Linear Algebra

Linear Transformations

Topics

- Preliminaries
- Definition and Examples
- Kernel and Range of a Linear Transformation
- Matrix of a Linear Transformation
- Vector Spaces of Matrices and Linear Transformations
- Similarity
- Homogeneous Coordinates

Topics

- ***Preliminaries***
- Definition and Examples
- Kernel and Range of a Linear Transformation
- Matrix of a Linear Transformation
- Vector Spaces of Matrices and Linear Transformations
- Similarity
- Homogeneous Coordinates

- Have pursued the following generalizations

  Vectors in R^2 & $R^3 \Rightarrow$ Vector Spaces

  Dot Product in R^2 & $R^3 \Rightarrow$ Inner Product

- Will now look at another generalization

  Matrices $\Rightarrow$ Linear Transformations

Topics

- Preliminaries
- ***Definition and Examples***
- Kernel and Range of a Linear Transformation
- Matrix of a Linear Transformation
- Vector Spaces of Matrices and Linear Transformations
- Similarity
- Homogeneous Coordinates

- Defn - Let V and W be vector spaces. A function $L:V \to W$ is called a linear transformation of V into W if

  a) $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$

  b) $L(c\mathbf{u}) = cL(\mathbf{u})$ for $\mathbf{u} \in V$ and real c

- If V = W, then L is called a linear operator

- Note: An m x n matrix takes a vector in R^n and maps it to a vector in R^m, so it can be viewed as a function from R^n to R^m

Example

- Define a mapping L: $R^3 \rightarrow R^2$ as $L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$

To verify, let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ be arbitrary

$$L(\mathbf{u}+\mathbf{v}) = L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = L\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$= L(\mathbf{u}) + L(\mathbf{v})$$

$$L(c\mathbf{u}) = L\left(c\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = L\left(\begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}\right) = \begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} = c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = cL(\mathbf{u})$$

Example

- Let $K(x,y)$ be continuous in x and y for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. Define L: C [0,1] → C [0,1] as

$$L(f) = \int_0^1 K(x,y) f(y) dy$$

From the properties of integrals, conditions (a) and (b) hold

Example

- Define a mapping L: $R^3 \rightarrow R^3$ as $L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = r\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$

  L is a linear operator on R^3. If $r > 1$, it is called a dilation. If $0 < r < 1$, it is called a contraction. General term is scaling

Example

- Consider the vector space $C^{\infty}[0,1]$ of infinitely differentiable functions defined on the interval $[0,1]$. Define a mapping $L: C^{\infty}[0,1] \to C^{\infty}[0,1]$ by

$$L(f) = f'$$

L is a linear operator on $C^{\infty}[0,1]$

Example

- Define a mapping L: $R^3 \rightarrow R^3$ as

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

L is a linear transformation. More generally, if **A** is an $m \times n$ matrix, then L(**x**)= **Ax** is a linear transformation from R^n to R^m

Example

- Define a mapping L: $R^2 \rightarrow R^2$ as $L\left(\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ -a_2 \end{bmatrix}$

  This is a linear operator, which is called a reflection in the *x*-axis

Example

- Define a mapping L: $R^2 \longrightarrow R^2$ as

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

L is a linear transformation. It is a counter-clockwise rotation by the angle ϕ

Example

- Define a mapping L: $R^3 \rightarrow R^3$ as $L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} a_1+1 \\ 2a_2 \\ a_3 \end{bmatrix}$

  Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$L(\mathbf{u}+\mathbf{v}) = L\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{bmatrix}\right) = \begin{bmatrix} u_1+v_1+1 \\ 2(u_2+v_2) \\ u_3+v_3 \end{bmatrix}$$

$\neq$

$$L(\mathbf{u})+L(\mathbf{v}) = \begin{bmatrix} u_1+1 \\ 2u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1+1 \\ 2v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} u_1+v_1+2 \\ 2(u_2+v_2) \\ u_3+v_3 \end{bmatrix}$$

Example

- Define a mapping L: $R_2 \to R_2$ as $L\left(\begin{bmatrix} a_1 & a_2 \end{bmatrix}\right) = \begin{bmatrix} a_1^2 & 2a_2 \end{bmatrix}$

Let $\mathbf{u} = [\ u_1 \ \ u_2\ ]$, $\mathbf{v} = [\ v_1 \ \ v_2\ ]$ be in R_2

$$L(\mathbf{u}+\mathbf{v}) = L\left(\begin{bmatrix} (u_1+v_1) & (u_2+v_2) \end{bmatrix}\right) = \begin{bmatrix} (u_1+v_1)^2 & 2(u_2+v_2) \end{bmatrix}$$

$$L(\mathbf{u}) + L(\mathbf{v}) = \begin{bmatrix} u_1^2 & 2u_2 \end{bmatrix} + \begin{bmatrix} v_1^2 & 2v_2 \end{bmatrix} = \begin{bmatrix} (u_1^2+v_1^2) & 2(u_2+v_2) \end{bmatrix}$$

$$L(\mathbf{u}+\mathbf{v}) \neq L(\mathbf{u}) + L(\mathbf{v})$$

So L is not a linear transformation

- Theorem - Let L: V → W be a linear transformation of an n dimensional vector space V into a vector space W. Let S = { $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ } be a basis for V. If $\mathbf{v}$ is any vector in V, then L($\mathbf{v}$) is completely determined by the set of vectors { $L(\mathbf{u}_1), L(\mathbf{u}_2), \ldots, L(\mathbf{u}_n)$ }
- Proof - Since S is a basis for V, can express $\mathbf{v}$ as $\mathbf{v} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$. Then

$$\begin{aligned} L(\mathbf{v}) &= L(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n) \\ &= L(a_1\mathbf{u}_1) + L(a_2\mathbf{u}_2) + \cdots + L(a_n\mathbf{u}_n) \\ &= a_1 L(\mathbf{u}_1) + a_2 L(\mathbf{u}_2) + \cdots + a_n L(\mathbf{u}_n) \end{aligned}$$

So L($\mathbf{v}$) can be expressed as a combination of the vectors { $L(\mathbf{u}_1), L(\mathbf{u}_2), \ldots, L(\mathbf{u}_n)$ }

QED

- Corollary - Let L: V $\rightarrow$ W and T: V $\rightarrow$ W be linear transformations. Let S = { $\mathbf{v}_1$, $\mathbf{v}_2$, …, $\mathbf{v}_n$ } be a basis for V. If L($\mathbf{v}_i$) = T($\mathbf{v}_i$) for $1 \leq i \leq n$, then L($\mathbf{v}$) = T($\mathbf{v}$) for all $\mathbf{v} \in$ V, i.e. L and T are identical linear transformations

Example

- Let L: $R^4 \rightarrow R^2$ be a linear transformation and let $S = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \}$ be a basis for R^4

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Let } L(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad L(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \quad L(\mathbf{v}_3) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad L(\mathbf{v}_4) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Example (continued)

$$\text{Let } \mathbf{v} = \begin{bmatrix} 3 \\ -5 \\ -5 \\ 0 \end{bmatrix} = 2\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3 + \mathbf{v}_4$$

$$\text{So } L(\mathbf{v}) = 2L(\mathbf{v}_1) + L(\mathbf{v}_2) - 3L(\mathbf{v}_3) + L(\mathbf{v}_4)$$

$$= 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

- Theorem - Let $L:R^n \to R^m$ be a linear transformation and **A** be the m x n matrix whose jth column is $L(\mathbf{e}_j)$, where $\{ \mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n \}$ is the natural basis for R^n. Then for every $\mathbf{x} \in R^n$, $L(\mathbf{x}) = \mathbf{Ax}$. Moreover, **A** is the only matrix with this property.
- Proof - Express **x** in terms of the natural basis as $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n$. By the properties of the linear transformation and the definition of **A**

$$\begin{aligned} L(\mathbf{x}) &= L\left(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n\right) \\ &= x_1 L(\mathbf{e}_1) + x_2 L(\mathbf{e}_2) + \cdots + x_n L(\mathbf{e}_n) \\ &= \begin{bmatrix} L(\mathbf{e}_1) & L(\mathbf{e}_2) & \cdots & L(\mathbf{e}_n) \end{bmatrix}\mathbf{x} = \mathbf{Ax} \end{aligned}$$

- Proof (continued) -

  To argue uniqueness, suppose that there is a matrix $\mathbf{B} \neq \mathbf{A}$ such that $L(\mathbf{x}) = \mathbf{Bx}$ for every $\mathbf{x} \in R^n$. Since $\mathbf{B} \neq \mathbf{A}$, **A** and **B** must differ in at least one column, call it j. By the definition of **A** and **B**, $L(\mathbf{e}_j) = \mathbf{Ae}_j = \mathbf{Be}_j$. $\mathbf{Ae}_j$ is just the jth column of **A**, $\mathbf{Be}_j$ is just the jth column of **B**, so the jth columns of **A** and **B** are the same, which is a contradiction. Therefore **A** is unique.

  QED

- Theorem - Let $L: V \to W$ be a linear transformation. Then

  a) $L(\mathbf{0}_V) = \mathbf{0}_W$

  b) $L(\mathbf{u} - \mathbf{v}) = L(\mathbf{u}) - L(\mathbf{v})$

- Proof -

  a) $\mathbf{0}_V = \mathbf{0}_V + \mathbf{0}_V$ then $L(\mathbf{0}_V) = L(\mathbf{0}_V) + L(\mathbf{0}_V)$, $L(\mathbf{0}_V) - L(\mathbf{0}_V) = L(\mathbf{0}_V) + L(\mathbf{0}_V) - L(\mathbf{0}_V)$. So $\mathbf{0}_W = L(\mathbf{0}_V)$

  b) $L(\mathbf{u} - \mathbf{v}) = L(\mathbf{u} + (-1)\mathbf{v}) = L(\mathbf{u}) + (-1)L(\mathbf{v})$ $= L(\mathbf{u}) - L(\mathbf{v})$

QED

- Preliminaries
- Definition and Examples
- ***Kernel and Range of a Linear Transformation***
- Matrix of a Linear Transformation
- Vector Spaces of Matrices and Linear Transformations
- Similarity
- Homogeneous Coordinates

- Defn - A linear transformation L: V $\longrightarrow$ W is one to one if it is a one to one function, i.e. if $\mathbf{v}_1 \neq \mathbf{v}_2$ implies $L(\mathbf{v}_1) \neq L(\mathbf{v}_2)$. (Equivalently, L is one to one if $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ implies $\mathbf{v}_1 = \mathbf{v}_2$.)

