ has first component 1, and the other two have first component 0, so $\mathbf{v}_1$ could not be a linear combination of $\mathbf{v}_2$ and $\mathbf{v}_3$. Similar arguments eliminate the other two possible relations.

In the above example, we could just as well have written

$$\mathbf{v}_1 = \mathbf{v}_2 + \mathbf{v}_3$$

and eliminated $\mathbf{v}_1$ from the spanning set without loss. In general, there are many possible ways to delete superfluous vectors from a spanning set.

There are a couple of slightly confusing points about the definition of linear independence. First, the set $\{\mathbf{v}\}$ consisting of a single *nonzero* vector $\mathbf{v}$ is linearly *independent*. (For, there aren't any other vectors in the set which could be linear combinations of it.) The set $\{\mathbf{0}\}$ consisting only of the zero vector is not covered by the definition, but, for technical reasons, we specify that it is linearly *dependent*. Also, for technical reasons, we specify that the empty set, that is, the set with *no* vectors, is linearly *independent*.

**Bases.** Let $V$ be a vector subspace of $\mathbf{R}^n$. A subset $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of $V$ which *spans* it and is also *linearly independent* is called a *basis* for $V$.

A simple example of a basis is the set

$$\left\{ \mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

which is a basis for $\mathbf{R}^3$. (These vectors are usually called $\mathbf{i}, \mathbf{j}$, and $\mathbf{k}$ in three dimensional vector algebra. They are the unit vectors pointing along the coordinate axes.) To see that this set is linearly independent, notice that the pattern of ones and zeroes precludes one of them being expressible as a linear combination of the others. Each is one where the others are zero. To see that the set spans $\mathbf{R}^3$, note that we can write any vector $\mathbf{x}$ in $\mathbf{R}^3$ as

$$\begin{aligned} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ x_3 \end{bmatrix} \\ &= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3. \end{aligned}$$

In general, let $\mathbf{e}_i$ be the vector in $\mathbf{R}^n$ with all entries zero except the $i$th entry which is one. (One may also describe this vector as the $i$th column of the $n \times n$

identity matrix.) Then, arguing as above, it is not hard to see that $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is a basis for $\mathbf{R}^n$. It is called the *standard basis* for $\mathbf{R}^n$.

There are many other bases for $\mathbf{R}^n$. Indeed, it turns out that *any linearly independent set in $\mathbf{R}^n$ with $n$ elements is necessarily a basis for $\mathbf{R}^n$*. (See the Appendix below for an explanation.)

It is more interesting, perhaps, to consider bases for proper subspaces of $\mathbf{R}^n$. Many algorithms for solving linear problems in mathematics and its applications yield bases.

Let $A$ be an $m \times n$ matrix, and let $W$ be null space of $A$, i.e., the solution space in $\mathbf{R}^n$ of the homogeneous system $A\mathbf{x} = \mathbf{0}$. The Gauss-Jordan reduction method always generates a basis the for null space $W$. We illustrate this with an example. (You should also go back and look at Example 2 in Section 6.)

**Example 3.** Consider

$$\begin{bmatrix} 1 & 1 & 0 & 3 & -1 \\ 1 & 1 & 1 & 2 & 1 \\ 2 & 2 & 1 & 5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0.$$

To solve it, apply Gauss-Jordan reduction

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 3 & -1 & 0 \\ 1 & 1 & 1 & 2 & 1 & 0 \\ 2 & 2 & 1 & 5 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 3 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \end{array}\right]$$
$$\to \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The last matrix is fully reduced with pivots in the $1,1$ and $2,3$ positions. The corresponding system is

$$\begin{aligned} x_1 + x_2 \quad + 3x_4 \quad &= 0 \\ x_3 - \; x_4 + 2x_5 &= 0 \end{aligned}$$

with $x_1, x_3$ bound and $x_2, x_4$, and $x_5$ free. Expressing the bound variables in terms of the free variables yields

$$\begin{aligned} x_1 &= -x_2 - 3x_4 \\ x_3 &= \qquad + \; x_4 - 2x_5. \end{aligned}$$

