Supermodular covering knapsack polytope[☆]

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ARTICLE INFO

ABSTRACT

The supermodular covering knapsack set is the discrete upper level set of a non-decreasing supermodular function. Submodular and supermodular knapsack sets arise naturally when modeling utilities, risk and probabilistic constraints on discrete variables. In a recent paper Atamtürk and Narayanan (2009) study the lower level set of a non-decreasing submodular function.

In this complementary paper we describe pack inequalities for the supermodular covering knapsack set and investigate their separation, extensions and lifting. We give sequence-independent upper bounds and lower bounds on the lifting coefficients. Furthermore, we present a computational study on using the polyhedral results derived for solving 0–1 optimization problems over conic quadratic constraints with a branch-and-cut algorithm.

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1. Introduction

A set function $f : 2^N \to \mathbb{R}$ is supermodular on finite ground set N [22] if

$$f(S) + f(T) \leq f(S \cup T) + f(S \cap T), \quad \forall S, T \subseteq N.$$

By abuse of notation, for $S \subseteq N$ we refer to $f(S)$ also as $f(\chi_S)$, where χ_S denotes the binary characteristic vector of S. Given a non-decreasing supermodular function f on N and $d \in \mathbb{R}$, the *supermodular covering knapsack* set is defined as

$$K := \left\{ \mathbf{x} \in \{0,1\}^N : f(\mathbf{x}) \geq d \right\},$$

that is, the discrete upper level set of f. Since $K \subseteq \{0,1\}^N$, its convex hull, conv(K), is a polyhedral set.

Our main motivation for studying the supermodular covering knapsack set is to address linear 0–1 covering constraints with uncertain coefficients. If the coefficients $\tilde{u}_i, i \in N$, of the constraint are random variables, then a probabilistic (chance) constraint

$$\textbf{Prob}(\tilde{\textbf{u}}'\textbf{x} \geq d) \geq 1 - \epsilon \qquad (1)$$

on $x \in \{0, 1\}^N$ with, $0 < \epsilon < 0.5$, can be modeled as a conic quadratic 0–1 covering knapsack

$$K_{CQ} := \left\{ x \in \{0, 1\}^N : \textbf{u}'\textbf{x} - \Omega\|\textbf{D}\textbf{x}\| \geq d \right\},$$

where u_i is a nominal value and d_i is a deviation statistic for $\tilde{u}_i, i \in N, \textbf{D} = \text{diag}(d_1, d_2, \ldots, d_{|N|}), \Omega > 0$. Indeed, the set of 0–1 solutions for the probabilistic covering constraint (1) is precisely K_{CQ} for normally distributed independent random variables $\tilde{u}_i$, by letting u_i and d_i be the mean and standard deviation of $\tilde{u}_i$, and $\Omega(\epsilon) = -\phi^{-1}(\epsilon)$ with $0 < \epsilon < 0.5$, where ϕ is the standard normal CDF [20]. On the other hand, if $\tilde{u}_i$'s are known only through their first two moments u_i and σ_i^2, then any point in K_{CQ} with $\Omega(\epsilon) = \sqrt{(1-\epsilon)/\epsilon}$ satisfies the probabilistic constraint (1) [15,23]. Alternatively, if $\tilde{u}_i$'s are only known to be symmetric with support $[u_i - d_i, u_i + d_i]$, then points in K_{CQ} with $\Omega(\epsilon) = \sqrt{\ln(1/\epsilon)}$ satisfy constraint (1) [13,14]. Therefore, under various models of uncertainty, one arrives at different instances of the conic quadratic covering knapsack set K_{CQ}.

