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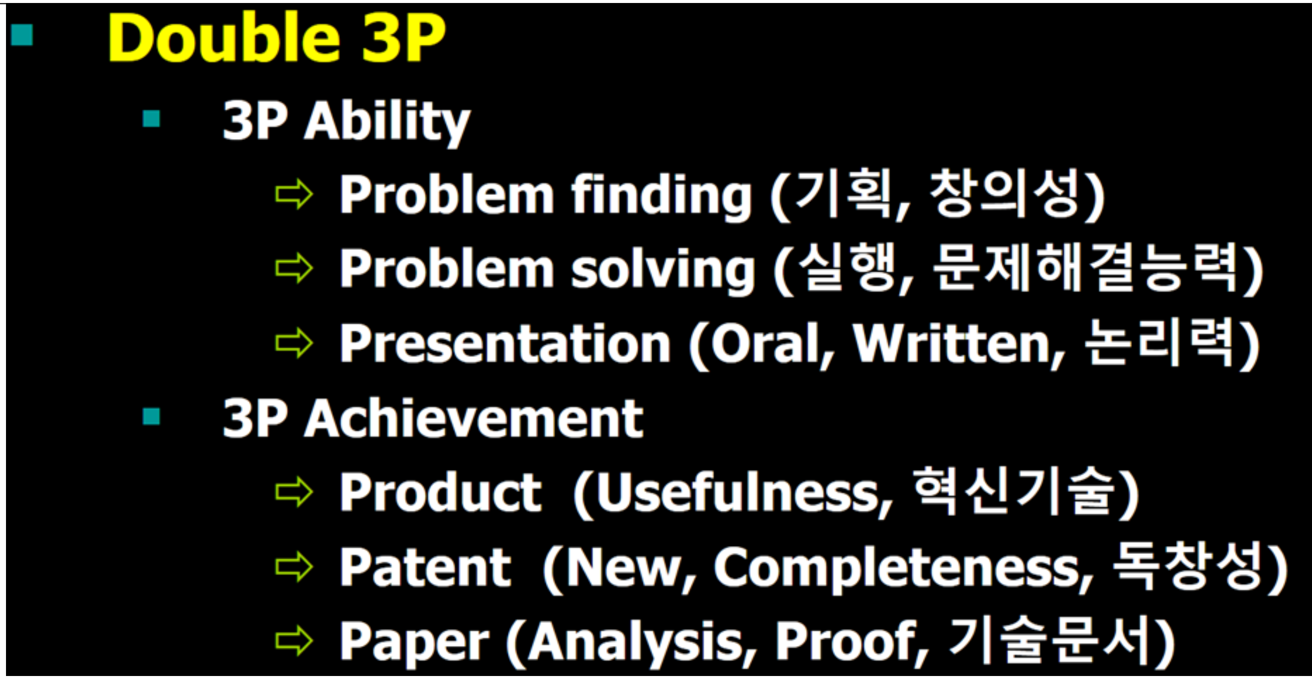

- Language, Tools, Knowledge, ...

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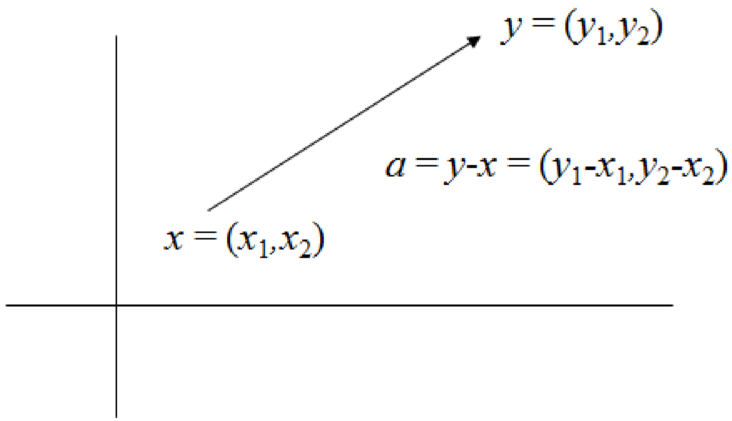
Chapter 1 Vector Spaces

mathematical spaces

- The physical space around us is called the 3-dimensional **Euclidean space**.
- It is commonly defined by real coordinates.
- A vector has a magnitude and a direction



- We will focus on linear **vector spaces**.
  - We will ”define” vectors and scalars using only sets and algebraic operations.
  - Forget about vectors you are accustomed to, with magnitude and direction, with arrow marks on the top, and with coordinates.
  - They are vectors in the Euclidean space and only an example, a special kind of vectors.
  - We want more general definitions that, to begin with, do not require magnitude, direction, or orthogonality.