The general solution vector, when expressed in terms of the free variables, is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 - 3x_4 \\ x_2 \\ x_4 - 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_4 \\ 0 \\ x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -2x_5 \\ 0 \\ x_5 \end{bmatrix}$$
$$= x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}.$$

If we put

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix},$$

and $c_1 = x_2$, $c_2 = x_4$, and $c_3 = x_5$, then the general solution takes the form

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

where the scalars $c_1, c_2, c_3$ (being new names for the free variables) can assume any values. Also, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. This is clear for the following reason. Each vector is associated with one of the free variables and has a 1 in that position where the other vectors necessarily have zeroes. Hence, none of the vectors can be linear combinations of the others. It follows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for the null space.

The above example illustrates all the important aspects of the solution process for a homogeneous system

$$A\mathbf{x} = 0.$$

We state the important facts about the solution without going through the general proofs since they are just the same as what we did in the example but with a lot more confusing notation. The general solution has the form

$$\mathbf{x} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k$$

where $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are *basic solutions* obtained *by successively setting each free variable equal to 1 and the other free variables equal to zero.* $s_1, s_2 \dots, s_k$ are just new names for the free variables. The set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is linearly independent because of the pattern of ones and zeroes at the positions of the free variables, and since it spans the null space, it is a basis for the null space of $A$.

*The dimension of the null space of $A$ is the nullity of $A$*, i.e., it is the number of free variables in the solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$.

There are some special cases which are a bit confusing. First, if $k = 1$, the basis consists of a single vector $\mathbf{v}_1$, and the set of solutions consists of all multiples of that vector. A much more confusing case is that in which the spanning set is the *empty set*, i.e., the set with no elements. That would arise if the zero solution were the unique solution of the homogeneous system, so there would be no free variables and no basic solutions. This is dealt with as follows. First, as noted earlier, the empty set is linearly independent by convention. Second, again by convention, every linear combination of *no vectors* is set to zero. It follows that the empty set spans the zero subspace $\{0\}$, and is a basis for it.

Let $V$ be a vector subspace of $\mathbf{R}^n$. If $V$ has a basis $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ with $k$ elements, then we say that $V$ is $k$-dimensional. That is, the dimension of a vector subspace is the number of elements in a basis.

Not too surprisingly, for the extreme case $V = \mathbf{R}^n$, the dimension is $n$. For, the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ has $n$ elements.

In this chapter we have defined the concept dimension only for vector subspaces or $\mathbf{R}^n$, but the notion is considerably more general. For example, a plane in $\mathbf{R}^3$ should be considered two dimensional even if it doesn't pass through the origin. Also, a surface in $\mathbf{R}^3$, e.g., a sphere or hyperboloid, should also be considered two dimensional. (People are often confused about curved objects because they seem to extend in extra dimensions. The point is that if you look at a small part of a surface, it normally looks like a piece of a plane, so it has the same dimension. Also, a surface can usually be represented parametrically with only two parameters.) Mathematicians have developed a very general theory of dimension which applies to almost any type of set. In cosmology, one envisions the entire universe as a certain type of four dimensional object. Certain bizarre sets can even have a fractional dimension, and that concept is useful in what is called 'chaos' theory.

**Coordinates.** Let $V$ be a subspace of $\mathbf{R}^n$ and suppose $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a basis for $V$. Suppose $\mathbf{v}$ is any vector of $V$. Then

$$\mathbf{v} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 + \cdots + s_k\mathbf{v}_k$$

for appropriate coefficients $s_1, s_2, \dots, s_k$. *The coefficients $s_1, s_2, \dots, s_k$ in such a linear combination are unique*, and are called the *coordinates* of the vector $\mathbf{v}$ with respect to the basis. We illustrate this with an example which shows how to find coordinates and why they are unique.

