Errata

Linear Algebra with Applications, 8th Ed.

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The following pages include all the items of errata that have been uncovered so far. In each case we include the entire page containing the errata and indicate the correction to be made. Help in uncovering additional errata would be greatly appreciated. Please send any errata you discover to

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Corrections will be made in later printings of the book.

Therefore, any solution of system **(b)** must also be a solution of system **(a)**. By a similar argument, it can be shown that any solution of **(a)** is also a solution of **(b)**. This can be done by subtracting the first equation from the second:

$$\begin{array}{rcl} x_2 \qquad\quad & = & 3 \\ 3x_1 + 2x_2 - x_3 & = & -2 \\ \hline -3x_1 - \ x_2 + x_3 & = & 5 \end{array}$$

Then add the first and third equations:

$$\begin{array}{rcl} 3x_1 + 2x_2 - \ x_3 & = & -2 \\ 2x_3 & = & 4 \\ \hline 3x_1 + 2x_2 + \ x_3 & = & 2 \end{array}$$

Thus, (x_1, x_2, x_3) is a solution of system **(b)** if and only if it is a solution of system **(a)**. Therefore, both systems have the same solution set, $\{(-2, 3, 2)\}$.

Definition

Two systems of equations involving the same variables are said to be **equivalent** if they have the same solution set.

Clearly, if we interchange the order in which two equations of a system are written, this will have no effect on the solution set. The reordered system will be equivalent to the original system. For example, the systems

$$\begin{aligned} x_1 + 2x_2 &= 4 \\ 3x_1 - \ x_2 &= 2 \\ 4x_1 + \ x_2 &= 6 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 4x_1 + \ x_2 &= 6 \\ 3x_1 - \ x_2 &= 2 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

both involve the same three equations and, consequently, they must have the same solution set.

If one equation of a system is multiplied through by a nonzero real number, this will have no effect on the solution set, and the new system will be equivalent to the original system. For example, the systems

$$\begin{aligned} x_1 + x_2 + \ x_3 &= 3 \\ -2x_1 - x_2 + 4x_3 &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 6 \\ -2x_1 - \ x_2 + 4x_3 &= 1 \end{aligned}$$

are equivalent.

If a multiple of one equation is added to another equation, the new system will be equivalent to the original system. This follows since the n-tuple $(x_1, \ldots, x_n)$ will satisfy the two equations

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ a_{j1}x_1 + \cdots + a_{jn}x_n &= b_j \end{aligned}$$

if and only if it satisfies the equations

$$\begin{aligned} a_{i1}x_1 + \cdots + a_{in}x_n &= b_i \\ (a_{j1} + \alpha a_{i1})x_1 + \cdots + (a_{jn} + \alpha a_{in})x_n &= b_j + \alpha b_i \end{aligned}$$

Returning to the example, we find that the first row is used to eliminate the elements in the first column of the remaining rows. We refer to the first row as the *pivotal row*. For emphasis, the entries in the pivotal row are all in bold type and the entire row is color shaded. The first nonzero entry in the pivotal row is called the *pivot*.

$$\begin{matrix} (\text{pivot } a_{11} = 1) \\ \left.\begin{matrix}\text{entries to be eliminated} \\ a_{21} = 3 \text{ and } a_{31} = 2\end{matrix}\right\} \rightarrow \end{matrix} \left[\begin{array}{rrr|r} \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{3} \\ 3 & -1 & -3 & -1 \\ 2 & 3 & 1 & 4 \end{array}\right] \leftarrow \text{pivotal row}$$

By using row operation III, 3 times the first row is subtracted from the second row and 2 times the first row is subtracted from the third. When this is done, we end up with the matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ \mathbf{0} & \mathbf{-7} & \mathbf{-6} & \mathbf{-10} \\ 0 & -1 & -1 & -2 \end{array}\right] \leftarrow \text{pivotal row}$$

At this step we choose the second row as our new pivotal row and apply row operation III to eliminate the last element in the second column. This time, the pivot is -7 and the quotient $\frac{-1}{-7} = \frac{1}{7}$ is the multiple of the pivotal row that is subtracted from the third row. We end up with the matrix

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 0 & -7 & -6 & -10 \\ 0 & 0 & -\frac{1}{7} & -\frac{4}{7} \end{array}\right]$$

This is the augmented matrix for the strictly triangular system, which is equivalent to the original system. The solution of the system is easily obtained by back substitution.

EXAMPLE 4 Solve the system

$$\begin{aligned} - x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 &= -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 &= 3 \end{aligned}$$

Solution

The augmented matrix for this system is

$$\left[\begin{array}{rrrr|r} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right]$$

Since it is not possible to eliminate any entries by using 0 as a pivot element, we will use row operation I to interchange the first two rows of the augmented matrix. The new first row will be the pivotal row and the pivot element will be 1:

$$(\text{pivot } a_{11} = 1) \quad \left[\begin{array}{rrrr|r} \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{6} \\ 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \leftarrow \text{pivot row}$$

8. Consider a linear system whose augmented matrix is of the form

$$\left(\begin{array}{rrr|r} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right)$$

For what values of a will the system have a unique solution?

9. Consider a linear system whose augmented matrix is of the form

$$\left(\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{array}\right)$$

(a) Is it possible for the system to be inconsistent? Explain.

(b) For what values of β will the system have infinitely many solutions?

10. Consider a linear system whose augmented matrix is of the form

$$\left(\begin{array}{rrr|r} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right)$$

(a) For what values of a and b will the system have infinitely many solutions?

(b) For what values of a and b will the system be inconsistent?