- Defn - Let L: V $\longrightarrow$ W be a linear transformation. The kernel of L, ker L, is the subset of V consisting of all $\mathbf{v} \in V$ such that $L(\mathbf{v}) = \mathbf{0}_W$

- Comment - Since $L(\mathbf{0}_V) = \mathbf{0}_W$, ker L is not empty

- Theorem - Let L: V $\rightarrow$ W be a linear transformation
  a) ker L is a subspace of V
  b) L is one to one if and only if ker L = { $\mathbf{0}_V$ }
- Proof -

  a) Use the theorem that tests for subspaces. Specifically, if U is a nonempty subset of V, it is a subspace if $\mathbf{v} + \mathbf{w} \in$ U and $c\mathbf{v} \in$ U for all $\mathbf{v}, \mathbf{w} \in$ U and all real c.
  So let $\mathbf{v}, \mathbf{w} \in$ ker L be arbitrary. Then $L(\mathbf{v}) = \mathbf{0}_W$ and $L(\mathbf{w}) = \mathbf{0}_W$. Since L is linear,
  $$L(\mathbf{v} + \mathbf{w}) = L(\mathbf{v}) + L(\mathbf{w}) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W$$
  So $\mathbf{v} + \mathbf{w} \in$ ker L

- Proof (continued)
  Let $\mathbf{v} \in$ ker L and real c be arbitrary. Since L is a linear transformation. $L(c\mathbf{v}) = cL(\mathbf{v}) = c\mathbf{0}_W = \mathbf{0}_W$
  So $c\mathbf{v} \in$ ker L

  b) $\Rightarrow\rfloor$ Let L be one to one. Let $\mathbf{v} \in$ ker L be arbitrary. Then $L(\mathbf{v}) = \mathbf{0}_W$. Also, $L(\mathbf{0}_V) = \mathbf{0}_W$. Since L is one to one, $L(\mathbf{v}) = L(\mathbf{0}_V)$ implies $\mathbf{v} = \mathbf{0}_V$ So ker L = $\{ \mathbf{0}_V \}$

  $\Leftarrow\rfloor$ Let ker L = $\{ \mathbf{0}_V \}$ and let $\mathbf{v}, \mathbf{w} \in V$ be such that $L(\mathbf{v}) = L(\mathbf{w})$. Need to show $\mathbf{v} = \mathbf{w}$. Since L is linear, $\mathbf{0}_W = L(\mathbf{v}) - L(\mathbf{w}) = L(\mathbf{v} - \mathbf{w})$. So $\mathbf{v} - \mathbf{w} \in$ ker L and $\mathbf{v} - \mathbf{w} = \mathbf{0}_V$ or $\mathbf{v} = \mathbf{w}$. So L is one to one

QED

- Note - Part (b) of the preceding theorem can be expressed as: L is one to one if and only if dim ker L = 0

- Corollary - Let $L: V \rightarrow W$ be a linear transformation. If $L(\mathbf{x}) = \mathbf{b}$ and $L(\mathbf{y}) = \mathbf{b}$, then $\mathbf{x} - \mathbf{y}$ belongs to ker L, i.e. any two solutions to $L(\mathbf{x}) = \mathbf{b}$ differ by an element of the kernel of L.
- Proof - Suppose that $L(\mathbf{x}) = \mathbf{b}$ and $L(\mathbf{y}) = \mathbf{b}$. Then $\mathbf{0}_W = \mathbf{b} - \mathbf{b} = L(\mathbf{x}) - L(\mathbf{y}) = L(\mathbf{x} - \mathbf{y})$. Therefore, $\mathbf{x} - \mathbf{y}$ belongs to ker L.

QED

Example

- Define L: $P_2 \longrightarrow R$ as

$$L\left(at^2+bt+c\right)=\int_0^1\left(at^2+bt+c\right)dt$$

i) Find ker L

ii) Find dim ker L

iii) Determine if L is one to one

$$\int_0^1\left(at^2+bt+c\right)dt=\frac{1}{3}a+\frac{1}{2}b+c$$

Example (continued)

i) $$\int_0^1 \left(at^2+bt+c\right)dt=0 \;\Rightarrow\; \frac{1}{3}a+\frac{1}{2}b+c=0$$

$$\Rightarrow\; c=-\frac{1}{3}a-\frac{1}{2}b$$

So ker L consists of polynomials of the form

$$at^2+bt+\left(-\frac{1}{3}a-\frac{1}{2}b\right)$$

Example (continued)

ii) $at^2 + bt + \left(-\frac{1}{3}a - \frac{1}{2}b\right) = a\left(t^2 - \frac{1}{3}\right) + b\left(t - \frac{1}{2}\right)$

So the vectors ($t^2 - 1/3$) and ($t - 1/2$) span ker L. Can argue that they are linearly independent. So the set { $t^2 - 1/3$, $t - 1/2$ } is a basis for ker L.

iii) Since dim ker L = 2, L is not one to one

- Defn - Let L: V → W be a linear transformation. The range of L, or image of V under L, denoted by range L, consists of all vectors **w** ∈ W such that **w** = L(**v**) for some **v** ∈ V

- Defn - The linear transformation L: V → W is onto if range L = W

- Theorem - Let $L: V \to W$ be a linear transformation. Then range L is a subspace of W.
- Proof - Let $\mathbf{w}_1, \mathbf{w}_2 \in$ range L be arbitrary. Then $\mathbf{w}_1 = L(\mathbf{v}_1)$ and $\mathbf{w}_2 = L(\mathbf{v}_2)$ for some $\mathbf{v}_1, \mathbf{v}_2 \in V$. $\mathbf{w}_1 + \mathbf{w}_2 = L(\mathbf{v}_1) + L(\mathbf{v}_2) = L(\mathbf{v}_1 + \mathbf{v}_2)$. So $\mathbf{w}_1 + \mathbf{w}_2 \in$ range L

  Let $\mathbf{w} \in$ range L be arbitrary. Then $\mathbf{w} = L(\mathbf{v})$ for some $\mathbf{v} \in V$. Let c be an arbitrary real number. $c\mathbf{w} = cL(\mathbf{v}) = L(c\mathbf{v})$. So $c\mathbf{w} \in$ range L.

  $\therefore$ range L is a subspace of W

QED

Example

- Consider L: $P_2 \rightarrow R$ defined as

$$L\left(at^2+bt+c\right)=\int_0^1\left(at^2+bt+c\right)dt=\frac{1}{3}a+\frac{1}{2}b+c$$

For an arbitrary real number r, then $0t^2 + 0t + r$ maps to r. So L is onto and dim range L = 1

- Note that dim ker L + dim range L = dim P_2

Example

- Let L: $R^3 \rightarrow R^3$ be defined by

$$L\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

a) Is L onto?

b) Find basis for range L

c) Find ker L

d) Is L one to one?

Example (continued)

a) Let $\mathbf{w} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3$ be arbitrary. Find $\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in R^3$

such that L($\mathbf{v}$) = $\mathbf{w}$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 2 & b \\ 2 & 1 & 3 & c \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 1 & 1 & c-2a \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b-a \end{array}\right]$$

Solution exists only if
$c - b - a = 0$
So, L is not onto

Example (continued)

b) Range of L is the span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

Can show that the first two vectors are linearly independent and the third is the sum of the first two. Alternatively, could take the transpose of the matrix and put it into row echelon form to get a basis for the row space of the transpose. Either way, basis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Example (continued)

c) Kernel of L consists of all vectors $\mathbf{v} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ such that L(**v**) = **0**

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} a_1 \quad\quad + a_3 &= 0 \\ a_1 + a_2 + 2a_3 &= 0 \\ 2a_1 + a_2 + 3a_3 &= 0 \end{aligned}$$

Set $a_3 = r$, then $a_1 = -r$ and $a_2 = -r$.
So, all vectors in the kernel look like $r\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

Basis for ker L is $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$

Example (continued)

d) To see if L is one to one, let $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ with $\mathbf{v} \neq \mathbf{w}$. Is it possible to have $L(\mathbf{v}) = L(\mathbf{w})$?

$$L(\mathbf{v}) = L(\mathbf{w}) \Rightarrow L(\mathbf{v}) - L(\mathbf{w}) = \mathbf{0} \Rightarrow L(\mathbf{v} - \mathbf{w}) = \mathbf{0}$$

So, $\mathbf{v} - \mathbf{w} \in \ker L$ (null space of the matrix) and

$$\mathbf{v} - \mathbf{w} = r\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Since it is possible to have $L(\mathbf{v}) = L(\mathbf{w})$ when $\mathbf{v} \neq \mathbf{w}$, L is not one to one

- Note that dim ker L + dim range L = dim domain L

- Theorem - If L: V $\rightarrow$ W is a linear transformation of an n-dimensional vector space V into a vector space W, then dim ker L + dim range L = dim V
- Proof - Let k = dim ker L. Then $0 \leq k \leq n$. Consider three cases: (1) $k = n$, (2) $1 \leq k < n$, and (3) $k = 0$

  Case 1 - $k = n$. Since ker L is a subspace of V and dim ker L = dim V, every basis for ker L is a basis for V. Since a vector space equals the span of its set of basis vectors, ker L = V. Now, $L(\mathbf{v}) = \mathbf{0}_W$ for all $\mathbf{v} \in V$. Consequently, range L = $\{ \mathbf{0}_W \}$ and dim range L = 0

- Proof (continued)

  Case 2 - $1 \leq k < n$. Show that dim range L = $n - k$.
  Let { $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ } be a basis for ker L. This is a linearly independent set in V and can be extended to a basis S = { $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n$ } for V.
  Strategy is to show that the set of vectors

  $$T = \{ L(\mathbf{v}_{k+1}), L(\mathbf{v}_{k+2}), \ldots, L(\mathbf{v}_n) \}$$

  is a basis for range L.