**Example 4.** Consider the plane $V$ in $\mathbf{R}^3$ spanned by the linearly independent pair of vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$$

and consider the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}.$$

If $\mathbf{v}$ is in $V$, then it can be written

$$\mathbf{v} = s_1\mathbf{v}_1 + s_2\mathbf{v}_2 = s_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} s_1 + \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} s_2.$$

Here, we have rewritten the linear combination with *the scalars on the right*. The advantage of so doing is that we may re-express it as a matrix product. Namely,

$$\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} s_1 + \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} s_2 = \begin{bmatrix} s_1 + s_2 \\ 2s_1 + s_2 \\ 4s_1 + 3s_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}.$$

Hence, asking if $\mathbf{v} = \mathbf{v}_1 s_1 + \mathbf{v}_2 s_2$ amounts to asking if

$$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}$$

has a solution $s_1, s_2$. This is a system of 3 equations in two unknowns (with the 'given' vector on the left instead of as usual on the right). It may be solved by Gauss-Jordan reduction as follows.

$$\begin{bmatrix} 1 & 1 & | & 1 \\ 2 & 1 & | & 3 \\ 4 & 3 & | & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & 1 \\ 0 & -1 & | & 1 \\ 0 & -1 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & -1 \\ 0 & 0 & | & 0 \end{bmatrix}.$$

Thus, it has the *unique* solution $s_1 = 2, s_2 = -1$. Thus,

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

is the *coordinate vector* giving the coordinates of $\mathbf{v}$ with respect to this basis, i.e., $\mathbf{v}$ can be written uniquely

$$\mathbf{v} = 2\mathbf{v}_1 - \mathbf{v}_2 = \mathbf{v}_1(2) + \mathbf{v}_2(-1) = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

.

There are a couple of points about the above example which merit some discussion. First, had the system not had a solution, that would just have meant that the vector $\mathbf{v}$ was not in fact in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2\}$. Second, the solution was unique because the rank was as large as possible, in this case two, and there were no free variables. If the rank had been smaller than two, then the corresponding homogeneous system

$$\mathbf{v}_1 s_1 + \mathbf{v}_2 s_2 = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] \begin{bmatrix} s_1 \\ s_2 \end{bmatrix} = \mathbf{0}$$

would necessarily have had non-zero solutions. However, any such solution with say $s_2 \neq 0$, would have allowed us to express

$$\mathbf{v}_2 = -\mathbf{v}_1 \frac{s_1}{s_2}$$

which would contradict the linear independence of the basis.

The general case is similar. If $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a basis for the subspace $V$ of $\mathbf{R}^n$, then $\mathbf{v}$ is in this subspace if and only if the system

$$\mathbf{v} = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_k\,] \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_k \end{bmatrix}$$

has a solution. (Note that the 'given' vector $\mathbf{v}$ is on the left rather than on the right as usual.) The coefficient matrix $[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_k\,]$ is an $n \times k$ matrix with columns the elements of the basis. It is necessarily of largest possible rank $k$, and the solution of the system is unique. Otherwise, the corresponding homogeneous system would have non-trivial solutions and that would contradict the linear independence of the basis.

Given a basis for a vector subspace $V$, one may think of the elements of the basis as unit vectors pointing along coordinate axes for $V$. The coordinates with respect to the basis then are the coordinates relative to these axes. The case $V = \mathbf{R}^n$ is specially enlightening. Implicitly at least one starts in $\mathbf{R}^n$ with the standard basis consisting of the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \mathbf{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}.$$

However, there are many other possible bases for $\mathbf{R}^n$ which might be useful in some applications. The axes associated with such a basis need not be mutually perpendicular, and also the units of length along these axes may differ.

**Appendix. Some subtleties.** We discuss here some of the subtleties of the theory. This should be of interest to mathematics majors and some others who enjoy theory, but it is not essential for understanding the subject matter.