11. Given the linear systems

(a)
$$\begin{aligned} x_1 + 2x_2 &= 2 \\ 3x_1 + 7x_2 &= 8 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 7x_2 &= 7 \end{aligned}$$

solve both systems by incorporating the right-hand sides into a 2×2 matrix B and computing the reduced row echelon form of

$$(A\,|\,B) = \left(\begin{array}{rr|rr} 1 & 2 & 2 & 1 \\ 3 & 7 & 8 & 7 \end{array}\right)$$

12. Given the linear systems

(a)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ -x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 + 3x_2 \qquad &= 0 \end{aligned}$$

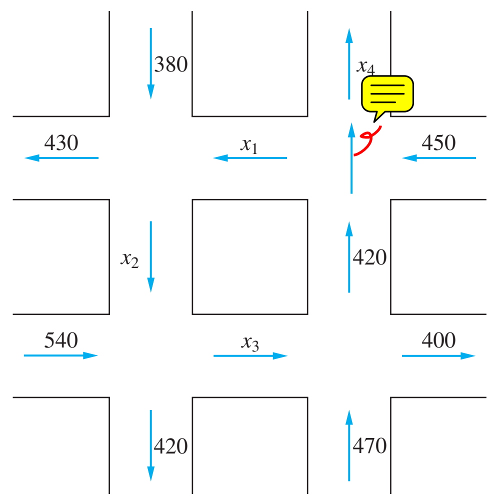
(b)
$$\begin{aligned} x_1 + 2x_2 + x_3 &= -1 \\ -x_1 - x_2 + 2x_3 &= 2 \\ 2x_1 + 3x_2 \qquad &= -2 \end{aligned}$$

solve both systems by computing the row echelon form of an augmented matrix $(A\,|\,B)$ and performing back substitution twice.

13. Given a homogeneous system of linear equations, if the system is overdetermined, what are the possibilities as to the number of solutions? Explain.

14. Given a nonhomogeneous system of linear equations, if the system is underdetermined, what are the possibilities as to the number of solutions? Explain.

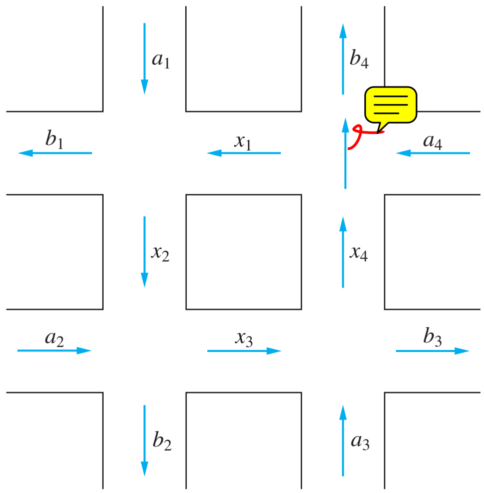
15. Determine the values of x_1, x_2, x_3, and x_4 for the following traffic flow diagram:



16. Consider the traffic flow diagram that follows, where $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4$ are fixed positive integers. Set up a linear system in the unknowns x_1, x_2, x_3, x_4 and show that the system will be consistent if and only if

$$a_1 + a_2 + a_3 + a_4 = b_1 + b_2 + b_3 + b_4$$

What can you conclude about the number of automobiles entering and leaving the traffic network?



17. Let (c_1, c_2) be a solution of the 2×2 system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned}$$

Show that, for any real number α, the ordered pair $(\alpha c_1, \alpha c_2)$ is also a solution.

18. In Application 3, the solution (6, 6, 6, 1) was obtained by setting the free variable $x_4 = 1$.

(a) Determine the solution corresponding to $x_4 = 0$. What information, if any, does this solution give about the chemical reaction? Is the term "trivial solution" appropriate in this case?

(b) Choose some other values of x_4, such as 2, 4, or 5, and determine the corresponding solutions. How are these nontrivial solutions related?

19. Liquid benzene burns in the atmosphere. If a cold object is placed directly over the benzene, water will condense on the object and a deposit of soot (carbon) will also form on the object. The chemical equation for this reaction is of the form

$$x_1 C_6H_6 + x_2 O_2 \rightarrow x_3 C + x_4 H_2O$$

Determine values of x_1, x_2, x_3, and x_4 to balance the equation.

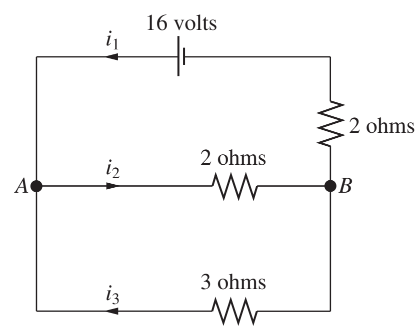
20. Nitric acid is prepared commercially by a series of three chemical reactions. In the first reaction, nitrogen (N_2) is combined with hydrogen (H_2) to form ammonia (NH_3). Next, the ammonia is combined with oxygen (O_2) to form nitrogen dioxide (NO_2) and water. Finally, the NO_2 reacts with some of the water to form nitric acid (HNO_3) and nitric oxide (NO). The amounts of each of the components of these reactions are measured in moles (a standard unit of measurement for chemical reactions). How many moles of nitrogen, hydrogen, and oxygen are necessary to produce 8 moles of nitric acid?

21. In Application 4, determine the relative values of x_1, x_2, and x_3 if the distribution of goods is as described in the following table:

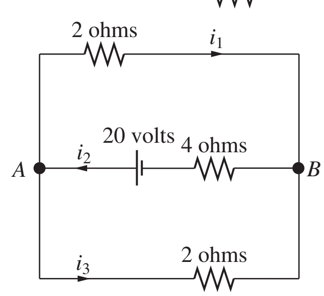
	F	M	C
F	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
M	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
C	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{2}$

22. Determine the amount of each current for the following networks:

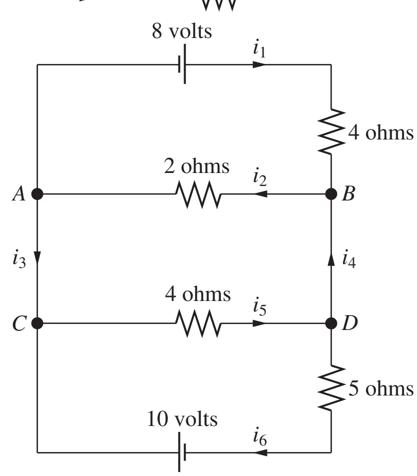
(a)



(b)



(c)



If we let e_i denote the average number of eggs laid by a member of stage i ($i = 2, 3, 4$) in 1 year and form the matrix

$$L = \begin{bmatrix} p_1 & e_2 & e_3 & e_4 \\ q_1 & p_2 & 0 & 0 \\ 0 & q_2 & p_3 & 0 \\ 0 & 0 & q_3 & p_4 \end{bmatrix} \qquad (3)$$

then L can be used to predict the turtle populations at each stage in future years. A matrix of the form (3) is called a *Leslie matrix*, and the corresponding population model is sometimes referred to as a *Leslie population model*. Using the figures from Table 1, the Leslie matrix for our model is

$$L = \begin{bmatrix} 0 & 0 & 127 & 79 \\ 0.67 & 0.7394 & 0 & 0 \\ 0 & 0.0006 & 0 & 0 \\ 0 & 0 & 0.81 & 0.8077 \end{bmatrix}$$