  Specifically, need to show

  a) T spans range L

  b) T is linearly independent

- Proof (continued)

  a) Let $\mathbf{w} \in$ range L be arbitrary. There exists a $\mathbf{v} \in$ V such that L($\mathbf{v}$) = $\mathbf{w}$. Express $\mathbf{v}$ in terms of the basis S. $\mathbf{v} = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_n\mathbf{v}_n$. Then

$$
\begin{aligned}
\mathrm{L}(\mathbf{v}) &= \mathrm{L}(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k + \alpha_{k+1}\mathbf{v}_{k+1} + \cdots + \alpha_n\mathbf{v}_n) \\
&= \mathrm{L}(\alpha_1\mathbf{v}_1) + \mathrm{L}(\alpha_2\mathbf{v}_2) + \cdots + \mathrm{L}(\alpha_k\mathbf{v}_k) + \mathrm{L}(\alpha_{k+1}\mathbf{v}_{k+1}) + \cdots + \mathrm{L}(\alpha_n\mathbf{v}_n) \\
&= \underbrace{\alpha_1\mathrm{L}(\mathbf{v}_1) + \alpha_2\mathrm{L}(\mathbf{v}_2) + \cdots + \alpha_k\mathrm{L}(\mathbf{v}_k)}_{\substack{=\mathbf{0} \\ \text{since } \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in \ker \mathrm{L}}} + \alpha_{k+1}\mathrm{L}(\mathbf{v}_{k+1}) + \cdots + \alpha_n\mathrm{L}(\mathbf{v}_n)
\end{aligned}
$$

So $\mathbf{w} = \alpha_{k+1}\mathrm{L}(\mathbf{v}_{k+1}) + \alpha_{k+2}\mathrm{L}(\mathbf{v}_{k+2}) + \cdots + \alpha_n\mathrm{L}(\mathbf{v}_n)$
and T spans range L

- <u>Proof</u> (continued)

b) To show that T is linearly independent, consider

$$\mathbf{0}_W = a_{k+1}L(\mathbf{v}_{k+1}) + a_{k+2}L(\mathbf{v}_{k+2}) + \cdots + a_n L(\mathbf{v}_n)$$

$$= L(a_{k+1}\mathbf{v}_{k+1}) + L(a_{k+2}\mathbf{v}_{k+2}) + \cdots + L(a_n\mathbf{v}_n)$$

$$= L(a_{k+1}\mathbf{v}_{k+1} + a_{k+2}\mathbf{v}_{k+2} + \cdots + a_n\mathbf{v}_n)$$

So $a_{k+1}\mathbf{v}_{k+1} + a_{k+2}\mathbf{v}_{k+2} + \cdots + a_n\mathbf{v}_n \in \ker L$ and can be written as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$

$$a_{k+1}\mathbf{v}_{k+1} + \cdots + a_n\mathbf{v}_n = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k$$

$$\mathbf{0} = b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + \cdots + b_k\mathbf{v}_k - a_{k+1}\mathbf{v}_{k+1} - \cdots - a_n\mathbf{v}_n$$

Since S is linearly independent, the *a* and *b* values are all zero. So T is linearly independent

- Proof (continued)

  Since T is a basis for range L, dim range L = $n - k$
  So dim V = dim ker L + dim range L

  Case 3 - $k = 0$. Since dim ker L = 0, ker L has no basis. Let S = { $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ } be a basis for V. Let T = { $L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_n)$ }. By an argument similar to Case 2, T is a basis for range L. So,

  dim range L = n = dim V

QED

- Corollary - Let $L: V \rightarrow W$ and let dim V = dim W. Then

  a) If L is one to one, then it is onto

  b) If L is onto, then it is one to one
- Defn - A linear transformation $L: V \rightarrow W$ is invertible if there exists a function $L^{-1}: W \rightarrow V$ such that $L \circ L^{-1} = I_W$ and $L^{-1} \circ L = I_V$

- Theorem - A linear transformation $L: V \to W$ is invertible if and only if L is one to one and onto. Also, L^{-1} is a linear transformation and $(L^{-1})^{-1} = L$
- Proof - $\Rightarrow$⌋ Let L be invertible. First show that L is one to one. Suppose that $L(\mathbf{v}_1) = L(\mathbf{v}_2)$ for some $\mathbf{v}_1, \mathbf{v}_2 \in V$. Then $L^{-1}(L(\mathbf{v}_1)) = L^{-1}(L(\mathbf{v}_2))$, implying $\mathbf{v}_1 = \mathbf{v}_2$. So, L is one to one. Now show L is onto. Let $\mathbf{w} \in W$ be arbitrary. L is invertible, so L^{-1} exists. $\mathbf{v} = L^{-1}(\mathbf{w}) \in V$. Then $L(\mathbf{v}) = \mathbf{w}$ and L is onto.

  $\Leftarrow$⌋ Let L be one to one and onto. By function theory, the inverse function L^{-1} exists.

- Proof (continued)

Now show that L^{-1} is a linear transformation.

a) Let $\mathbf{w}_1, \mathbf{w}_2 \in W$ be arbitrary. Show that

$$L^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = L^{-1}(\mathbf{w}_1) + L^{-1}(\mathbf{w}_2).$$

Since L is onto, there exist $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $\mathbf{w}_1 = L(\mathbf{v}_1)$ and $\mathbf{w}_2 = L(\mathbf{v}_2)$. Need to show that

$$L^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v}_1 + \mathbf{v}_2.$$

Since L is linear,

$$L(\mathbf{v}_1+\mathbf{v}_2) = L(\mathbf{v}_1) + L(\mathbf{v}_2) = \mathbf{w}_1 + \mathbf{w}_2.$$

So $L^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{v}_1 + \mathbf{v}_2$

- Proof (continued)

  b) Let $\mathbf{w} \in$ W and $c \neq 0$ be an arbitrary real . Show that $L^{-1}(c\mathbf{w}) = cL^{-1}(\mathbf{w})$. Since $c\mathbf{w} \in$ W, there exists $\mathbf{v} \in$ V such that $L(\mathbf{v}) = c\mathbf{w}$. Since L is linear, $L((1/c)\mathbf{v}) = \mathbf{w}$. Then

  $$L^{-1}(\mathbf{w}) = (1/c)\mathbf{v} = (1/c)L^{-1}(c\mathbf{w}).$$

  So $L^{-1}(c\mathbf{w}) = cL^{-1}(\mathbf{w})$.

  Thus L^{-1} is a linear transformation.

QED

- Theorem - A linear transformation L: V $\rightarrow$ W is one to one if and only if the image of every linearly independent set of vectors in V is a linearly independent set of vectors in W
- Proof - Let S = { $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ } be a linearly independent set of vectors in V and let T = { $L(\mathbf{v}_1), L(\mathbf{v}_2), \ldots, L(\mathbf{v}_k)$ }

 $\Rightarrow\rfloor$ Let L be one to one. Consider

$$a_1 L(\mathbf{v}_1) + a_2 L(\mathbf{v}_2) + \cdots + a_k L(\mathbf{v}_k) = \mathbf{0}_W$$

 Need to argue that $a_1 = a_2 = \cdots = a_k = 0$

- Proof (continued)

$a_1\mathrm{L}(\mathbf{v}_1) + a_2\mathrm{L}(\mathbf{v}_2) + \cdots + a_k\mathrm{L}(\mathbf{v}_k) =$

$\mathrm{L}(a_1\mathbf{v}_1) + \mathrm{L}(a_2\mathbf{v}_2) + \cdots + \mathrm{L}(a_k\mathbf{v}_k) =$

$\mathrm{L}(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k) = \mathbf{0}_\mathrm{W}$

Since L is one to one $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}_\mathrm{V}$.

Since S is linearly independent,

$a_1 = a_2 = \cdots = a_k = 0$. So T is linearly independent

$\Leftarrow\rfloor$ Let the image of every set of linearly independent vectors in V be an independent set of vectors in W. Let $\mathbf{u}, \mathbf{v} \in \mathrm{V}$ with $\mathbf{u} \neq \mathbf{v}$. Need to show that $\mathrm{L}(\mathbf{u}) \neq \mathrm{L}(\mathbf{v})$. Suppose $\mathrm{L}(\mathbf{u}) = \mathrm{L}(\mathbf{v})$.

- <u>Proof</u> (continued)

  Let { $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_n$ } be a basis for V. Can express **u** and **v** as

  $\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$

  $\mathbf{v} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \cdots + b_n\mathbf{u}_n$

  $\mathrm{L}(\mathbf{u}) = a_1\mathrm{L}(\mathbf{u}_1) + a_2\mathrm{L}(\mathbf{u}_2) + \cdots + a_n\mathrm{L}(\mathbf{u}_n)$

  $\mathrm{L}(\mathbf{v}) = b_1\mathrm{L}(\mathbf{u}_1) + b_2\mathrm{L}(\mathbf{u}_2) + \cdots + b_n\mathrm{L}(\mathbf{u}_n)$

  $\mathbf{0}_\mathrm{W} = \mathrm{L}(\mathbf{u}) - \mathrm{L}(\mathbf{v})$

  $= (a_1-b_1)\mathrm{L}(\mathbf{u}_1) + (a_2-b_2)\mathrm{L}(\mathbf{u}_2) + \cdots + (a_n-b_n)\mathrm{L}(\mathbf{u}_n)$

  By hypothesis, T is linearly independent. So $a_1 = b_1$, $a_2 = b_2$, … , $a_n = b_n$. So, **u** = **v**, which is a contradiction. Thus L(**u**) ≠ L(**v**).