- According to the new definition,
  - a continuous function can be a vector;
  - an infinite sequence can be a vector;
  - a matrix can be a vector.
  - So a vector will now be a more abstract, flexible thing than what you are used to.
  - So be ready to accept new concepts that seem at first strange, and you will feel comfortable with them later in this course.
- This will be useful for applications such as machine learning, least square approximation, regression, electric circuits, graph theory, and cryptography etc.

Field and vector space

- **field**: $(F, +, \cdot)$ such that

  1. $\forall a, b \in F$, $a + b$ and $a \cdot b$ are unique in F
  2. $\forall a, b, c \in F$, the following hold:

  F1 commutativity: $a + b = b + a$; $a \cdot b = b \cdot a$

  F2 associativity: $(a + b) + c = a + (b + c)$; $(a \cdot b) \cdot c = a \cdot (b \cdot c)$

  F3 identity:

  additive: $\exists 0 \in F$ such that $a + 0 = a$

  multiplicative: $\exists 1 \in F$ such that $a \cdot 1 = a$

F4 inverse:

additive: $\exists d \in F$ such that $a + d = 0$

multiplicative: $\exists e \in F$ such that, for $a \neq 0$, $a \cdot e = 1$

F5 distributivity: $a \cdot (b + c) = a \cdot b + a \cdot c$

- Commonly, the operations + and $\cdot$ become implicit, and the set F is called the field, ex, $1 + 1 = 0$ for binary(prime-2) field.
- An element of a field is called a **scalar**.
- The additive inverse d of a is commonly denoted by $-a$.
- The multiplicative inverse e of a is commonly denoted by a^{-1}.

- example: field
  - set of real numbers with the ordinary operations
  - set of rational numbers with the ordinary operations
  - set of complex numbers with the ordinary operations
  - A field with a finite number of elements are called a **finite field**.
  - For a prime number n, $\{0, 1, \cdots, n-1\}$ with modulo-n operations is a prime-n field.

- example: field
  - $\{0, 1\}$ with the modulo-2 operations: prime-2 field
  - $\{0, 1, 2, 3, 4\}$ with the modulo-5 operations: prime-5 field

prime-2 field

a	b	$+$	$\cdot$
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

a	$-a$	a^{-1}
0	0	
1	1	1

prime-5 field

a	$-a$	a^{-1}
0	0	
1	4	1
2	3	3
3	2	2
4	1	4

- How about the set of integers or the set of irrational numbers?
  $\rightarrow$ no multiplicative inverse, no multiplicative identity.

- **vector space** over a field $F : (V(F), +, \cdot)$ such that

1. $\forall a \in F, \forall x, y \in V$, $x + y$ and $a \cdot x = ax$ are unique in V.
2. $\forall a, b \in F, \forall x, y, z \in V$, the following hold:

VS1 commutativity: $x + y = y + x$

VS2 associativity: $(x + y) + z = x + (y + z)$

VS3 identity: $\exists \mathit{0} \in V$ such that $x + \mathit{0} = x$

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VS5 identity: $1x = x$

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- Commonly, the operations + and $\cdot$(and F) are defined, and the set $V(F)$(or V) is called the vector space.

- An element of a vector space is called a **vector**.
- The additive inverse u of x is commonly denoted by $-x$.
- Note that multiplication is always between a scalar and a vector, so the inverse is always additive.

example: vector space

- the set of n-tuples $F^n = \{(a_1, \cdots, a_n) : a_i \in F\}$
- the set of $m \times n$ matrices

$$M_{m\times n}(F) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} : a_{ij} \in F \right\}$$

- the set of functions $\mathcal{F}(S, F) = \{f : S \to F\}$
- the set of polynomials

$$P(F) = \{a_0 + a_1 x + \cdots + a_n x^n : n = 0, 1, 2, \cdots, a_i \in F\}$$

  - the set of infinite sequences $F^\infty = \{(a_1, a_2, a_3, \cdots) : a_i \in F\}$

- example: not a vector space

  - $(\mathbb{R}^2, +, \cdot)$ where + and $\cdot$ are defined by
    $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 - b_2); c(a_1, a_2) = (ca_1, ca_2)$
    not commutative [VS1], not associative [VS2]
  - $(\mathbb{R}^2, +, \cdot)$ where + and $\cdot$ are defined by
    $(a_1, a_2) + (b_1, b_2) = (0, a_2 + b_2); c(a_1, a_2) = (0, ca_2)$
    no identity [VS3], no inverse [VS4], $1(a_1, a_2) \neq (a_1, a_2)$ [VS5]
    lack of uniqueness of $+$

- What is your motivation to take the linear algebra course?
- Explain why the set of integers and the set of irrational numbers can not be a Field?
- Determine whether the set of functions can be a vector space or not? Why?