Suppose that the initial populations at each stage were 200,000, 300,000, 500, and 1500, respectively. If we represent these initial populations by a vector $\mathbf{x}_0$, the populations at each stage after 1 year are determined with the matrix equation

$$\mathbf{x}_1 = L\mathbf{x}_0 = \begin{bmatrix} 0 & 0 & 127 & 79 \\ 0.67 & 0.7394 & 0 & 0 \\ 0 & 0.0006 & 0 & 0 \\ 0 & 0 & 0.81 & 0.8077 \end{bmatrix} \begin{bmatrix} 200{,}000 \\ 300{,}000 \\ 500 \\ 1{,}500 \end{bmatrix} = \begin{bmatrix} 182{,}000 \\ 355{,}820 \\ 180 \\ 1{,}617 \end{bmatrix}$$

(The computations have been rounded to the nearest integer.) To determine the population vector after 2 years, we multiply again by the matrix L:

$$\mathbf{x}_2 = L\mathbf{x}_1 = L^2\mathbf{x}_0$$

In general, the population after k years is determined by computing $\mathbf{x}_k = L^k\mathbf{x}_0$. To see longer range trends, we compute $\mathbf{x}_{10}$, $\mathbf{x}_{25}$, and $\mathbf{x}_{50}$. The results are summarized in Table 2. The model predicts that the total number of breeding-age turtles will decrease by 80 percent over a 50-year period.

Table 2 Loggerhead Sea Turtle Population Projections

Stage Number	Initial population	10 years	25 years	50 years
1	200,000	114,264	74,039	35,966
2	300,000	329,212	213,669	103,795
3	500	214	139	68
4	1,500	1,061	687	334

A seven-stage model describing the population dynamics is presented in reference [1] to follow. We will use the seven-stage model in the computer exercises at the end of this chapter. Reference [2] is the original paper by Leslie.

Symmetric Matrices and Networks

Recall that a matrix A is symmetric if $A^T = A$. One type of application that leads to symmetric matrices is problems involving networks. These problems are often solved with the techniques of an area of mathematics called *graph theory*.

APPLICATION 3 Networks and Graphs

Graph theory is an important areas of applied mathematics. It is used to model problems in virtually all the applied sciences. Graph theory is particularly useful in applications involving communication networks.

A *graph* is defined to be a set of points called *vertices*, together with a set of unordered pairs of vertices, which are referred to as *edges*. Figure 1.4.2 gives a geometrical representation of a graph. We can think of the vertices V_1, V_2, V_3, V_4, and V_5 as corresponding to the nodes in a communication network.

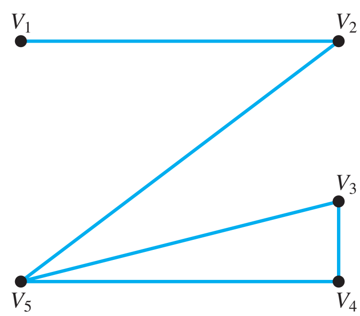


Figure 1.4.2.

The line segments joining the vertices correspond to the edges:

$$\{V_1, V_2\},\ \{V_2, V_5\},\ \{V_3, V_4\},\ \{V_3, V_5\},\ \{V_4, V_5\}$$

Each edge represents a direct communication link between two nodes of the network.

An actual communication network could involve a large number of vertices and edges. Indeed, if there are millions of vertices, a graphical picture of the network would be quite confusing. An alternative is to use a matrix representation for the network. If the graph contains a total of n vertices, we can define an $n \times n$ matrix A by

$$a_{ij} = \begin{cases} 1 & \text{if } \{V_i, V_j\} \text{ is an edge of the graph} \\ 0 & \text{if there is no edge joining } V_i \text{ and } V_j \end{cases}$$

The matrix A is called the *adjacency matrix* of the graph. The adjacency matrix for the graph in Figure 1.4.2 is given by

$$A = \begin{Bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{Bmatrix}$$

(g) $\begin{bmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}$ **(h)** $\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & -2 & -3 \end{bmatrix}$

11. Given

$$A = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

compute A^{-1} and use it to

(a) find a 2×2 matrix X such that $AX = B$.

(b) find a 2×2 matrix Y such that $YA = B$.

12. Let

$$A = \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}, B = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}, C = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$$

Solve each of the following matrix equations:

(a) $AX + B = C$ **(b)** $XA + B = C$

(c) $AX + B = X$ **(d)** $XA + C = X$

13. Is the transpose of an elementary matrix an elementary matrix of the same type? Is the product of two elementary matrices an elementary matrix?

14. Let U and R be $n \times n$ upper triangular matrices and set $T = UR$. Show that T is also upper triangular and that $t_{jj} = u_{jj}r_{jj}$ for $j = 1, \ldots, n$.

15. Let A be a 3×3 matrix and suppose that

$$2\mathbf{a}_1 + \mathbf{a}_2 - 4\mathbf{a}_3 = \mathbf{0}$$

How many solutions will the system $A\mathbf{x} = \mathbf{0}$ have? Explain. Is A nonsingular? Explain.

16. Let A be a 3×3 matrix and suppose that

$$\mathbf{a}_1 = 3\mathbf{a}_2 - 2\mathbf{a}_3$$

Will the system $A\mathbf{x} = \mathbf{0}$ have a nontrivial solution? Is A nonsingular? Explain your answers.

17. Let A and B be $n \times n$ matrices and let $C = A - B$. Show that if $A\mathbf{x}_0 = B\mathbf{x}_0$ and $\mathbf{x}_0 \neq \mathbf{0}$, then C must be singular.

18. Let A and B be $n \times n$ matrices and let $C = AB$. Prove that if B is singular, then C must be singular. [*Hint*: Use Theorem 1.5.2.]

19. Let U be an $n \times n$ upper triangular matrix with nonzero diagonal entries.

(a) Explain why U must be nonsingular.

(b) Explain why U^{-1} must be upper triangular.

20. Let A be a nonsingular $n \times n$ matrix and let B be an $n \times r$ matrix. Show that the reduced row echelon form of $(A|B)$ is $(I|C)$, where $C = A^{-1}B$.

21. In general, matrix multiplication is not commutative (i.e., $AB \neq BA$). However, in certain special cases the commutative property does hold. Show that

(a) if D_1 and D_2 are $n \times n$ diagonal matrices, then $D_1D_2 = D_2D_1$.