  QED

Topics

- Preliminaries
- Definition and Examples
- Kernel and Range of a Linear Transformation
- ***Matrix of a Linear Transformation***
- Vector Spaces of Matrices and Linear Transformations
- Similarity
- Homogeneous Coordinates

- Theorem - Let L: V $\longrightarrow$ W be a linear transformation of an n-dimensional vector space V into an m-dimensional vector space W ($n \neq 0$, $m \neq 0$) and let S = { $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ } be an ordered basis for V and T = { $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ } be an ordered basis for W. Then the $m \times n$ matrix $\mathbf{A}$ whose jth column is the coordinate vector $[L(\mathbf{v}_j)]_T$ of $L(\mathbf{v}_j)$ with respect to T has the following property: If $\mathbf{y} = L(\mathbf{x})$ for some $\mathbf{x} \in V$, then $[\mathbf{y}]_T = \mathbf{A}[\mathbf{x}]_S$. Also, $\mathbf{A}$ is unique.

- Proof - Consider L($\mathbf{v}_j$) for $1 \le j \le n$. L($\mathbf{v}_j$) $\in$ W, so it can be expanded in terms of T

$$\mathrm{L}\left(\mathbf{v}_j\right) = c_{1j}\mathbf{w}_1 + c_{2j}\mathbf{w}_2 + \cdots + c_{mj}\mathbf{w}_m \quad \Rightarrow \quad \left[\mathrm{L}\left(\mathbf{v}_j\right)\right]_{\mathrm{T}} = \begin{bmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{mj} \end{bmatrix}$$

Define **A** as the matrix whose jth column is $[\,\mathrm{L}(\mathbf{v}_j)\,]_{\mathrm{T}}$
Let $\mathbf{x} \in$ V be arbitrary and let $\mathbf{y} = \mathrm{L}(\mathbf{x})$.

$$\left[\mathbf{x}\right]_{\mathrm{S}} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad \left[\mathbf{y}\right]_{\mathrm{T}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

- Proof (continued)

$$\begin{aligned} \mathrm{L}(\mathbf{x}) &= \mathrm{L}(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n) \\ &= a_1\,\mathrm{L}(\mathbf{v}_1) + a_2\,\mathrm{L}(\mathbf{v}_2) + \cdots + a_n\,\mathrm{L}(\mathbf{v}_n) \end{aligned}$$

$$\mathbf{y} = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + \cdots + b_m\mathbf{w}_m = \mathrm{L}(\mathbf{x})$$

- Proof (continued)

$$
\begin{aligned}
L(\mathbf{x}) = {} & a_1\left(c_{11}\mathbf{w}_1 + c_{21}\mathbf{w}_2 + \cdots + c_{m1}\mathbf{w}_m\right) + \\
& a_2\left(c_{12}\mathbf{w}_1 + c_{22}\mathbf{w}_2 + \cdots + c_{m2}\mathbf{w}_m\right) + \\
& \vdots \\
& a_n\left(c_{1n}\mathbf{w}_1 + c_{2n}\mathbf{w}_2 + \cdots + c_{mn}\mathbf{w}_m\right) \\
= {} & \left(a_1c_{11} + a_2c_{12} + \cdots + a_nc_{1n}\right)\mathbf{w}_1 + \\
& \left(a_1c_{21} + a_2c_{22} + \cdots + a_nc_{2n}\right)\mathbf{w}_2 + \\
& \vdots \\
& \left(a_1c_{m1} + a_2c_{m2} + \cdots + a_nc_{mn}\right)\mathbf{w}_m
\end{aligned}
$$

- Proof (continued)

  Comparing coefficients of the **w** vectors gives

$$
\begin{aligned}
b_1 &= a_1c_{11} + a_2c_{12} + \cdots + a_nc_{1n} \\
b_2 &= a_1c_{21} + a_2c_{22} + \cdots + a_nc_{2n} \\
&\vdots \\
b_m &= a_1c_{m1} + a_2c_{m2} + \cdots + a_nc_{mn}
\end{aligned}
$$

- Proof (continued)

  In matrix form

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

or $[\mathbf{y}]_T = \mathbf{A}[\mathbf{x}]_S$. So the effect of L may be accomplished by letting **A** operate on the coordinate vector of **x**

- Proof (continued)

  To show uniqueness, suppose there is a second matrix $\mathbf{A}^* = \left[c_{ij}^* \right]$, which has the same properties as $\mathbf{A}$ but $\mathbf{A}^* \neq \mathbf{A}$. Since $\mathbf{A}^* \neq \mathbf{A}$ some of the elements of $\mathbf{A}^*$ are different from the elements of $\mathbf{A}$. So, suppose some elements in column k are different. $[L(\mathbf{v}_k)]_T = \mathbf{A} [\mathbf{v}_k]_S$ and $[L(\mathbf{v}_k)]_T = \mathbf{A}^* [\mathbf{v}_k]_S$. So $\mathbf{A} [\mathbf{v}_k]_S = \mathbf{A}^* [\mathbf{v}_k]_S$

- <u>Proof</u> (continued)

$$[\mathbf{v}_k]_S = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \leftarrow k \text{ th position}$$

$\mathbf{A}\,[\mathbf{v}_k]_S$ is just the kth column of $\mathbf{A}$

$\mathbf{A}^*\,[\mathbf{v}_k]_S$ is just the kth column of $\mathbf{A}^*$

Since $\mathbf{A}\,[\mathbf{v}_k]_S = \mathbf{A}^*\,[\mathbf{v}_k]_S$, $\mathbf{A} = \mathbf{A}^*$ and $\mathbf{A}$ is unique

QED

Comments

- Matrix **A** is called the representation of L with respect to the ordered bases S and T
- If L: V $\rightarrow$ V, can have two bases, S and T, and get a representation of L with respect to S and T. If S = T, then L has a representation with respect to S

Example

- Let L: $P_2 \to P_1$ be defined by $L\big(p(t)\big) = p'(t)$ and let $S = \{ t^2, t, 1 \}$ and $T = \{ t, 1 \}$ be bases for P_2 and P_1 respectively.

a) Find the matrix **A** associated with L

b) If $p(t) = 5t^2 - 3t + 2$, compute $L(p(t))$ using **A**

Example (continued)

a) Let $\mathbf{v}_1 = t^2$, $\mathbf{v}_2 = t$, $\mathbf{v}_3 = 1$, $\mathbf{w}_1 = t$, $\mathbf{w}_2 = 1$

$$L(\mathbf{v}_1) = 2t = 2\mathbf{w}_1 \qquad \Rightarrow \left[L(\mathbf{v}_1)\right]_T = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$L(\mathbf{v}_2) = 1 = \mathbf{w}_2 \qquad \Rightarrow \left[L(\mathbf{v}_2)\right]_T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L(\mathbf{v}_3) = \mathbf{0} = 0\mathbf{w}_1 + 0\mathbf{w}_2 \Rightarrow \left[L(\mathbf{v}_3)\right]_T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Example (continued)

b) $L(p(t)) = 10t - 3$

$$\left[p(t)\right]_S = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} \qquad \mathbf{A}\left[p(t)\right]_S = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$$

$$\Rightarrow L\big(p(t)\big) = 10\mathbf{w}_1 + (-3)\mathbf{w}_2 = 10t + (-3)1 = 10t - 3$$

Example

- Let L: $R^3 \to R^2$ be defined as $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\text{Let } S = \left\{ \underset{\mathbf{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\mathbf{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}, \underset{\mathbf{v}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\} \qquad T = \left\{ \underset{\mathbf{w}_1}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}, \underset{\mathbf{w}_2}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} \right\}$$

Example (continued)

$$L\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2\\3\end{bmatrix}=3\begin{bmatrix}1\\2\end{bmatrix}-\begin{bmatrix}1\\3\end{bmatrix} \Rightarrow \left[L(\mathbf{v}_1)\right]_T=\begin{bmatrix}3\\-1\end{bmatrix}$$

$$L\left(\begin{bmatrix}0\\1\\1\end{bmatrix}\right)=\begin{bmatrix}2\\5\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}1\\3\end{bmatrix} \Rightarrow \left[L(\mathbf{v}_2)\right]_T=\begin{bmatrix}1\\1\end{bmatrix}$$

$$L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\3\end{bmatrix}=0\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}1\\3\end{bmatrix} \Rightarrow \left[L(\mathbf{v}_3)\right]_T=\begin{bmatrix}0\\1\end{bmatrix}$$

$$\mathbf{A}=\begin{bmatrix}3&1&0\\-1&1&1\end{bmatrix}$$

- Preliminaries
- Definition and Examples
- Kernel and Range of a Linear Transformation
- Matrix of a Linear Transformation
- ***Vector Spaces of Matrices and Linear Transformations***
- Similarity
- Homogeneous Coordinates

Comments

- Have shown that the set of $m \times n$ matrices ${}_m\mathrm{R}_n$ is a vector space
- Want to show that the set U of all linear transformations from V to W forms a vector space
- Need to define the operations for the vector space

  a) sum of two linear transformations

  b) scalar times a linear transformation

- Defn - Let L_1: V → W and L_2: V → W. Define the sum of L_1 and L_2, $L = L_1 \oplus L_2$ as follows

$$L(\mathbf{x}) = L_1(\mathbf{x}) + L_2(\mathbf{x}) \quad \forall \mathbf{x} \in V$$