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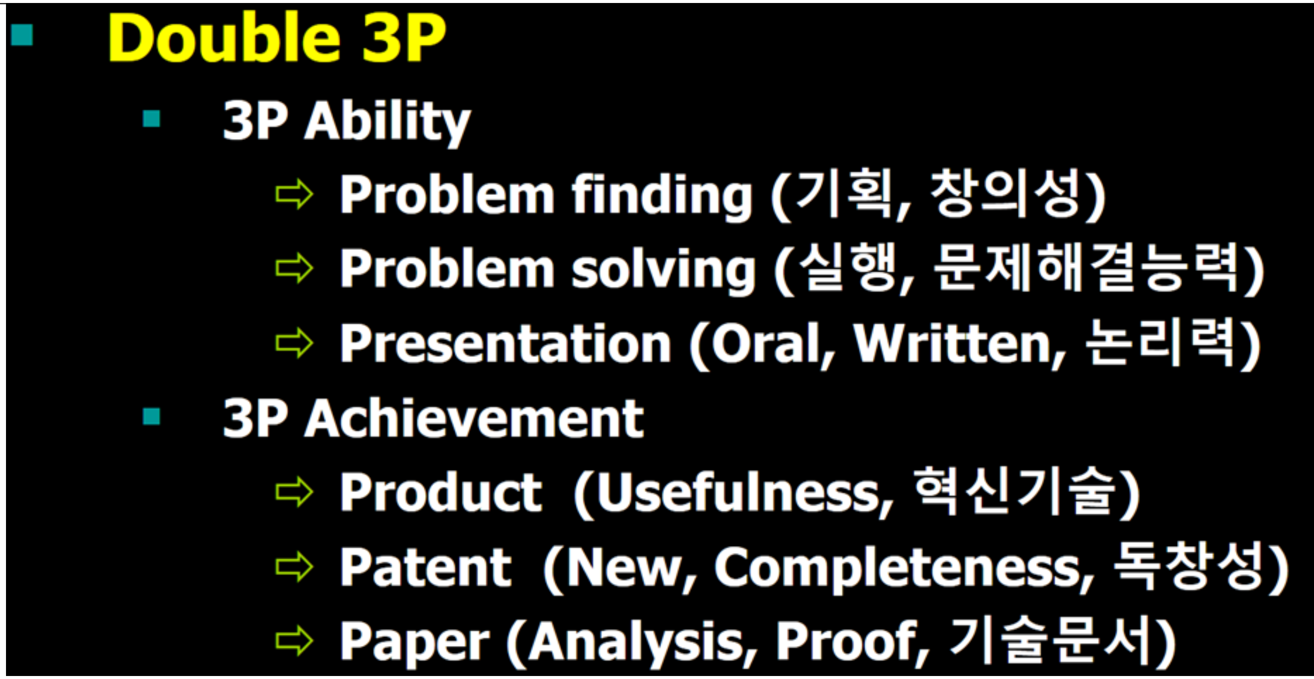

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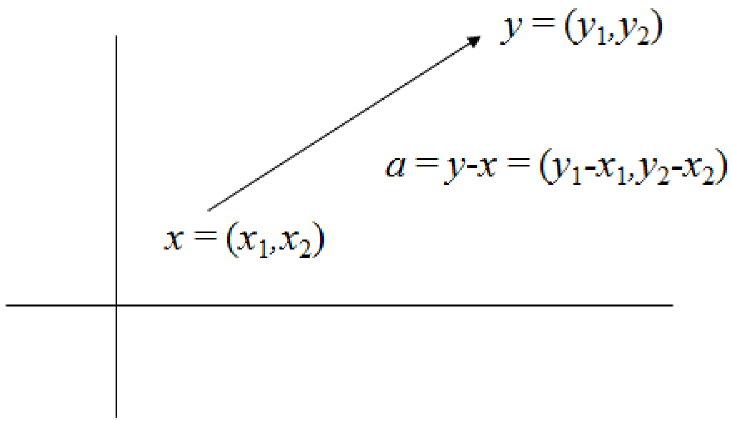

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