(b) if A is an $n \times n$ matrix and

$$B = a_0I + a_1A + a_2A^2 + \cdots + a_kA^k$$

where $a_0, a_1, \ldots, a_k$ are scalars, then $AB = BA$.

22. Show that if A is a symmetric nonsingular matrix, then A^{-1} is also symmetric.

23. Prove that if A is row equivalent to B, then B is row equivalent to A.

24. **(a)** Prove that if A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

(b) Prove that any two nonsingular $n \times n$ matrices are row equivalent.

25. Let A and B be $m \times n$ matrices. Prove that if B is row equivalent to A and U is any row echelon form A, then B is row equivalent to U.

26. Prove that B is row equivalent to A if and only if there exists a nonsingular matrix M such that $B = MA$.

27. Is it possible for a singular matrix B to be row equivalent to a nonsingular matrix A? Explain.

28. Given a vector $\mathbf{x} \in \mathbb{R}^{n+1}$, the $(n+1) \times (n+1)$ matrix V defined by

$$v_{ij} = \begin{cases} 1 & \text{if } j = 1 \\ x_i^{j-1} & \text{for } j = 2, \ldots, n+1 \end{cases}$$

is called the Vandermonde matrix.

(a) Show that if

$$V\mathbf{c} = \mathbf{y}$$

and

$$p(x) = c_1 + c_2x + \cdots + c_{n+1}x^n$$

then

$$p(x_i) = y_i, \qquad i = 1, 2, \ldots, n+1$$

(b) Suppose that $x_1, x_2, \ldots, x_{n+1}$ are all distinct. Show that if $\mathbf{c}$ is a solution to $V\mathbf{x} = \mathbf{0}$, then the coefficients $c_1, c_2, \ldots, c_n$ must all be zero and hence V must be nonsingular.

then

$$AB = \begin{bmatrix} 4 & 5 \\ 7 & 10 \end{bmatrix} \quad \textit{and} \quad BA = \begin{bmatrix} 11 & 7 \\ 4 & 3 \end{bmatrix}$$

This proves th reduced *is false.*

1. If the row echelon form of A involves free variables, then the system $A\mathbf{x} = \mathbf{b}$ will have infinitely many solutions.
2. Every homogeneous linear system is consistent.
3. An $n \times n$ matrix A is nonsingular if and only if the reduced row echelon form of A is I (the identity matrix).
4. If A is nonsingular, then A can be factored into a product of elementary matrices.
5. If A and B are nonsingular $n \times n$ matrices, then $A + B$ is also nonsingular and
$$(A + B)^{-1} = A^{-1} + B^{-1}.$$
6. If $A = A^{-1}$, then A must be equal to either I or $-I$.
7. If A and B are $n \times n$ matrices, then
$$(A - B)^2 = A^2 - 2AB + B^2.$$
8. If $AB = AC$ and $A \neq O$ (the zero matrix), then $B = C$.
9. If $AB = O$, then $BA = O$.
10. If A is a 3×3 matrix and $\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$, then A must be singular.
11. If A is a 4×3 matrix and $\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_3$, then the system $A\mathbf{x} = \mathbf{b}$ must be consistent.
12. Let A be a 4×3 matrix with $\mathbf{a}_2 = \mathbf{a}_3$. If $\mathbf{b} = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$, then the system $A\mathbf{x} = \mathbf{b}$ will have infinitely many solutions.
13. If E is an elementary matrix, then E^T is also an elementary matrix.
14. The product of two elementary matrices is an elementary matrix.
15. If $\mathbf{x}$ and $\mathbf{y}$ are nonzero vectors in $\mathbb{R}^n$ and $A = \mathbf{x}\mathbf{y}^T$, then the row echelon form of A will have exactly one nonzero row.

CHAPTER TEST B

1. Find all solutions of the linear system
$$\begin{aligned} x_1 - x_2 + 3x_3 + 2x_4 &= 1 \\ -x_1 + x_2 - 2x_3 + x_4 &= -2 \\ 2x_1 - 2x_2 + 7x_3 + 7x_4 &= 1 \end{aligned}$$
2. **(a)** A linear equation in two unknowns corresponds to a line in the plane. Give a similar geometric interpretation of a linear equation in three unknowns.
   (b) Given a linear system consisting of two equations in three unknowns, what is the possible number of solutions? Give a geometric explanation of your answer.
   (c) Given a homogeneous linear system consisting of two equations in three unknowns, how many solutions will it have? Explain.
3. Let $A\mathbf{x} = \mathbf{b}$ be a system of n linear equations in n unknowns, and suppose that $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions and $\mathbf{x}_1 \neq \mathbf{x}_2$.
   (a) How many solutions will the system have? Explain.
   (b) Is the matrix A nonsingular? Explain.
4. Let A be a matrix of the form
$$A = \begin{bmatrix} \alpha & \beta \\ 2\alpha & 2\beta \end{bmatrix}$$
where α and β are fixed scalars not both equal to 0.
   (a) Explain why the system
$$A\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$
must be inconsistent.
   (b) How can one choose a nonzero vector $\mathbf{b}$ so that the system $A\mathbf{x} = \mathbf{b}$ will be consistent? Explain.
5. Let
$$A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 7 \\ 1 & 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 5 \\ 4 & 2 & 7 \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 7 \\ -5 & 3 & 5 \end{bmatrix}$$
   (a) Find an elementary matrix E such that $EA = B$.
   (b) Find an elementary matrix F such that $AF = C$.
6. Let A be a 3×3 matrix and let
$$\mathbf{b} = 3\mathbf{a}_1 + \mathbf{a}_2 + 4\mathbf{a}_3$$
Will the system $A\mathbf{x} = \mathbf{b}$ be consistent? Explain.

the transition matrix from $[1, 2x, 4x^2 - 2]$ to $[1, x, x^2]$. Since

$$\begin{aligned} 1 &= 1 \cdot 1 + 0x + 0x^2 \\ 2x &= 0 \cdot 1 + 2x + 0x^2 \\ 4x^2 - 2 &= -2 \cdot 1 + 0x + 4x^2 \end{aligned}$$

the transition matrix is

$$S = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The inverse of S will be the transition matrix from $[1, x, x^2]$ to $[1, 2x, 4x^2 - 2]$:

$$S^{-1} = \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$$

Given any $p(x) = a + bx + cx^2$ in P_3, to find the coordinates of $p(x)$ with respect to $[1, 2x, 4x^2 - 2]$, we simply multiply

$$\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ \frac{1}{2}b \\ \frac{1}{4}c \end{pmatrix}$$