First, we explain why any linearly independent subset $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ with exactly $n$ elements is necessarily a basis for $\mathbf{R}^n$. Namely, we saw that the linear independence of the set assures us that the $n \times n$ matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots \mathbf{v}_n \end{bmatrix}$$

has rank $n$. Hence, it follows from our general theory that

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots \mathbf{v}_n \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

will have a solution

$$\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_n \end{bmatrix}$$

for every $\mathbf{v}$ in $\mathbf{R}^n$. That is, every vector in $\mathbf{R}^n$ is expressible as a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$.

Next, we investigate some subtle points involved in the definition of dimension.

**Invariance of dimension.**

The dimension of $V$ is the number of elements in a basis for $V$, but it is at least conceivable that two different bases have different numbers of elements. If that were the case, $V$ would have two different dimensions, and that does not square with our idea of how such words should be used.

In fact it can never happen that two different bases have different numbers of elements. To see this, we shall prove something slightly different. Suppose $V$ has a basis with $k$ elements. We shall show that *any linearly independent subset of $V$ has at most $k$ elements.*

This would suffice for what we want because if we had two bases one with $k$ and the other with $m$ elements, either could play the role of the basis and the other the role of the linearly independent set. (Any basis is also linearly independent!) Hence, on the one hand we would have $k \le m$ and on the other hand $m \le k$, whence it follows that $m = k$.

Here is the proof of the above assertion about linearly independent subsets.

Let $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m\}$ be a linearly independent subset. Each $\mathbf{u}_i$ can be expressed uniquely in terms of the basis

$$\mathbf{u}_1 = \sum_{j=1}^{k} \mathbf{v}_j p_{j1} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \\ \vdots \\ p_{k1} \end{bmatrix}$$

$$\mathbf{u}_2 = \sum_{j=1}^{k} \mathbf{v}_j p_{j2} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \end{bmatrix} \begin{bmatrix} p_{21} \\ p_{22} \\ \vdots \\ p_{k2} \end{bmatrix}$$

$$\vdots$$

$$\mathbf{u}_m = \sum_{j=1}^{k} \mathbf{v}_j p_{jm} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \end{bmatrix} \begin{bmatrix} p_{1m} \\ p_{2k} \\ \vdots \\ p_{km} \end{bmatrix}.$$

Each of these equations represents one column of the complete matrix equation

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1m} \\ p_{21} & p_{22} & \dots & p_{2m} \\ \vdots & \vdots & \dots & \vdots \\ p_{k1} & p_{k2} & \dots & p_{km} \end{bmatrix}.$$

Note that the matrix on the right is an $k \times m$ matrix. Consider the homogeneous system

$$\begin{bmatrix} p_{11} & p_{12} & \dots & p_{1m} \\ p_{21} & p_{22} & \dots & p_{2m} \\ \vdots & \vdots & \dots & \vdots \\ p_{k1} & p_{k2} & \dots & p_{km} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = 0.$$

Assume, contrary to what we hope, that $m > k$. Then, we know by the theory of homogeneous linear systems, that there is a non-trivial solution to this system, i.e., one with at least one $x_i$ not zero. Then

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} =$$

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1m} \\ p_{21} & p_{22} & \dots & p_{2m} \\ \vdots & \vdots & \dots & \vdots \\ p_{k1} & p_{k2} & \dots & p_{km} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = 0.$$

Thus, $\mathbf{0}$ has a non-trivial representation

$$\mathbf{0} = \mathbf{u}_1 x_1 + \mathbf{u}_2 x_2 + \dots + \mathbf{u}_m x_m$$

which we know can never happen for a linearly independent set. Thus, the only way out of this contradiction is to believe that $m \le k$ as claimed.

One consequence of this argument is the following fact. If $V$ and $W$ are subspaces of $\mathbf{R}^n$ with $V \subseteq W$, then the dimension of $V$ is less than or equal to the dimension of $W$, i.e., larger subspaces have larger dimension. The reasoning is that a basis for $V$ is necessarily a linearly independent set and so it cannot have more elements than the dimension of $W$.

It is important to note that two different bases of the same vector space might have no elements whatsoever in common. All we can be sure of is that they have the same size.