Note: This is just the definition of the sum of two functions

- Defn - Let L: V → W be a linear transformation and let *c* be real. Define the scalar multiple of *c* and L, $c \circ L$, as

$$(c \circ L)(\mathbf{x}) = c\,L(\mathbf{x}) \quad \forall \mathbf{x} \in V$$

- Verification of vector space properties for U, the set of all linear transformations from V to W with the operations $\oplus$ and $\circ$, is straightforward except for

  a) zero vector - define $\mathbf{0}(\mathbf{x}) = \mathbf{0}_W \quad \forall\, \mathbf{x} \in V$

  b) additive inverse - Let $L \in U$, define –L as $(-1) \circ L$

- Since U is a vector space, what is its dimension? Answer question via a basis

- Defn - Let S = { L_1, L_2, …, L_k } be a set of linear transformations. S is linearly dependent if there exist scalars a_1, a_2, …, a_k, not all zero, such that

$$\left(a_1 \circ L_1\right) \oplus \left(a_2 \circ L_2\right) \oplus \cdots \oplus \left(a_k \circ L_k\right) = \mathbf{0}$$

where **0** is the zero linear transformation

Example

- Consider linear transformations L_1, L_2, L_3 from R_2 to R_3 defined as

  $L_1([x_1, x_2]) = [\, x_1 + x_2 , 2x_1 , x_2 \,]$

  $L_2([x_1, x_2]) = [\, x_2 - x_1 , 2x_1 + x_2 , x_1 \,]$

  $L_3([x_1, x_2]) = [\, 3x_1 , -2x_2 , x_1 + 2x_2 \,]$

  Determine if $S = \{ L_1 , L_2 , L_3 \}$ is linearly dependent

  Suppose $\left(a_1 \circ L_1\right) \oplus \left(a_2 \circ L_2\right) \oplus \left(a_3 \circ L_3\right) = \mathbf{0}$ where a_1, a_2, a_3 are real. This equation means

$$a_1 L_1(\mathbf{x}) + a_2 L_2(\mathbf{x}) + a_3 L_3(\mathbf{x}) = \mathbf{0}_{R_3} \quad \forall \mathbf{x} \in R_2, \mathbf{x} = \left[x_1, x_2 \right]$$

Example (continued)

$$\mathbf{0}_{\mathrm{R}_3} = \left[0,0,0\right]$$

$$= a_1 \mathrm{L}_1(\mathbf{x}) + a_2 \mathrm{L}_2(\mathbf{x}) + a_3 \mathrm{L}_3(\mathbf{x})$$

$$= a_1\left[x_1 + x_2, 2x_1, x_2\right] + a_2\left[x_2 - x_1, 2x_1 + x_2, x_1\right] +$$

$$a_3\left[3x_1, -2x_2, x_1 + 2x_2\right]$$

$$= \left[a_1\left(x_1 + x_2\right) + a_2\left(x_2 - x_1\right) + a_3\left(3x_1\right),\right.$$

$$a_1\left(2x_1\right) + a_2\left(2x_1 + x_2\right) + a_3\left(-2x_2\right),$$

$$\left. a_1 x_2 + a_2 x_1 + a_3\left(x_1 + 2x_2\right)\right]$$

Example (continued)

$$a_1(x_1+x_2)+a_2(x_2-x_1)+3a_3x_1=0$$

$$2a_1x_1+a_2(2x_1+x_2)-2a_3x_2=0$$

$$a_1x_2+a_2x_1+a_3(x_1+2x_2)=0$$

This must be true $\forall\ x_1, x_2$. Pick particular values

$x_1 = 1, x_2 = 0$

$$\begin{aligned} a_1 - a_2 + 3a_3 &= 0 \\ 2a_1 + 2a_2 &= 0 \\ a_2 + a_3 &= 0 \end{aligned}$$

The only solution is $a_1 = 0$, $a_2 = 0$, $a_3 = 0$. So S = { L_1 , L_2 , L_3 } is linearly independent

- Theorem - Let U be the set of all linear transformations of V into W where dim V = n and dim W = m, $n \neq 0$, $m \neq 0$, and operations in U are $\oplus$ and $\circ$. U is isomorphic to the vector space ${}_mR_n$ of all $m \times n$ matrices
- Proof - Strategy is to pick a basis for V and for W and map L to its matrix representation with respect to these bases. This gives a mapping from U to ${}_mR_n$. Need to show that the mapping

  1) is one to one

  2) is onto

  3) preserves vector operations

- Proof (continued)

  Let S = { $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ } be a basis for V and let T = { $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ } be a basis for W. Define a mapping M: U $\rightarrow$ ${}_mR_n$ as M(L) = matrix representing L with respect to S and T.

  1) Show M is one to one. Let $L_1, L_2 \in$ U with $L_1 \neq L_2$. Need to show $M(L_1) \neq M(L_2)$. Since $L_1 \neq L_2$, $\exists\, \mathbf{v} \in V$ such that $L_1(\mathbf{v}) \neq L_2(\mathbf{v})$. $\mathbf{v}$ can be expressed as a linear combination of elements of S. So, must have $L_1(\mathbf{v}_j) \neq L_2(\mathbf{v}_j)$ for some $1 \leq j \leq n$. The jth column of $M(L_1)$ is $[L_1(\mathbf{v}_j)]_T$. The jth column of $M(L_2)$ is $[L_2(\mathbf{v}_j)]_T$. Since $L_1(\mathbf{v}_j) \neq L_2(\mathbf{v}_j)$, $[L_1(\mathbf{v}_j)]_T \neq [L_2(\mathbf{v}_j)]_T$. So $M(L_1) \neq M(L_2)$.

- Proof (continued)

2) Show M is onto. Let **A** = [a_{ij}] be an arbitrary $m \times n$ matrix. Define a linear transformation

L: V $\rightarrow$ W by $\mathrm{L}(\mathbf{v}_i) = \sum_{k=1}^{m} a_{ki}\mathbf{w}_k$ for $1 \le i \le n$

Note: it is sufficient to define L on S since for any $\mathbf{x} \in$ V, $\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$, then

$$\mathrm{L}(\mathbf{x}) = \sum_{i=1}^{n} c_i \mathrm{L}(\mathbf{v}_i)$$

L is a linear transformation and the matrix of L with respect to bases S and T is **A**. So M is onto

- <u>Proof</u> (continued)

3) Show that M preserves vector addition and scalar multiplication. Let $L_1, L_2 \in U$ be arbitrary. Let $M(L_1) = \mathbf{A} = [\, a_{ij} \,]$ and $M(L_2) = \mathbf{B} = [\, b_{ij} \,]$. First show that $M(L_1 \oplus L_2) = \mathbf{A} + \mathbf{B}$. The jth column of $M(L_1 \oplus L_2)$ is

$$\left[(L_1 \oplus L_2)(\mathbf{v}_j) \right]_T = \left[L_1(\mathbf{v}_j) + L_2(\mathbf{v}_j) \right]_T$$
$$= \left[L_1(\mathbf{v}_j) \right]_T + \left[L_2(\mathbf{v}_j) \right]_T$$

So jth column of $M(L_1 \oplus L_2)$ is sum of jth columns of $M(L_1) = \mathbf{A}$ and $M(L_2) = \mathbf{B}$. So $M(L_1 \oplus L_2) = \mathbf{A} + \mathbf{B}$

- Proof (continued)

  Consider scalar multiplication. Let $M(L) = \mathbf{A}$ and real c be arbitrary. The jth column of $M(c \circ L)$ is

$$\left[(c \circ L)(\mathbf{v}_j)\right]_T = \left[cL(\mathbf{v}_j)\right]_T = c\left[L(\mathbf{v}_j)\right]_T$$

So $M(c \circ L) = c\mathbf{A}$

$\therefore$ U and ${}_m R_n$ are isomorphic

QED

Vector Spaces of Matrices and Linear Transformations

- Corollary - dim U = mn

- Since linear transformations are just functions, can form composition of those functions. Following theorem shows that matrix of composition is simply related to matrices of individual transformations

Linear Algebra

Vector Spaces of Matrices and Linear Transformations

- Theorem - Let V_1, V_2, V_3 be vector spaces with $\dim V_1 = n$, $\dim V_2 = m$, $\dim V_3 = p$. Let $L_1: V_1 \to V_2$, $L_2: V_2 \to V_3$ be linear transformations. Let P, S, T be bases for V_1, V_2, V_3 respectively. Then $M(L_2 \circ L_1) = M(L_2)M(L_1)$ where $\circ$ is the composition of functions

- <u>Proof</u> **-** Let $M(L_1) = \mathbf{A}$, with respect to bases P and S. Let $M(L_2) = \mathbf{B}$, with respect to bases S and T. Let $\mathbf{x} \in V_1$ be arbitrary. Then $[L_1(\mathbf{x})]_S = \mathbf{A}[\mathbf{x}]_P$. For any $\mathbf{y} \in V_2$, $[L_2(\mathbf{y})]_T = \mathbf{B}[\mathbf{y}]_S$

$$\left[\left(L_2 \circ L_1\right)(\mathbf{x})\right]_T = \left[L_2\left(L_1(\mathbf{x})\right)\right]_T$$

$$= \mathbf{B}\left[L_1(\mathbf{x})\right]_S = \mathbf{B}\left(\mathbf{A}\left[\mathbf{x}\right]_P\right) = (\mathbf{BA})\left[\mathbf{x}\right]_P$$

Have proved that matrix of a linear transformation with respect to a particular basis is unique. So

$$M\left(L_2 \circ L_1\right) = \mathbf{AB} = M\left(L_2\right)M\left(L_1\right)$$

QED

Linear Algebra

Topics

- Preliminaries
- Definition and Examples
- Kernel and Range of a Linear Transformation
- Matrix of a Linear Transformation
- Vector Spaces of Matrices and Linear Transformations
- ***Similarity***
- Homogeneous Coordinates

Comments

- Let L: V $\rightarrow$ W be a linear transformation, where dim V = $n \neq 0$ and dim W = $m \neq 0$ and the spaces have ordered bases S = { $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ } for V and T = { $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ } for W. Have seen how to construct a matrix **A** that represents L with respect to these bases. Specifically, the jth column of **A** is $[L(\mathbf{v}_j)]_T$
- Also know that picking a different basis in either V or W gives a different matrix
- Since all of these matrices represent L, they ought to be related, i.e. we ought to be able to get one matrix from another

- Theorem - Let L: V $\longrightarrow$ W be a linear transformation, where dim V = $n \neq 0$ and dim W = $m \neq 0$. Let S = $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \}$ and $S' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ be ordered bases for V with transition matrix **P** from S$'$ to S. Let T = $\{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m \}$ and $T' = \{\mathbf{w}'_1, \mathbf{w}'_2, \ldots, \mathbf{w}'_m\}$ be ordered bases for W with transition matrix **Q** from T$'$ to T. If **A** is the representation of L with respect to S and T, then $\mathbf{Q}^{-1}\mathbf{AP}$ is the representation of L with respect to S$'$ and T$'$

- Proof - Recall the definition of the transition matrices:

$$[\mathbf{x}]_S = \mathbf{P}[\mathbf{x}]_{S'} \quad \forall \mathbf{x} \in V$$

$$[\mathbf{y}]_T = \mathbf{Q}[\mathbf{y}]_{T'} \quad \forall \mathbf{y} \in W$$

jth column of $\mathbf{P}$ is coordinate vector $[\mathbf{v}'_j]_S$ of $\mathbf{v}'_j$ with respect to S

jth column of $\mathbf{Q}$ is coordinate vector $[\mathbf{w}'_j]_T$ of $\mathbf{w}'_j$ with respect to T

- Proof (continued)

  Let **A** be the representation of L with respect to S and T, then $[L(\mathbf{x})]_T = \mathbf{A}[\mathbf{x}]_S$.