Thus,

$$p(x) = (a + \tfrac{1}{2}c) \cdot 1 + (\tfrac{1}{2}b) \cdot 2x + \tfrac{1}{4}c \cdot (4x^2 - 2)$$ ■

We have seen that each transition matrix is nonsingular. Actually, any nonsingular matrix can be thought of as a transition matrix. If S is an $n \times n$ nonsingular matrix and $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an ordered basis for V, then define $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ by (4). To see that the $\mathbf{w}_j$'s are linearly independent, suppose that

$$\sum_{j=1}^{n} x_j \mathbf{w}_j = \mathbf{0}$$

It follows from (4) that

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{n} s_{ij} x_j \right) \mathbf{v}_i = \mathbf{0}$$

By the linear independence of the $\mathbf{v}_i$'s, it follows that

$$\sum_{j=1}^{n} s_{ij} x_j = 0 \qquad i = 1, \ldots, n$$

or, equivalently,

$$S\mathbf{x} = \mathbf{0}$$

Theorem 5.5.7 Let S be a subspace of an inner product space V and let $\mathbf{x} \in V$. Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ be an orthonormal basis for S. If

$$\mathbf{p} = \sum_{i=1}^{n} c_i \mathbf{u}_i \tag{3}$$

where

$$c_i = \langle \mathbf{x}, \mathbf{u}_i \rangle \qquad \text{for each } i \tag{4}$$

then $\mathbf{p} - \mathbf{x} \in S^{\perp}$ (see Figure 5.5.2).

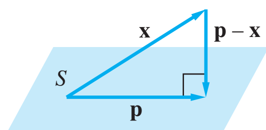


Figure 5.5.2.

Proof We will show first that $(\mathbf{p} - \mathbf{x}) \perp \mathbf{u}_i$ for each i:

$$\begin{aligned}
\langle \mathbf{u}_i, \mathbf{p} - \mathbf{x} \rangle &= \langle \mathbf{u}_i, \mathbf{p} \rangle - \langle \mathbf{u}_i, \mathbf{x} \rangle \\
&= \langle \mathbf{x}_i, \sum_{j=1}^{n} c_j \mathbf{u}_j \rangle - c_i \\
&= \sum_{j=1}^{n} c_j \langle \mathbf{u}_i, \mathbf{u}_j \rangle - c_i \\
&= 0
\end{aligned}$$

So $\mathbf{p} - \mathbf{x}$ is orthogonal to all the $\mathbf{u}_i$'s. If $\mathbf{y} \in S$, then

$$\mathbf{y} = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i$$

and hence

$$\langle \mathbf{p} - \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{p} - \mathbf{x}, \sum_{i=1}^{n} \alpha_i \mathbf{u}_i \rangle = \sum_{i=1}^{n} \alpha_i \langle \mathbf{p} - \mathbf{x}, \mathbf{u}_i \rangle = 0 \qquad \blacksquare$$

If $\mathbf{x} \in S$, the preceding result is trivial, since, by Theorem 5.5.2, $\mathbf{p} - \mathbf{x} = \mathbf{0}$. If $\mathbf{x} \notin S$, then $\mathbf{p}$ is the element in S closest to $\mathbf{x}$.

Theorem 5.5.8 Under the hypothesis of Theorem 5.5.7, $\mathbf{p}$ is the element of S that is closest to $\mathbf{x}$; that is,

$$\|\mathbf{y} - \mathbf{x}\| > \|\mathbf{p} - \mathbf{x}\|$$

for any $\mathbf{y} \neq \mathbf{p}$ in S.

2. Let

$$\mathbf{z}_1 = \begin{pmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{pmatrix} \quad \text{and} \quad \mathbf{z}_2 = \begin{pmatrix} \frac{i}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$$

   (a) Show that $\{\mathbf{z}_1, \mathbf{z}_2\}$ is an orthonormal set in $\mathbb{C}^2$.

   (b) Write the vector $\mathbf{z} = \begin{pmatrix} 2+4i \\ -2i \end{pmatrix}$ as a linear combination of $\mathbf{z}_1$ and $\mathbf{z}_2$.

3. Let $\{\mathbf{u}_1, \mathbf{u}_2\}$ be an orthonormal basis for $\mathbb{C}^2$, and let $\mathbf{z} = (4+2i)\mathbf{u}_1 + (6-5i)\mathbf{u}_2$.

   (a) What are the values of $\mathbf{u}_1^H\mathbf{z}$, $\mathbf{z}^H\mathbf{u}_1$, $\mathbf{u}_2^H\mathbf{z}$, and $\mathbf{z}^H\mathbf{u}_2$?

   (b) Determine the value of $\|\mathbf{z}\|$.

4. Which of the matrices that follow are Hermitian? Normal?

   (a) $\begin{pmatrix} 1-i & 2 \\ 2 & 3 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 2-i \\ 2+i & -1 \end{pmatrix}$

   (c) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

   (d) $\begin{pmatrix} \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}i \end{pmatrix}$

   (e) $\begin{pmatrix} 0 & i & 1 \\ i & 0 & -2+i \\ -1 & 2+i & 0 \end{pmatrix}$

   (f) $\begin{pmatrix} 3 & 1+i & i \\ 1-i & 1 & 3 \\ -i & 3 & 1 \end{pmatrix}$

5. Find an orthogonal or unitary diagonalizing matrix for each of the following:

   (a) $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ **(b)** $\begin{pmatrix} 1 & 3+i \\ 3-i & 4 \end{pmatrix}$

   (c) $\begin{pmatrix} 2 & i & 0 \\ -i & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ **(d)** $\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{pmatrix}$

   (e) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ **(f)** $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

   (g) $\begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix}$

6. Show that the diagonal entries of a Hermitian matrix must be real.

7. Let A be a Hermitian matrix and let $\mathbf{x}$ be a vector in $\mathbb{C}^n$. Show that if $c = \mathbf{x}A\mathbf{x}^H$, then c is real.

