## 15. Polarization

## Linear, circular, and elliptical polarization

Mathematics of polarization

Polarizers

Birefringence


## Notation: polarization near an interface

You cannot define these unless the propagating wave is near an interface.


But even when there is no interface around, we still need to consider the polarization of light waves.

## Polarization of a light wave

We describe the polarization of a light wave (without any interface nearby) according to how the E-field vector varies in a projection onto a plane perpendicular to the propagation direction.

For convenience, the propagation direction is generally assumed to be along the positive $z$ axis.

Here are two possibilities:


In these diagrams, the propagation direction is out of the screen at you.

## $45^{\circ}$ Polarization

$$
\begin{aligned}
& E_{x}(z, t)=\operatorname{Re}\{\underset{\sim}{E} \exp [j(k z-\omega t)]\} \\
& E_{y}(z, t)=\operatorname{Re}\left\{{\underset{\sim}{0}}^{E} \exp [j(k z-\omega t)]\right\}
\end{aligned}
$$


and the total wave is:

$$
\vec{E}(z, t)=E_{x} \hat{x}+E_{y} \hat{y}
$$

Here, the complex amplitudes of the $x$ component and the $y$ component are equal.

So the components are always in phase.

## Arbitrary-Angle Linear Polarization

$$
\begin{gathered}
E_{x}(z, t)=\operatorname{Re}\{\underset{\sim}{E} \underset{0}{E} \cos (\alpha) \exp [j(k z-\omega t)]\} \\
E_{y}(z, t)=\operatorname{Re}\left\{\underset{\underset{\sim}{E}}{\left.E_{0} \sin (\alpha) \exp [j(k z-\omega t)]\right\}}\right. \\
\text { where } \alpha \text { is a constant. }
\end{gathered}
$$

and (as always) the total wave is:

$$
\vec{E}(z, t)=E_{x} \hat{x}+E_{y} \hat{y}
$$



## Circular (or Helical) Polarization

The x and y components of the E-field need not be in phase.
For example:

$$
\begin{aligned}
& E_{x}(z, t)=E_{0} \cos (k z-\omega t) \\
& E_{y}(z, t)=E_{0} \sin (k z-\omega t)
\end{aligned}
$$

or, in complex notation:
$E_{x}(z, t)=\operatorname{Re}\{\underset{\sim}{E} \exp [j(k z-\omega t)]\}$
$E_{y}(z, t)=\operatorname{Re}\left\{-j{\underset{\sim}{0}}_{0} \exp [j(k z-\omega t)]\right\}$
Here, the complex amplitude of the $y$-component is $-j$ times the complex amplitude of the $x$ component.

So the components are always $90^{\circ}$ out of phase.


The resulting E-field rotates clockwise around the $k$-vector (looking along $k$ ). This is called a right-handed rotation.

## Right vs. Left Circular (or Helical) Polarization

$E_{x}(z, t)=E_{0} \cos (k z-\omega t)$
$E_{y}(z, t)=-E_{0} \sin (k z-\omega t)$
or, more generally:
$E_{x}(z, t)=\operatorname{Re}\{\underset{\sim}{E} \exp [j(k z-\omega t)]\}$
$E_{y}(z, t)=\operatorname{Re}\left\{+j \underset{\sim}{E} E_{0} \exp [j(k z-\omega t)]\right\}$
Here, the complex amplitude of the y-component is $+j$ times the complex amplitude of the x-component.

So the components are always $90^{\circ}$ out of phase, but in the other direction.


The resulting E-field rotates counterclockwise around the k -vector (looking along $k$ ).
This is a left-handed rotation.

## Circular Polarization - the movie



Question: is this cartoon showing right-handed or lefthanded circular polarization?