  Also,

  $$\left[L(\mathbf{x})\right]_T = \mathbf{Q}\left[L(\mathbf{x})\right]_{T'}, \quad \left[\mathbf{x}\right]_S = \mathbf{P}\left[\mathbf{x}\right]_{S'}$$

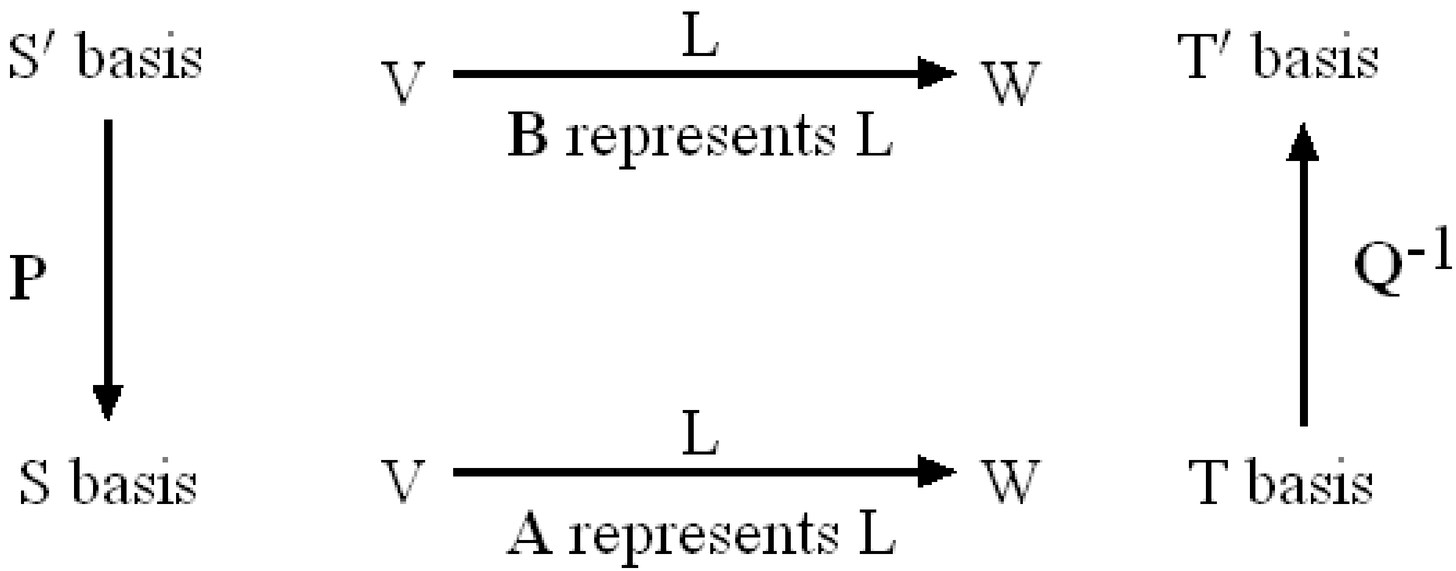
  $$\Rightarrow \mathbf{Q}\left[L(\mathbf{x})\right]_{T'} = \mathbf{AP}\left[\mathbf{x}\right]_{S'} \Rightarrow \left[L(\mathbf{x})\right]_{T'} = \mathbf{Q}^{-1}\mathbf{AP}\left[\mathbf{x}\right]_{S'}$$

  So $\mathbf{Q}^{-1}\mathbf{AP}$ is the representation of L with respect to S' and T'

QED

Comments

- Let $\mathbf{B} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{P}$. Can calculate the effect of L on $\mathbf{x} \in V$ two ways in terms of S′ and T′



- Note that **A** and **B** are equivalent matrices

Example

- Define L: $R^3 \to R^2$ by $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 \\ x_2 - x_3 \end{bmatrix}$

Bases for R^3 $S = \left\{ \underset{\mathbf{v}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}, \underset{\mathbf{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}, \underset{\mathbf{v}_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$ $S' = \left\{ \underset{\mathbf{v}'_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}, \underset{\mathbf{v}'_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}, \underset{\mathbf{v}'_3}{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} \right\}$

Bases for R^2 $T = \left\{ \underset{\mathbf{w}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{\mathbf{w}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$ $T' = \left\{ \underset{\mathbf{w}'_1}{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}, \underset{\mathbf{w}'_2}{\begin{bmatrix} 1 \\ 3 \end{bmatrix}} \right\}$

Example (continued)

$$\mathbf{A} = \left[\left[L(\mathbf{v}_1) \right]_T \quad \left[L(\mathbf{v}_2) \right]_T \quad \left[L(\mathbf{v}_3) \right]_T \right]$$

$$L(\mathbf{v}_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \left[L(\mathbf{v}_1) \right]_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ Since T is the natural basis

$$L(\mathbf{v}_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \left[L(\mathbf{v}_2) \right]_T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$ Since T is the natural basis

$$L(\mathbf{v}_3) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \left[L(\mathbf{v}_3) \right]_T = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$ Since T is the natural basis

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

Example (continued)

Calculate representation of L with respect to S' and T' two ways

Calculate **P** - Columns of **P** are $\left[\mathbf{v}_1'\right]_S, \quad \left[\mathbf{v}_2'\right]_S, \quad \left[\mathbf{v}_3'\right]_S$

Since S is a natural basis $\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

Calculate **Q** - Columns of **Q** are $\left[\mathbf{w}_1'\right]_T, \quad \left[\mathbf{w}_2'\right]_T$

Since T is a natural basis $\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$

Example (continued)

$$\mathbf{Q}^{-1} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$\mathbf{B} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 3/2 & 2 \\ 0 & -1/2 & -1 \end{bmatrix}$$

Example (continued)

Compute **B** directly

Columns of **B** are $\left[L(\mathbf{v}'_1)\right]_{T'},\quad \left[L(\mathbf{v}'_2)\right]_{T'},\quad \left[L(\mathbf{v}'_3)\right]_{T'}$

$$L(\mathbf{v}'_1) = \begin{bmatrix}1\\1\end{bmatrix} = 1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\3\end{bmatrix} \qquad \left[L(\mathbf{v}'_1)\right]_{T'} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$L(\mathbf{v}'_2) = \begin{bmatrix}1\\0\end{bmatrix} = \frac{3}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\3\end{bmatrix} \qquad \left[L(\mathbf{v}'_2)\right]_{T'} = \begin{bmatrix}3/2\\-1/2\end{bmatrix}$$

$$L(\mathbf{v}'_3) = \begin{bmatrix}1\\-1\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix} - 1\begin{bmatrix}1\\3\end{bmatrix} \qquad \left[L(\mathbf{v}'_3)\right]_{T'} = \begin{bmatrix}2\\-1\end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix}1 & 3/2 & 2\\0 & -1/2 & -1\end{bmatrix}$$

- Corollary - Let L: V $\rightarrow$ V be a linear operator on an *n*-dimensional vector space V. Let S = { $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ } and $S' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ be ordered bases for V, with **P** being the transition matrix from S' to S. If **A** is the representation of L with respect to S, then $\mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ is the representation of L with respect to S'

- Defn - The rank of L: V $\rightarrow$ W, notation rank L, is the rank of any matrix representing L

- Note: rank L is well defined since any two matrices representing L are equivalent and thus have the same rank

- <u>Theorem</u> - Let L: V $\longrightarrow$ W be a linear transformation. Then rank L = dim range L
- <u>Proof</u> - Let dim V = n, dim W = m and dim range L = r. We have proved a theorem that says dim ker L + dim range L = n. Then dim ker L = $n - r$. Let $\mathbf{v}_{r+1}, \mathbf{v}_{\mathrm{r+2}}, \ldots, \mathbf{v}_n$ be a basis for ker L. This can be extended to a basis S = $\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r , \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n \}$ for V.

  The vectors $\mathbf{w}_1 = \mathrm{L}(\mathbf{v}_1), \mathbf{w}_2 = \mathrm{L}(\mathbf{v}_2), \ldots, \mathbf{w}_r = \mathrm{L}(\mathbf{v}_r)$ span range L. Since there are r = dim range L of them, they form a basis for range L. This can be extended to a basis T = $\{ \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_r, \mathbf{w}_{r+1}, \ldots, \mathbf{w}_m \}$ for W.

- Proof (continued)

  Now let **A** be the matrix that represents L with respect to S and T. The columns of **A** are

  $$\left[\mathrm{L}(\mathbf{v}_i) \right]_{\mathrm{T}} = \left[\mathbf{w}_i \right] = \mathbf{e}_i \qquad i = 1, 2, \ldots, r$$

  $$\left[\mathrm{L}(\mathbf{v}_i) \right]_{\mathrm{T}} = \left[\mathbf{0}_{\mathrm{W}} \right]_{\mathrm{T}} = \mathbf{0}_{\mathrm{R}^m} \quad i = r+1, r+2, \ldots, n$$

  So $\mathbf{A} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$

  Thus rank L = rank **A** = r = dim range L

  QED

- Defn **-** If **A** and **B** are are $n \times n$ matrices, then **B** is similar to A if there is a nonsingular matrix **P** such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$

- Theorem - Let V be any n-dimensional vector space and let **A** and **B** be any $n \times n$ matrices. Then **A** and **B** are similar if and only if **A** and **B** represent the same linear transformation $L: V \rightarrow V$ with respect to two ordered bases for V.
- Proof - $\Rightarrow\rfloor$ Let **A** and **B** be similar. Then there is a nonsingular matrix $\mathbf{P} = [\ p_{ij}\ ]$ such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for V and define a linear transformation on V by $[L(\mathbf{x})]_S = \mathbf{A}[\mathbf{x}]_S$ for all $\mathbf{x}$ in V.