8. Let A be a Hermitian matrix and let $B = iA$. Show that B is skew Hermitian.

9. Let A and C be matrices in $\mathbb{C}^{m\times n}$ and let $B \in \mathbb{C}^{n\times r}$. Prove each of the following rules:

   (a) $(A^H)^H = A$
   (b) $(\alpha A + \beta C)^H = \overline{\alpha}A^H + \overline{\beta}C^H$
   (c) $(AB)^H = B^H A^H$

10. Let A and B be Hermitian matrices. Answer *true* or *false* for each of the statements that follow. In each case, explain or prove your answer.

   (a) The eigenvalues of AB are all real.
   (b) The eigenvalues of ABA are all real.

11. Show that

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^H \mathbf{z}$$

defines an inner product on $\mathbb{C}^n$.

12. Let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be vectors in $\mathbb{C}^n$ and let α and β be complex scalars. Show that

$$\langle \mathbf{z}, \alpha\mathbf{x} + \beta\mathbf{y} \rangle = \overline{\alpha}\langle \mathbf{z}, \mathbf{x} \rangle + \overline{\beta}\langle \mathbf{z}, \mathbf{y} \rangle$$

13. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be an orthonormal basis for a complex inner product space V, and let

$$\mathbf{z} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + \cdots + a_n\mathbf{u}_n$$
$$\mathbf{w} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + \cdots + b_n\mathbf{u}_n$$

Show that

$$\langle \mathbf{z}, \mathbf{w} \rangle = \sum_{i=1}^{n} \overline{b_i} a_i$$

14. Given that

$$A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{pmatrix}$$

find a matrix B such that $B^H B = A$.

15. Let U be a unitary matrix. Prove that

   (a) U is normal.
   (b) $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$.
   (c) if λ is an eigenvalue of U, then $|\lambda| = 1$.

16. Let $\mathbf{u}$ be a unit vector in $\mathbb{C}^n$ and define $U = I - 2\mathbf{u}\mathbf{u}^H$. Show that U is both unitary and Hermitian and, consequently, is its own inverse.

17. Show that if a matrix U is both unitary and Hermitian, then any eigenvalue of U must equal either 1 or -1.

18. Let A be a 2×2 matrix with Schur decomposition UTU^H and suppose that $t_{12} \neq 0$. Show that

color-blind men is p and, over a number of generations, no outsiders have entered the population, there is justification for assuming that the proportion of genes for color blindness in the female population is also p. Since color blindness is recessive, we would expect the proportion of color-blind women to be about p^2. Thus, if 1 percent of the male population is color blind, we would expect about 0.01 percent of the female population to be color blind.

The Exponential of a Matrix

Given a scalar a, the exponential e^a can be expressed in terms of a power series

$$e^a = 1 + a + \frac{1}{2!}a^2 + \frac{1}{3!}a^3 + \cdots$$

Similarly, for any $n \times n$ matrix A, we can define the *matrix exponential* e^A in terms of the convergent power series

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \tag{6}$$

The matrix exponential (6) occurs in a wide variety of applications. In the case of a diagonal matrix

$$D = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

the matrix exponential is easy to compute:

$$\begin{aligned} e^D &= \lim_{m\to\infty}\left(I + D + \frac{1}{2!}D^2 + \cdots + \frac{1}{m!}D^m\right) \\ &= \lim_{m\to\infty}\begin{bmatrix} \sum_{k=1}^{m}\frac{1}{k!}\lambda_1^k & & \\ & \ddots & \\ & & \sum_{k=1}^{m}\frac{1}{k!}\lambda_n^k \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix} \end{aligned}$$

It is more difficult to compute the matrix exponential for a general $n \times n$ matrix A. If, however, A is diagonalizable, then

$$\begin{aligned} A^k &= XD^kX^{-1} \qquad \text{for} \quad k = 1, 2, \ldots \\ e^A &= X\left(I + D + \frac{1}{2!}D^2 + \frac{1}{3!}D^3 + \cdots\right)X^{-1} \\ &= Xe^DX^{-1} \end{aligned}$$

EXAMPLE 6 Compute e^A for

$$A = \begin{bmatrix} -2 & -6 \\ 1 & 3 \end{bmatrix}$$

then

$$\|A - A'\|_F = \left(\sigma_{k+1}^2 + \cdots + \sigma_n^2\right)^{1/2} = \min_{S \in \mathcal{M}} \|A - S\|_F$$

Proof Let X be a matrix in $\mathcal{M}$ satisfying (7). Since $A' \in \mathcal{M}$, it follows that

$$\|A - X\|_F \leq \|A - A'\|_F = \left(\sigma_{k+1}^2 + \cdots + \sigma_n^2\right)^{1/2} \tag{8}$$

We will show that

$$\|A - X\|_F \geq \left(\sigma_{k+1}^2 + \cdots + \sigma_n^2\right)^{1/2}$$

and hence that equality holds in (8). Let $Q\Omega P^T$ be the singular value decomposition of X, where

$$\Omega = \left[\begin{array}{cccc|c} \omega_1 & & & & \\ & \omega_2 & & & \\ & & \ddots & & O \\ & & & \omega_k & \\ \hline & & O & & O \end{array}\right] = \begin{bmatrix} \Omega_k & O \\ O & O \end{bmatrix}$$

If we set $B = Q^T AP$, then $A = QBP^T$, and it follows that

$$\|A - X\|_F = \|Q(B - \Omega)P^T\|_F = \|B - \Omega\|_F$$

Let us partition B in the same manner as Ω:

$$B = \left[\begin{array}{c|c} \overbrace{B_{11}}^{k \times k} & \overbrace{B_{12}}^{k \times (n-k)} \\ \hline \underbrace{B_{21}}_{(m-k) \times k} & \underbrace{B_{22}}_{(m-k) \times (n-k)} \end{array}\right]$$

It follows that

$$\|A - X\|_F^2 = \|B_{11} - \Omega_k\|_F^2 + \|B_{12}\|_F^2 + \|B_{21}\|_F^2 + \|B_{22}\|_F^2$$

We claim that $B_{12} = O$. If not, then define

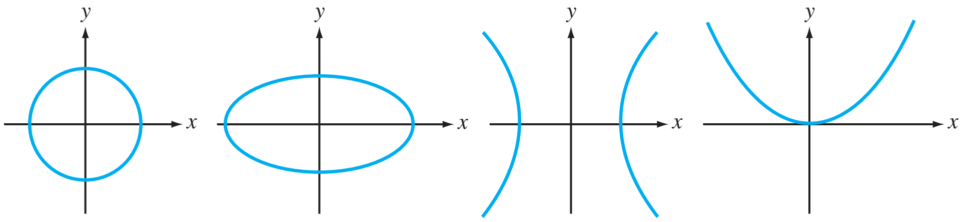
$$Y = Q \begin{bmatrix} B_{11} & B_{12} \\ O & O \end{bmatrix} P^T$$

The matrix Y is in $\mathcal{M}$ and

$$\|A - Y\|_F^2 = \|B_{21}\|_F^2 + \|B_{22}\|_F^2 < \|A - X\|_F^2$$

But this contradicts the definition of X. Therefore, $B_{12} = O$. In a similar manner, it can be shown that B_{21} must equal O. If we set

$$Z = Q \begin{bmatrix} B_{11} & O \\ O & O \end{bmatrix} P^T$$



(i) Circle (ii) Ellipse (iii) Hyperbola (iv) Parabola

Figure 6.6.1.