**Existence of Bases.**

We assumed implicitly in our discussion of subspaces that every subspace $V$ does in fact have a basis. The following arguments show that this is true.

Start by choosing a sequence of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \dots$ in $V$, but make sure that at each stage the next vector $\mathbf{v}_p$ that you choose is not a linear combination of the previous vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{p-1}$. It is possible to show that the finite set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ is always linearly independent. (The vector $\mathbf{v}_p$ is not a linear combination of the others by construction, but you have to fiddle a bit to show the none of the previous ones are linear combinations involving $\mathbf{v}_p$.) The only question then is whether or not this sequence can go on forever. It can't do that since eventually we would get a linearly independent subset of $\mathbf{R}^n$ with $n+1$ elements, and since $\mathbf{R}^n$ has dimension $n$, that is impossible. Hence, the sequence stops, since, at some stage, we can't choose any vector in $V$ which is not a linear combination of the set of vectors so far chosen. Thus, that set spans $V$, and since, as just noted, it is linearly independent, it is a basis.

**Exercises for Section 8.**

1. In each of the following cases, determine if the indicated set is linearly independent or not.

(a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}.$

(b) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$

2. Find a basis for the subspace of $\mathbf{R}^4$ consisting of solutions of the homogeneous system

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 2 & -1 & 1 \\ 0 & 3 & -2 & 2 \end{bmatrix} \mathbf{x} = 0.$$

3. Find the dimension of the nullspace of $A$ in each of the following cases. (See the Exercises for Section 6.)

$$\text{(a) } A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 0 \\ -1 & 4 & -1 & -2 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} 1 & 3 & 4 & 0 & 2 \\ 2 & 7 & 6 & 1 & 1 \\ 4 & 13 & 14 & 1 & 3 \end{bmatrix}$$

4. Can the zero vector be an element of a linearly independent set?

5. (Optional) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a subset of $\mathbf{R}^n$ Show that the set is linearly independent if and only if the equation

$$0 = c_1\mathbf{v}_2 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

has only the trivial solution, i.e., all the coefficients $c_1 = c_2 = c_3 = 0$.

The generalization of this to $n$ vectors is very convenient to use when proving a set is linearly independent. It is often taken as the *definition* of linear independence in books on linear algebra.

6. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(a) Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set. Hint. If one of the three vectors were a linear combination of the other two, what relation would it have to the cross product of those two?

(b) Why can you conclude that it is a basis for $\mathbf{R}^3$?

(c) Find the coordinates of $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ with respect to this basis.

7. Show that

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

form a linearly independent pair in $\mathbf{R}^2$. It follows that they form a basis for $\mathbf{R}^2$. Why? Find the coordinates of $\mathbf{e}_1$ and $\mathbf{e}_2$ with respect to this new basis. Hint. You need to solve

$$\begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \mathbf{e}_1 \qquad \text{and} \qquad \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \mathbf{e}_2.$$

You can solve these simultaneously by solving

$$\begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2\end{bmatrix} X = I$$

for an appropriate $2 \times 2$ matrix $X$. What does this have to do with inverses?

8. Let

$$\mathbf{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

(a) Show that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for the subspace $W$ of $\mathbf{R}^3$ that it spans.

(b) Is $\mathbf{v} = \begin{bmatrix}1\\-1\\-2\end{bmatrix}$ in this subspace? If so, find its coordinates with respect to the basis.

9. (Optional) It is possible to consider infinite sequences of the form

$$\mathbf{x} = (x_1, x_2, \dots, x_n, \dots)$$

to be 'infinite dimensional' vectors. The set $\mathbf{R}^\infty$ of all of these is a generalization of a vector space, and many of the concepts we developed for $\mathbf{R}^n$ apply to it. Such sequences are added by adding corresponding entries and a sequence is multiplied by a scalar by multiplying each entry by that scalar. Let $\mathbf{e}_i$ be the vector in $\mathbf{R}^\infty$ with $x_i = 1$ and all other entries zero.