## Unequal Arbitrary-Relative-Phase Components yield "Elliptical Polarization"

$E_{x}(z, t)=E_{0 x} \cos (k z-\omega t)$
$E_{y}(z, t)=E_{0 y} \cos (k z-\omega t-\theta)$
where $E_{0 x} \neq E_{0 y}$
or, in complex notation:
$E_{x}(z, t)=\operatorname{Re}\left\{E_{0 . x} \exp [j(k z-\omega t)]\right\}$
$E_{y}(z, t)=\operatorname{Re}\left\{\underset{{\underset{V}{0 y}}^{E}}{E_{0}} \exp [j(k z-\omega t)]\right\}$
where ${\underset{\sim}{\sim}}_{0 x}$ and ${\underset{\sim}{0}}_{0 y}$ are arbitrary complex numbers.

The resulting E-field can rotate clockwise or counter-clockwise around the k -vector.

## The Mathematics of Polarization

Define the polarization state of a field as a 2D vector-
"Jones vector" -containing the two complex amplitudes: $\left.E=\left[\begin{array}{c}E_{x} \\ E_{y}\end{array}\right].\right] ~$
For many purposes, we only care about the relative values:

$$
\frac{E}{E_{x}}=\left[\begin{array}{c}
1 \\
E_{y} / E_{x}
\end{array}\right]
$$

Some specific examples:
$0^{\circ}$ linear $(x)$ polarization: $E_{y} / E_{x}=0$
$90^{\circ}$ linear $(y)$ polarization: $E_{y} / E_{x}=\infty$$\longrightarrow\left[\begin{array}{l}0 \\ 1\end{array}\right]$
$45^{\circ}$ linear polarization: $E_{y} / E_{x}=1$
Arbitrary linear polarization: $\frac{E_{y}}{E_{x}}=\frac{\sin \alpha}{\cos \alpha}=\tan \alpha$

## The Mathematics of Circular and Elliptical Polarization

Circular polarization has an imaginary Jones vector y-component:

$$
E=\left[\begin{array}{c}
E_{x} \\
E_{y}
\end{array}\right]=\left[\begin{array}{c}
1 \\
\pm j
\end{array}\right]
$$

Right circular polarization: $E_{y} / E_{x}=-j$ when Iooking along the
Left circular polarization: $\quad E_{y} / E_{x}=+j \quad$ counterclockwise rotation.

For elliptical polarization, the two components have different phases and maybe also different amplitudes, so their ratio is complex:

$$
E_{y} / E_{x}=a+j b
$$

We can calculate the eccentricity and tilt of the ellipse if we feel like it.

## Jones vectors - a common mistake

NOTE: the Jones vector contains the complex amplitudes only. Its components do not include the rapidly varying dependence on position and time.

$$
E=\left[\begin{array}{c}
E_{x} e^{j(k z-\omega t)} \\
E_{y} e^{j(k z-\omega t)}
\end{array}\right] \quad \text { This is wrong! }
$$

## An example

$$
E=\left[\begin{array}{c}
1 \\
1+j
\end{array}\right]
$$

What is the polarization of this wave?
This Jones vector is equivalent to: $\vec{E}(z, t)=E_{0}[\hat{x}+(1+j) \hat{y}] e^{j(k z-\omega t)}$

$$
\begin{aligned}
& \text { Using }(1+j)=\sqrt{2} e^{j \pi / 4} \text { we find, at } \mathrm{z}=0 \text { : } \\
& E_{x}=E_{0} \cos (\omega t) \\
& E_{y}=\sqrt{2} E_{0} \cos (\omega t+\pi / 4) \\
& \begin{array}{rcc} 
& E_{x} & E_{y} \\
\cline { 2 - 3 } \omega \mathrm{t}=0 & E_{0} & E_{0} \\
\omega \mathrm{t}=\pi / 4 & E_{0} \cdot \sqrt{2} / 2 & 0 \\
\omega \mathrm{t}=\pi / 2 & 0 & -E_{0} \\
\omega \mathrm{t}=3 \pi / 4 & -E_{0} \cdot \sqrt{2} / 2 & -\sqrt{2} E_{0} \\
\omega \mathrm{t}=\pi & -E_{0} & -E_{0}
\end{array}
\end{aligned}
$$

## A polarizer is a device which filters out one polarization component

Output light contains only $x$-polarization.