Case 3. The conic section has been rotated from the standard position by an angle θ that is not a multiple of 90°. This occurs when the coefficient of the xy term is nonzero (i.e., $b \neq 0$).

In general, we may have any one or any combination of these three cases. To graph a conic section that is not in standard position, we usually find a new set of axes x' and y' such that the conic section is in standard position with respect to the new axes. This is not difficult if the conic has only been translated horizontally or vertically, in which case the new axes can be found by completing the squares. The following example illustrates how this is done:

EXAMPLE 1 Sketch the graph of the equation

$$9x^2 - 18x + 4y^2 + 16y - 11 = 0$$

Solution

To see how to choose our new axis system, we complete the squares:

$$9(x^2 - 2x + 1) + 4(y^2 + 4y + 4) - 11 = 9 + 16$$

This equation can be simplified to the form

$$\frac{(x-1)^2}{2^2} + \frac{(y+2)^2}{3^2} = 1$$

If we let

$$x' = x - 1 \quad \text{and} \quad y' = y + 2$$

the equation becomes

$$\frac{(x')^2}{2^2} + \frac{(y')^2}{3^2} = 1$$

which is in standard form with respect to the variables x' and y'. Thus, the graph, as shown in Figure 6.6.2, will be an ellipse that is in standard position in the $x'y'$-axis system. The center of the ellipse will be at the origin of the $x'y'$-plane [i.e., at the point $(x, y) = (1, -2)$]. The equation of the x'-axis is simply $y' = 0$, which is the equation of the line $y = -2$ in the xy-plane. Similarly, the y'-axis coincides with the line $x = 1$. ■

11. Let $A = \mathbf{w}\mathbf{y}^T$, where $\mathbf{w} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$. Show that

(a) $\dfrac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2} \le \|\mathbf{y}\|_2\|\mathbf{w}\|_2$ for all $\mathbf{x} \ne \mathbf{0}$ in $\mathbb{R}^n$.

(b) $\|A\|_2 = \|\mathbf{y}\|_2\|\mathbf{w}\|_2$

12. Let

$$A = \begin{Bmatrix} 3 & -1 & -2 \\ -1 & 2 & -7 \\ 4 & 1 & 4 \end{Bmatrix}$$

(a) Determine $\|A\|_\infty$.

(b) Find a vector $\mathbf{x}$ whose coordinates are each ± 1 such that $\|A\mathbf{x}\|_\infty = \|A\|_\infty$. (Note that $\|\mathbf{x}\|_\infty = 1$, so $\|A\|_\infty = \|A\mathbf{x}\|_\infty / \|\mathbf{x}\|_\infty$.)

13. Theorem 7.4.2 states that

$$\|A\|_\infty = \max_{1 \le i \le m} \left(\sum_{j=1}^n |a_{ij}| \right)$$

Prove this in two steps.

(a) Show first that

$$\|A\|_\infty \le \max_{1 \le i \le m} \left(\sum_{j=1}^n |a_{ij}| \right)$$

(b) Construct a vector $\mathbf{x}$ whose coordinates are each ± 1 such that

$$\frac{\|A\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} = \|A\mathbf{x}\|_\infty = \max_{1 \le i \le m} \left(\sum_{j=1}^n |a_{ij}| \right)$$

14. Show that $\|A\|_F = \|A^T\|_F$.

15. Let A be a symmetric $n \times n$ matrix. Show that $\|A\|_\infty = \|A\|_1$.

16. Let A be a 5×4 matrix with singular values $\sigma_1 = 5$, $\sigma_2 = 3$, and $\sigma_3 = \sigma_4 = 1$. Determine the values of $\|A\|_2$ and $\|A\|_F$.

17. Let A be an $m \times n$ matrix.

(a) Show that $\|A\|_2 \le \|A\|_F$.

(b) Under what circumstances will $\|A\|_2 = \|A\|_F$?

18. Let $\|\cdot\|$ denote the family of vector norms and let $\|\cdot\|_M$ be a subordinate matrix norm. Show that

$$\|A\|_M = \max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|$$

19. Let A be an $m \times n$ matrix and let $\|\cdot\|_{\mathrm{v}}$ and $\|\cdot\|_{\mathrm{w}}$ be vector norms on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Show that

$$\|A\|_{\mathrm{v,w}} = \max_{\mathbf{x} \ne \mathbf{0}} \frac{\|A\mathbf{x}\|_{\mathrm{w}}}{\|\mathbf{x}\|_{\mathrm{v}}}$$

defines a matrix norm on $\mathbb{R}^{m \times n}$.

20. Let A be an $m \times n$ matrix. The 1,2-norm of A is given by

$$\|A\|_{1,2} = \max_{\mathbf{x} \ne \mathbf{0}} \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_1}$$

(See Exercise 19.) Show that

$$\|A\|_{1,2} = \max\left(\|\mathbf{a}_1\|_2, \|\mathbf{a}_2\|_2, \ldots, \|\mathbf{a}_n\|_2\right)$$

21. Let A be an $m \times n$ matrix. Show that $\|A\|_{1,2} \le \|A\|_2$

22. Let A be an $m \times n$ matrix and let $B \in \mathbb{R}^{n \times r}$. Show that

(a) $\|A\mathbf{x}\| \le \|A\|_{1,2}\|\mathbf{x}\|_1$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

(b) $\|AB\|_{1,2} \le \|A\|_2\|B\|_{1,2}$

23. Let A be an $n \times n$ matrix and let $\|\cdot\|_M$ be a matrix norm that is compatible with some vector norm on $\mathbb{R}^n$. Show that if λ is an eigenvalue of A, then $|\lambda| \le \|A\|_M$.