(a) Show that the set $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ of the first $n$ of these is a linearly independent set for each $n$. Thus there is no upper bound on the size of a linearly independent subset of $\mathbf{R}^\infty$.

(b) Does the set of all possible $\mathbf{e}_i$ span $\mathbf{R}^\infty$? Explain.

## 9. Calculations in $\mathbf{R}^n$

Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a collection of vectors in $\mathbf{R}^n$. It is a consequence of our discussion of coordinates in the previous section that *the set is linearly independent if and only if the $n \times k$ matrix $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \dots \mathbf{v}_k\end{bmatrix}$ has rank $k$.* In that case, the set is a basis for the subspace $W$ that is spans.

**Example 1.** Is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

a linearly independent set? To test this, find the rank of the matrix with these vectors as columns:

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix}.$$

It is clear without proceeding further that the rank is three, so the set is linearly independent.

More generally, suppose $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ is a set of vectors in $\mathbf{R}^n$ which may or may not be linearly independent. It is often useful to have a way to pick out a linearly independent *subset* of the set which spans the same subspace $W$ as the original set. Then that subset will be a basis for $W$. The basic idea (no pun intended) is to throw away superfluous vectors until that is no longer possible, but there is a systematic way to do this all at once. Since the vectors $\mathbf{v}_i$ are elements of $\mathbf{R}^n$, each may be specified as a $n \times 1$ column vector. Put these vectors together to form an $n \times k$ matrix

$$A = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_k\,].$$

To find a basis, apply Gaussian reduction to the matrix $A$, and pick out the columns of $A$ which in the transformed reduced matrix end up with pivots.

**Example 2.** Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

Form the matrix $A$ with these columns and apply Gaussian reduction

$$\begin{aligned} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} &\to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & -1 & 0 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

This completes the Gaussian reduction, and the pivots are in the first, second, and fourth columns. Hence, the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 4 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

form a basis for the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

Let's look more closely at this example to see why the subset is linearly independent and also spans the same subspace as the original set. The proof that the algorithm works in the general case is more complicated to write down but just elaborates the ideas exhibited in the example. Consider the homogeneous system $A\mathbf{x} = 0$. This may also be written

$$A\mathbf{x} = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4\,] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \mathbf{v}_1 x_1 + \mathbf{v}_2 x_2 + \mathbf{v}_3 x_3 + \mathbf{v}_4 x_4 = 0.$$

In the general solution, $x_1, x_2$, and $x_4$ will be bound variables (from the pivot positions) and $x_3$ will be free. That means we can set $x_3$ equal to anything, say $x_3 = -1$ and the other variables will be determined. For this choice, the relation becomes

$$\mathbf{v}_1 x_1 + \mathbf{v}_2 x_2 - \mathbf{v}_3 + \mathbf{v}_4 x_4 = 0$$

which may be rewritten

$$\mathbf{v}_3 = x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_4 \mathbf{v}_4.$$

Thus, $\mathbf{v}_3$ is superfluous and may be eliminated from the set without changing the subspace spanned by the set. On the other hand, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ is linearly independent, since if we were to apply Gaussian reduction to the matrix

$$A' = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_4\,]$$

the reduced matrix would have a pivot in every column, i.e., it would have rank 3. Thus, the system

$$[\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_4\,] \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \mathbf{v}_1 x_1 + \mathbf{v}_2 x_2 + \mathbf{v}_4 x_4 = 0$$

has only the trivial solution. That means that no one of the three vectors can be expressed as a linear combination of the other two. For example, if $\mathbf{v}_2 = c_1 \mathbf{v}_1 + c_4 \mathbf{v}_4$, we have

$$\mathbf{v}_1 c_1 + \mathbf{v}_2(-1) + \mathbf{v}_4 c_4 = 0.$$

It follows that the set is linearly independent.