  Now define a new basis for V by taking appropriate linear combinations of vectors in S.

- Proof (continued) -

  Define a set of vectors T = { $\mathbf{w}_1$, $\mathbf{w}_2$, …, $\mathbf{w}_n$ } as $\mathbf{w}_j = \sum_{i=1}^{n} p_{ij}\mathbf{v}_i, \quad 1 \le j \le n,$ and show that T is a basis for V by showing that it is linearly independent and appealing to an earlier theorem that says that a set of *n* linearly independent vectors in an *n*-dimensional space is a basis.

  Consider

$$\begin{aligned}\mathbf{0} &= a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots + a_n\mathbf{w}_n \\ &= a_1\sum_{i=1}^{n} p_{i1}\mathbf{v}_i + a_2\sum_{i=1}^{n} p_{i2}\mathbf{v}_i + \cdots + a_n\sum_{i=1}^{n} p_{in}\mathbf{v}_i \\ &= \left(\sum_{j=1}^{n} p_{1j}a_j\right)\mathbf{v}_1 + \left(\sum_{j=1}^{n} p_{2j}a_j\right)\mathbf{v}_2 + \cdots + \left(\sum_{j=1}^{n} p_{nj}a_j\right)\mathbf{v}_n\end{aligned}$$

- Proof (continued) -

Since S is linearly independent $\left(\sum_{j=1}^{n} p_{ij} a_j\right) = 0, \quad 1 \le i \le n$ or equivalently $\mathbf{Pa} = \mathbf{0}$, where $\mathbf{a} = [\ a_1\ \ a_2\ \ \dots\ \ a_n\ ]^T$. Since $\mathbf{P}$ is nonsingular, the only solution is $\mathbf{a} = \mathbf{0}$. Thus T is linearly independent and is a basis for V.

The definition of T, $\mathbf{w}_j = \sum_{i=1}^{n} p_{ij} \mathbf{v}_i, \quad 1 \le j \le n$, implies that $\mathbf{P}$ is the transition matrix from T to S, i.e. $[\mathbf{y}]_S = \mathbf{P}[\mathbf{y}]_T$. Then, recalling the definition of L,

$$\left[\mathrm{L}(\mathbf{x})\right]_S = \mathbf{P}\left[\mathrm{L}(\mathbf{x})\right]_T \rightarrow \mathbf{A}\left[\mathbf{x}\right]_S = \mathbf{P}\left[\mathrm{L}(\mathbf{x})\right]_T \rightarrow$$

$$\left[\mathrm{L}(\mathbf{x})\right]_T = \mathbf{P}^{-1}\mathbf{A}\left[\mathbf{x}\right]_S = \mathbf{P}^{-1}\mathbf{AP}\left[\mathbf{x}\right]_T$$

So, the matrix of L with respect to T is $\mathbf{P}^{-1}\mathbf{AP} = \mathbf{B}$

- Proof (continued) -

  $\Leftarrow\rfloor$ By the preceding corollary, any two matrix representations of a linear transformation are similar.

  QED

- Theorem - If **A** and **B** are similar $n \times n$ matrices, then rank **A** = rank **B**.
- Proof - By the preceding theorem, **A** and **B** represent the same linear transformation L: $R^n \to R^n$ with respect to different bases. Since the rank of L is defined uniquely as the rank of any matrix representing it, rank **A** = rank L = rank **B**.

QED

Linear Algebra

Topics

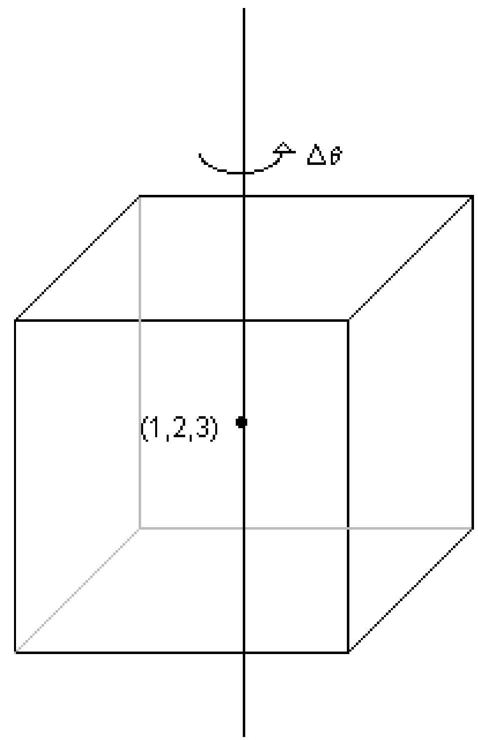
- Preliminaries
- Definition and Examples
- Kernel and Range of a Linear Transformation
- Matrix of a Linear Transformation
- Vector Spaces of Matrices and Linear Transformations
- Similarity
- ***Homogeneous Coordinates***

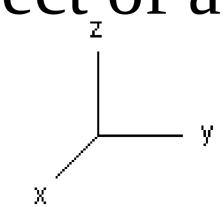
Homogeneous Coordinates

- A commonly used technique in computer graphics is the homogeneous coordinate transformation, which combines a sequence of translations, scalings and rotations into a single matrix which is then applied to the vertices of a geometric object.
  - This allows a compact representation of the combined operations that is easy to apply.
  - Also, the individual transformations can be implemented in hardware in a high-end workstation to permit the rotation of an object on the screen by means of turning a knob.

Motivating Example

- Rotate the cube about an axis parallel to the *z* axis passing through the point (1, 2, 3), by angles of $\Delta\theta$, $2\Delta\theta$, $3\Delta\theta$, etc. from its original position (i.e. successive rotations by angles of $\Delta\theta$). After each rotation, display the rotated cube to give the visual effect of a spinning cube.





Vertices at (1±1, 2±1, 3±1). Cube's faces are parallel to coordinate planes

Basic Coordinate Operations

- The application of any of these operations to a cube is accomplished by applying the operation to each vertex of the cube.
- Translation: The translation of (x, y, z) by the translation vector (t_x, t_y, t_z) yields the point whose coordinates are ($x + t_x, y + t_y, z + t_z$), i.e. (x, y, z) is moved to ($x + t_x, y + t_y, z + t_z$).
- Scaling: The scaling of (x, y, z) by the scaling vector (s_x, s_y, s_z), with $s_x > 0$, $s_y > 0$ and $s_z > 0$, yields the point with coordinates (xs_x, ys_y, zs_z), i.e. the point's coordinates are scaled by these amounts.

Basic Coordinate Operations

- Rotation: Simple rotations are done about the *x*-axis, *y*-axis and *z*-axis.
  - *x*-axis: If (x, y, z) is rotated about the *x*-axis by an angle θ to a new point (x', y', z'), the coordinates are related by

$$x' = x$$

$$y' = y \cos\theta - z \sin\theta$$

$$z' = y \sin\theta + z \cos\theta$$

  - *y*-axis: If (x, y, z) is rotated about the *y*-axis by an angle θ to a new point (x', y', z'), the coordinates are related by

$$x' = z \sin\theta + x \cos\theta$$

$$y' = y$$

$$z' = z \cos\theta - x \sin\theta$$

Basic Coordinate Operations

– z-axis: If (x, y, z) is rotated about the x-axis by an angle θ to a new point (x', y', z'), the coordinates are related by

$$x' = x \cos \theta - y \sin \theta$$

$$y' = x \sin \theta + y \cos \theta$$

$$z' = z$$

- In the motivating example, rotation of the cube by $\Delta\theta$ about a line through (1, 2, 3) parallel to the *z*-axis can be expressed in terms of the coordinate operations defined on the previous slides

  (1) Translate each vertex by (–1, –2, –3) to place the center of the cube at the origin and cause the axis of rotation to coincide with the *z*-axis.

  (2) Rotate the cube about the *z*-axis by an angle of $\Delta\theta$

  (3) Translate the rotated cube by (1, 2, 3) to put it back in position.

- The three steps above will perform the rotation and successive applications of the process will perform subsequent rotations.