24. Use the result from Exercise 23 to show that if λ is an eigenvalue of a stochastic matrix, then $|\lambda| \le 1$.

25. Sudoku is a popular puzzle involving matrices. In this puzzle, one is given some of the entries of a 9×9 matrix A and asked to fill in the missing entries. The matrix A has block structure

$$A = \begin{Bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{Bmatrix}$$

where each submatrix A_{ij} is 3×3. The rules of the puzzle are that each row, each column, and each of the submatrices of A must be made up of all of the integers 1 through 9. We will refer to such a matrix as a *sudoku matrix*. Show that if A is a sudoku matrix, then $\lambda = 45$ is its dominant eigenvalue.

26. Let A_{ij} be a submatrix of a sudoku matrix A (see Exercise 25). Show that if λ is an eigenvalue of A_{ij}, then $|\lambda| \le 22$.

27. Let A be an $n \times n$ matrix and $\mathbf{x} \in \mathbb{R}^n$. Prove:

(a) $\|A\mathbf{x}\|_\infty \le n^{1/2}\|A\|_2\|\mathbf{x}\|_\infty$

(b) $\|A\mathbf{x}\|_2 \le n^{1/2}\|A\|_\infty\|\mathbf{x}\|_2$

(c) $n^{-1/2}\|A\|_2 \le \|A\|_\infty \le n^{1/2}\|A\|_2$

28. Let A be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$. Let $\mathbf{x} \in \mathbb{R}^n$ and let $c_i = \mathbf{u}_i^T\mathbf{x}$ for $i = 1, 2, \ldots, n$. Show that

(a) $\|A\mathbf{x}\|_2^2 = \displaystyle\sum_{i=1}^n (\lambda_i c_i)^2$

(b) If $\mathbf{x} \ne \mathbf{0}$, then

$$\min_{1 \le i \le n} |\lambda_i| \le \frac{\|A\mathbf{x}\|_2}{\|\mathbf{x}\|_2} \le \max_{1 \le i \le n} |\lambda_i|$$

5. (a) $\det(A) = 0$, so A is singular.

   (b) $\operatorname{adj} A = \begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$ and

   $$A \operatorname{adj} A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

9. (a) $\det(\operatorname{adj}(A)) = 8$ and $\det(A) = 2$

   (b) $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & -1 & 1 \\ 0 & -6 & 2 & -2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

14. DO YOUR HOMEWORK.

CHAPTER TEST A

1. True 2. False 3. False 4. True 5. False
6. True 7. True 8. True 9. False 10. True

Chapter 3

3.1

1. (a) $\|\mathbf{x}_1\| = 10$, $\|\mathbf{x}_2\| = \sqrt{17}$

   (b) $\|\mathbf{x}_3\| = 13 < \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$

2. (a) $\|\mathbf{x}_1\| = \sqrt{5}$, $\|\mathbf{x}_2\| = 3\sqrt{5}$

   (b) $\|\mathbf{x}_3\| = 4\sqrt{5} = \|\mathbf{x}_1\| + \|\mathbf{x}_2\|$

7. If $\mathbf{x} + \mathbf{y} = \mathbf{x}$ for all $\mathbf{x}$ in the vector space, then $\mathbf{0} = \mathbf{0} + \mathbf{y} = \mathbf{y}$.

8. If $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$, then

   $$-\mathbf{x} + (\mathbf{x} + \mathbf{y}) = -\mathbf{x} + (\mathbf{x} + \mathbf{z})$$

   and the conclusion follows from axioms 1, 2, 3, and 4.

11. V is not a vector space. Axiom 6 does not hold.

3.2

1. (a) and (c) are subspaces; (b), (d), and (e) are not.

2. (b) and (c) are subspaces; (a) and (d) are not.

3. (a), (c), (e), and (f) are subspaces; (b), (d), and (g) are not.

4. (a) $\{(0, 0)^T\}$

   (b) $\operatorname{Span}((-2, 1, 0, 0)^T, (3, 0, 1, 0)^T)$

   (c) $\operatorname{Span}((1, 1, 1)^T)$

   (d) $(-1, 1, 0, 0)^T \operatorname{Span}(-5, 0, -3, 1)^T)$

5. Only the set in part (c) is a subspace of P_4.

6. (a), (b), and (d) are subspaces.

11. (a), (c), and (e) are spanning sets.

12. (a) and (b) are spanning sets.

16. (b) and (c)

3.3

1. (a) and (e) are linearly independent (b), (c), and (d) are linearly dependent.

2. (a) and (e) are linearly independent (b), (c), and (d) are not.

3. (a) and (b) are all of 3-space

   (c) a plane through (0, 0, 0)

   (d) a line through (0, 0, 0)

   (e) a plane through (0, 0, 0)

4. (a) linearly independent

   (b) linearly independent

   (c) linearly dependent

8. (a) and (b) are linearly dependent while (c) and (d) are linearly independent.

11. When α is an odd multiple of $\pi/2$. If the graph of $y = \cos x$ is shifted to the left or right by an odd multiple of $\pi/2$, we obtain the graph of either $\sin x$ or $-\sin x$.

3.4

1. Only in parts (a) and (e) do they form a basis.

2. Only in part (a) do they form a basis.

3. (c) 2

4. 1

5. (c) 2

   (d) a plane through (0, 0, 0) in 3-space

6. (b) $\{(1, 1, 1)^T\}$, dimension 1

   (c) $\{(1, 0, 1)^T, (0, 1, 1)^T\}$, dimension 2

7. $\{(1, 1, 0, 0)^T, (1, -1, 1, 0)^T, (0, 2, 0, 1)^T\}$

11. $\{x^2 + 2, x + 3\}$

12. (a) $\{E_{11}, E_{22}\}$ (c) $\{E_{11}, E_{21}, E_{22}\}$

    (e) $\{E_{12}, E_{21}, E_{22}\}$

    (f) $\{E_{11}, E_{22}, E_{21} + E_{12}\}$

13. 2

14. (a) 3 (b) 3 (c) 2 (d) 2

15. (a) $\{x, x^2\}$ (b) $\{x - 1, (x - 1)^2\}$

    (c) $\{x(x - 1)\}$

3.5

1. (a) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$

   (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2. (a) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

   (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3. (a) $\begin{bmatrix} \frac{5}{2} & \frac{7}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$ (b) $\begin{bmatrix} 11 & 14 \\ -4 & -5 \end{bmatrix}$