**Column Space and Row Space.** Let $A$ be an $m \times n$ matrix. The columns $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ of $A$ are vectors in $\mathbf{R}^m$, and $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ spans a subspace of $\mathbf{R}^m$ called the *column space of* $A$. The column space plays a role in the the theory of inhomogeneous systems $A\mathbf{x} = \mathbf{b}$ in the following way. A vector $\mathbf{b}$ is in the column space if and only if it is expressible as a linear combination

$$\mathbf{b} = \mathbf{v}_1 x_1 + \mathbf{v}_2 x_2 + \dots + \mathbf{v}_n x_n = [\,\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n\,] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = A\mathbf{x}.$$

Thus, *the column space of $A$ consists of all vectors $\mathbf{b}$ in $\mathbf{R}^m$ for which the system $A\mathbf{x} = \mathbf{b}$ has a solution.*

**Example 3, continued.** We wish to determine if

$$\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

is in the column space of

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix}.$$

This will be true if and only if

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

has a solution. Reduce the augmented matrix

$$\begin{aligned}
\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 1 & 4 & 0 & 1 & 1 \\ 1 & 0 & -2 & 0 & 0 \end{array}\right] &\to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & -2 & -1 & 0 & -1 \end{array}\right] \\
&\to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \\
&\to \left[\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 1 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].
\end{aligned}$$

It is clear at this point that the system will have a solution, so we need not go any farther. We can conclude that $\mathbf{b}$ is in the column space of $A$.

Note that the method outlined in the beginning of this section gives a basis for the column space, and the number of elements in this basis is the rank of $A$. (The rank is the number of pivots!) Hence, *the rank of an $m \times n$ matrix $A$ is the dimension of its column space.*

The column space of a matrix $A$ is often called the *range* of $A$. That is because it describes all possible vectors in $\mathbf{R}^m$ of the form $A\mathbf{x}$.

There is a similar concept for rows; the row space of an $m \times n$ matrix $A$ is the subspace of $\mathbf{R}^n$ spanned by the rows of $A$. It is not hard to see that *the dimension of the row space of $A$ is also the rank of $A$.* For, since each row operation is reversible, applying a row operation does not change the subspace spanned by the rows. Hence, the row space of the matrix $A'$ obtained by Gauss-Jordan reduction from $A$ is the same as the row space of $A$. However, the set of non-zero rows of the reduced matrix is a basis for this subspace. To see this, note first that it certainly spans (since leaving out zero rows doesn't cost us anything). Moreover, it is also a linearly independent set because each non-zero row has a 1 in a pivot position where all the other rows are zero.

The fact that both the column space and the row space have the same dimension is sometimes expressed by saying "the column rank equals the row rank". Of course, there is no particular reason why the row space and the column space should be identical. For example, unless the matrix is square, the vectors in them won't even have the same number of components.

**A Note on the Definition of Rank.** The rank of $A$ is defined as the number of pivots in the reduced matrix obtained from $A$ by an appropriate sequence of elementary row operations. Since we can specify a standard procedure for performing such row operations, that means the rank is a well defined number. On the other hand, it is natural to wonder what might happen if $A$ were reduced by an alternative, perhaps less systematic, sequence of row operations. The above analysis shows that we would still get the same answer for the rank. Namely, the rank is the dimension of the column space of $A$, and that number depends only on the column space itself, not on any particular basis for it. (Or you could use the same argument using the row space.)

The rank is also the number of non-zero rows in the reduced matrix, so it follows that this number does not depend on the particular sequence of row operations used to reduce $A$ to Gauss-Jordan reduced form. In fact, the entire matrix obtained at the end (as long as it is in Gauss-Jordan reduced form) depends only on the original matrix $A$ and not on the particular sequence or row operations used to obtain it. The proof of this fact is not so easy, and we omit it here.

**Exercises for Section 9.**

1. Find a subset of the following set of vectors which is a basis for the subspace

it spans.

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -3 \\ 1 \end{bmatrix} \right\}$$

2. Let $A = \begin{bmatrix} 1 & 0 & 2 & 1 & 1 \\ -1 & 1 & 3 & 0 & 1 \\ 1 & 1 & 7 & 2 & 3 \end{bmatrix}$.

(a) Find a basis for the column space of $A$.