Input light contains both polarizations.

The light can excite electrons to move along the wires. These moving charges then emit light that cancels the input light. This cannot happen if the E-field is perpendicular to the wires, since the current can only flow along the wires.

Such polarizers are commonly used for long-wave infrared radiation, because the wire spacing has to be much smaller than the wavelength (and so it is easier to manufacture if the wavelength is longer).

## Polymer-based polarizers

A polymer is a long chain molecule. Some polymers can conduct electricity (i.e., they can respond to electric fields similar to the way a wire does).

The light can excite electrons to move along the wires, just as in the case of the polymer chains.


This is how polarized sunglasses work.

## Crossed polarizers block light

Blocking both x and y polarizations means that you have blocked everything.


Inserting a third polarizer between the two crossed ones can allow some light to leak through.

## Why sunglasses are polarized


no sunglasses

sunglasses


## Birefringence

The molecular "spring constant" can be different for different directions.



The $x$ - and $y$-polarizations can see different refractive index curves.

Hence, the refractive index of a material can depend on the orientation of the material relative to the polarization axis!

## Uniaxial crystals have an optic axis



Uniaxial crystals have one refractive index for light polarized along the optic axis ( $\mathrm{n}_{\mathrm{e}}$ ) and another for light polarized in either of the two directions perpendicular to it $\left(\mathrm{n}_{0}\right)$.

Light polarized along the optic axis is called the extraordinary ray, and light polarized perpendicular to it is called the ordinary ray. These polarization directions are the crystal "principal axes."

Light with any other polarization must be broken down into its ordinary and extraordinary components, considered individually, and added back together afterward.

## Birefringence can separate the two polarizations into separate beams

Due to Snell's Law, light of different polarizations will refract by different amounts at an interface.


## Birefringent Materials

Refractive Indices of Uniaxial
Crystals


$$
\left(20^{\circ} \mathrm{C} ; \lambda=589.3 \mathrm{~nm}\right)
$$

| Material | $n_{o}$ | $n_{e}$ |
| :--- | :---: | :---: |
| Tourmaline | 1.669 | 1.638 |
| Calcite | 1.6584 | 1.4864 |
| Quartz | 1.5443 | 1.5534 |
| Sodium nitrate | 1.5854 | 1.3369 |
| Ice | 1.309 | 1.313 |
| Rutile | 2.616 | 2.903 |

Calcite is one of the most birefringent materials known. It is particularly useful because it's also transparent over the entire visible spectrum and even into the UV (~300nm).

## Some polarizers use birefringence.

The Wollaston polarizing beam splitter uses refraction in two birefringent prisms, the second rotated by $90^{\circ}$ with respect to the first.


The x-polarization goes from e (low-n) to o (high- $n$ ) and so bends toward the normal (upward).

The y-polarization goes from o (high- $n$ ) to e (low-n) and so bends away from the normal (downward).

This effectively splits the two orthogonal polarization components into two separate (diverging) output beams.

## The measure of a polarizer

The ideal (perfect) polarizer passes 100\% of the desired polarization and $0 \%$ of the undesired polarization. Real polarizers aren't perfect.

The ratio of the transmitted irradiance through polarizers oriented parallel and then crossed is the Extinction ratio or Extinction coefficient.

We'd like it to be $\infty$.


The second polarizer is often called an "analyzer."

| Type of polarizer | Ext. Ratio | Transmission | Cost |
| :---: | :---: | :---: | :---: |
| Calcite: | $10^{6}$ | > 95\% | \$1000-\$2000 |
| Dielectric/wire grid: | $10^{3}$ | > 99\% | \$100-\$200 |
| Polaroid sheet: | $10^{3}$ | ~ $50 \%$ | \$1-\$2 |