Matrix Procedure

- Represent the point (x, y, z) as a 4 x 1 matrix $\mathbf{X} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$

- Translation: Translation by (t_x, t_y, t_z) can be accomplished as

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix} \Rightarrow \mathbf{T}\left(t_x, t_y, t_z\right)\mathbf{X} = \mathbf{X}'$$

Matrix Procedure

- Scaling: Scaling by (s_x, s_y, s_z) can be accomplished as

$$\begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} xs_x \\ ys_y \\ zs_z \\ 1 \end{bmatrix} \Rightarrow \mathbf{S}\left(s_x, s_y, s_z\right)\mathbf{X} = \mathbf{X}'$$

Matrix Procedure

- Rotation: Rotation about the x-axis by θ can be accomplished as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y\cos\theta - z\sin\theta \\ y\sin\theta + z\cos\theta \\ 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{R}_x(\theta)\mathbf{X} = \mathbf{X}'$$

Matrix Procedure

- Rotation: Rotation about the y-axis by θ can be accomplished as

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta + z\sin\theta \\ y \\ -x\sin\theta + z\cos\theta \\ 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{R}_y\left(\theta\right)\mathbf{X} = \mathbf{X}'$$

Matrix Procedure

- Rotation: Rotation about the *z*-axis by θ can be accomplished as

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \\ z \\ 1 \end{bmatrix}$$

$$\Rightarrow \mathbf{R}_z(\theta)\mathbf{X} = \mathbf{X}'$$

Matrix Procedure

- Inverses of the matrices are easy to compute

$$\mathbf{T}^{-1}\left(t_x, t_y, t_z\right) = \mathbf{T}\left(-t_x, -t_y, -t_z\right)$$

$$\mathbf{S}^{-1}\left(s_x, s_y, s_z\right) = \mathbf{S}\left(1/s_x, 1/s_y, 1/s_z\right)$$

$$\mathbf{R}_x^{-1}\left(\theta\right) = \mathbf{R}_x\left(-\theta\right)$$

$$\mathbf{R}_y^{-1}\left(\theta\right) = \mathbf{R}_y\left(-\theta\right)$$

$$\mathbf{R}_z^{-1}\left(\theta\right) = \mathbf{R}_z\left(-\theta\right)$$

Matrix Procedure

- Note that any sequence of coordinate operations may be performed by multiplying by the appropriate matrices
- The sequence of operations may be inverted by multiplying by the inverse matrices in reverse order

Motivating Example (continued)

- The operations in the example can be accomplished as

  (1) Translate by $(-1, -2, -3) \rightarrow \mathbf{T}(-1, -2, -3)$

  (2) Rotate about the z-axis by $\Delta\theta \rightarrow \mathbf{R}_z(\Delta\theta)$

  (3) Translate by $(1, 2, 3) \rightarrow \mathbf{T}(1, 2, 3)$

Motivating Example (continued)

- Define $\mathbf{M}(\Delta\theta)$ as

$$\mathbf{M}(\Delta\theta) = \mathbf{T}(1,2,3)\mathbf{R}_z(\Delta\theta)\mathbf{T}(-1,-2,-3)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\Delta\theta) & -\sin(\Delta\theta) & 0 & 0 \\ \sin(\Delta\theta) & \cos(\Delta\theta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Motivating Example (continued)

$$\mathbf{M}^2\left(\Delta\theta\right)$$

$$=\mathbf{T}\left(1,2,3\right)\mathbf{R}_z\left(\Delta\theta\right)\mathbf{T}\left(-1,-2,-3\right)\mathbf{T}\left(1,2,3\right)\mathbf{R}_z\left(\Delta\theta\right)\mathbf{T}\left(-1,-2,-3\right)$$

$$=\mathbf{T}\left(1,2,3\right)\mathbf{R}_z\left(\Delta\theta\right)\mathbf{R}_z\left(\Delta\theta\right)\mathbf{T}\left(-1,-2,-3\right)$$

$$=\mathbf{T}\left(1,2,3\right)\mathbf{R}_z^2\left(\Delta\theta\right)\mathbf{T}\left(-1,-2,-3\right)$$

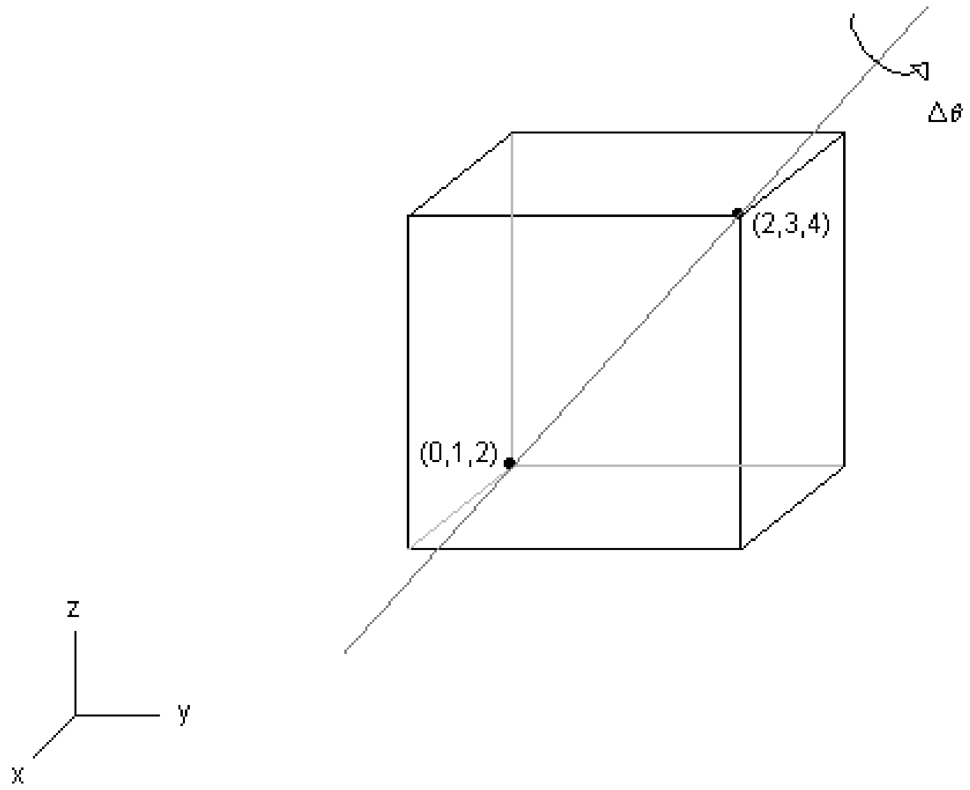
$$=\mathbf{T}\left(1,2,3\right)\mathbf{R}_z\left(2\Delta\theta\right)\mathbf{T}\left(-1,-2,-3\right)$$

By an inductive argument, can show

$$\mathbf{M}^n\left(\Delta\theta\right)=\mathbf{T}\left(1,2,3\right)\mathbf{R}_z\left(n\Delta\theta\right)\mathbf{T}\left(-1,-2,-3\right)=\mathbf{M}\left(n\Delta\theta\right)$$

Another Example

- Consider problem of rotating the cube by $\Delta\theta$ about an axis passing through the vertices (0, 1, 2) and (2, 3, 4)



Another Example (continued)

- Other than simplifying the discussion, there is nothing special about the points $(0, 1, 2)$ and $(2, 3, 4)$ or the fact that they are vertices of the cube. One could just as readily talk about rotation about an axis through the points (x_1, y_1, z_1) and (x_2, y_2, z_2)
- It does make a difference whether one considers the axis of rotation as going from (0, 1, 2) to (2, 3, 4) or from (2, 3, 4) to (0, 1, 2). The second choice reverses the sense of the rotation from the first choice.

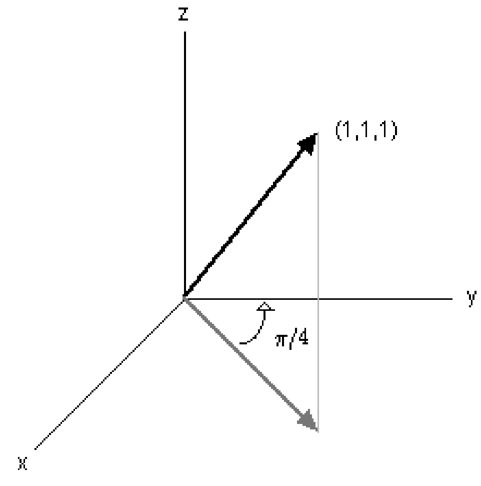
Another Example (continued)

- Procedure is

(1) Translate the cube by the translation vector (–1,–2,–3) This places the points which determine the rotation axis at (–1,–1,–1) and (1,1,1)

(2) Rotate the axis of rotation into the *z*-axis by the following steps

(a) Rotate by π/4 about the *z*-axis to put (1,1,1) and (–1,–1,–1) in the *yz*-plane



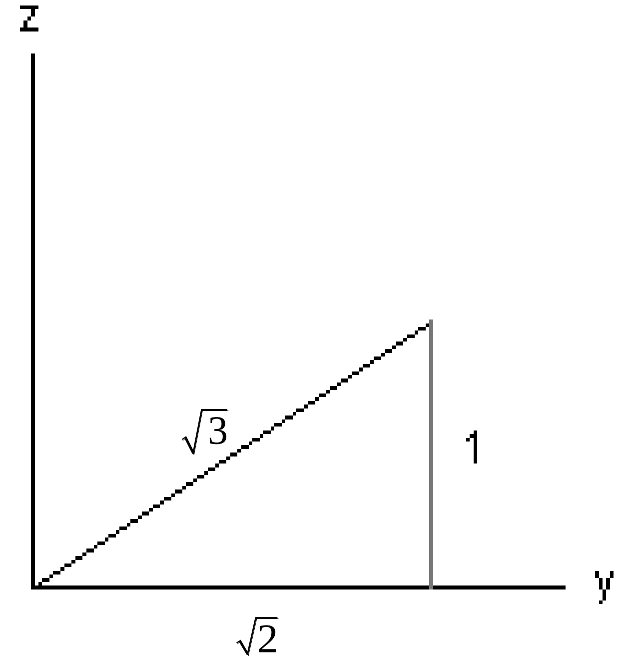
Another Example (continued)

(b) Rotate by arctan(√2) about the *x*-axis to put (1, 1, 1) and ((–1,–1,–1) onto the *z*-axis

(3) Rotate about the z axis by $\Delta\theta$

(4) Undo steps 1 and 2

- The matrix $\mathbf{M}(\Delta\theta)$ to do this is



$$\mathbf{M}(\Delta\theta) =$$

$$\mathbf{T}^{-1}(-1,-2,-3)\,\mathbf{R}_z^{-1}(\pi/4)\,\mathbf{R}_x^{-1}\left(\arctan\sqrt{2}\right)\mathbf{R}_z(\Delta\theta)\,\mathbf{R}_x\left(\arctan\sqrt{2}\right)\mathbf{R}_z(\pi/4)\,\mathbf{T}(-1,-2,-3)$$

$$= \mathbf{T}(1,2,3)\,\mathbf{R}_z(-\pi/4)\,\mathbf{R}_x\left(-\arctan\sqrt{2}\right)\mathbf{R}_z(\Delta\theta)\,\mathbf{R}_x\left(\arctan\sqrt{2}\right)\mathbf{R}_z(\pi/4)\,\mathbf{T}(-1,-2,-3)$$