(b) Find a basis for the row space of $A$.

3. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Find a basis for $\mathbf{R}^3$ by finding a third vector $\mathbf{v}_3$ such that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. Hint. You may find an easier way to do it, but the following method should work. Use the method suggested in Section 9 to pick out a linearly independent subset from $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

4. (Optional) Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ be a linearly independent subset of $\mathbf{R}^n$. Apply the method in section 9 to the set $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k, \mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$. It will necessarily yield a basis for $\mathbf{R}^n$. Why? Show that this basis will include $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k\}$ as a subset. That is show that none of the $\mathbf{v}_i$ will be eliminated by the process.

5. (a) Find a basis for the column space of

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 6 & 7 & 10 \end{bmatrix}.$$

(b) Is $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ in the column space of $A$?

6. (a) Suppose $A$ is a $7 \times 12$ matrix and $W$ is its range or column space. If $\dim W = 7$, what can you say about the general solvability of systems of the form $A\mathbf{x} = \mathbf{b}$?

(b) Suppose instead that $A$ is $12 \times 7$. What if anything can you say about the general solvability of systems of the form $A\mathbf{x} = \mathbf{b}$?

## 10. Review Problems

**Exercises for Section 10.**

1. Find $A^{-1}$ for $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 3 & 1 & -1 \end{bmatrix}$. Check your answer by computing $AA^{-1}$.

2. Let $A = \begin{bmatrix} 2 & -1 & 6 & 1 & 2 \\ 1 & 3 & 4 & 0 & 0 \\ 4 & 5 & 14 & 1 & 2 \end{bmatrix}$.

(a) Find the dimension of the nullspace of $A$.
(b) Find the dimension of the column space of $A$.
(c) How are these two numbers related?

3. What is wrong with the following statement? If $A$ and $B$ are invertible $n \times n$ matrices, then $AB$ is invertible and $(AB)^{-1} = A^{-1}B^{-1}$.

4. Suppose $A$ is a $15 \times 23$ matrix. In which circumstances will each of the following statements be true?

(a) A system $A\mathbf{x} = \mathbf{b}$ of 15 equations in 23 unknowns has a solution for every $\mathbf{b}$ in $\mathbf{R}^{15}$.
(b) The homogeneous system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

5. Let $A = \begin{bmatrix} 1 & 3 & 4 & 0 & 2 \\ 2 & 7 & 6 & 1 & 1 \\ 4 & 13 & 14 & 1 & 3 \end{bmatrix}$.

(a) Find a basis for the nullspace of the $A$.
(b) Find the dimensions of the nullspace and the column space of $A$.
(c) Does $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ belong to the column space of $A$? Explain.

6. Find the inverse of $A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 8 \\ 0 & 2 & 2 \end{bmatrix}$.

7. Let $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 3 & 6 & 7 & 10 \end{bmatrix}$.

(a) Find a basis for the nullspace of $A$.
(b) Find a basis for the column space of $A$.
(c) Do the columns of $A$ form a linearly independent set? Explain.
(d) Does $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ belong to the column space of $A$? Explain your answer.

8. In each case tell if the indicated subset is a vector subspace of $\mathbf{R}^3$ and give a reason for your answer.

(a) The plane defined by the equation $x_1 - 2x_2 + 3x_3 = 0$.
(b) The sphere defined by the equation ${x_1}^2 + {x_2}^2 + {x_3}^2 = 16$.

9. All the parts of this question refer to the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 3 & 1 \\ 2 & 1 & 3 & 3 & 2 \\ 1 & -1 & 2 & 0 & 1 \end{bmatrix}$.

(a) What is the rank of $A$?
(b) What is the dimension of the nullspace of $A$?

10. Consider the subset

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

of $\mathbf{R}^4$.
(a) Is this set linearly independent? Explain?
(b) Is it a basis for $\mathbf{R}^4$? Explain.