For a vector $\textbf{v} \in \mathbb{R}^N$ and $S \subseteq N$ we use $\textbf{v}(S)$ to denote $\sum_{i \in S} v_i$. Now, consider $f : 2^N \to R$ defined as

$$f(S) = \textbf{u}(S) - g(\textbf{c}(S)), \qquad (2)$$

where $g : \mathbb{R} \to \mathbb{R}$ is a concave function and $\textbf{u}, \textbf{c} \in \mathbb{R}^N$. It is easily checked that if $\textbf{c} \geq \textbf{0}$, then f is supermodular on N (e.g. Ahmed and Atamtürk [2]). Letting $c_i = \Omega^2 d_i^2$ for $i \in N$, we see that

$$f(S) = \textbf{u}(S) - \sqrt{\textbf{c}(S)} \geq d \qquad (3)$$

if and only if $\chi_S \in K_{CQ}$. Moreover, f is non-decreasing if $u_i \geq \Omega d_i$ for $i \in N$.

Although the polyhedral results in this paper are for the more general supermodular covering knapsack polytope $\text{conv}(K)$, we give examples and a separation algorithm for a specific set function of form (2). Because K reduces to the linear 0–1 covering knapsack set when f is modular, optimization over K is $\mathcal{NP}$-hard.

For notational simplicity, we denote a singleton set $\{i\}$ with its unique element i. For an ordered set $\{s_1, s_2, \ldots, s_n\}$, the subset $\{s_i, s_{i+1}, \ldots, s_j\} = \emptyset, \forall i > j$. For a set function f on N and $i \in N$, let its difference function be

$$\rho_i(S) := f(S \cup i) - f(S) \quad \text{for } S \subseteq N \setminus i.$$

Note that f is supermodular if and only if $\rho_i(S) \leq \rho_i(T), \ \forall S \subseteq T \subseteq N \setminus i$ and $i \in N$; that is, the difference function ρ_i is non-decreasing on $N \setminus i$ (e.g. Schrijver [37]). Furthermore, f is non-decreasing on N if and only if $\rho_i(\cdot) \geq 0$ for all $i \in N$.

Relevant literature. In a closely related paper, Atamtürk and Narayanan [6] study the lower level set of a non-decreasing submodular function. Negating inequality (3) yields a knapsack set with non-increasing submodular function $-f$, and, therefore, their results are not applicable here. Indeed, as the upper level set of a non-decreasing supermodular function is equivalent to the lower level set of a non-increasing submodular function, the current paper closes a gap by covering the case complementary to the one treated in Atamtürk and Narayanan [6].

Although there is a rich body of literature in approximation algorithms for submodular or supermodular functions, polyhedral results are scarce. Nemhauser et al. [36], Sviridenko [38], Iwata and Nagano [28], Lee et al. [32] give approximation algorithms for optimizing submodular/supermodular functions over various constraints. There is an extensive literature on the polyhedral analysis of the linear knapsack set. The polyhedral analysis of the linear knapsack set was initiated by Balas [9], Hammer et al. [25], and Wolsey [39]. For

a recent review of the polyhedral results on the linear knapsack set we refer the reader to Atamtürk [4,5]. Martello and Toth [33] present a survey of solution procedures for linear knapsack problems. Covering knapsack has also been extensively studied in the purview of approximation algorithms and heuristics [21,16]. Carnes and Shmoys [18] study the flow cover inequalities (Aardal, Pochet and Wolsey [1]) in the context of the deterministic minimum knapsack problem. The majority of the research on the nonlinear knapsack problem is devoted to the case with separable nonlinear functions (Morin [35]). Hochbaum [27] maximizes a separable concave objective function, subject to a packing constraint. There are fewer studies on the nonseparable knapsack problem, most notably on the knapsack problem with quadratic objective and linear constraint. Helmberg et al. [26] give semidefinite programming relaxations of knapsack problems with quadratic objective. Ahmed and Atamtürk [2] consider maximizing a submodular function over a linear knapsack constraint. We refer the reader to also Bretthauer et al. [17], Kellerer [29] for a survey of nonlinear knapsack problems.

On the conic integer optimization Atamtürk and Narayanan [7] describe a general mixed-integer rounding approach for conic quadratic mixed-integer sets. Atamtürk and Narayanan [8] describe lifting techniques for conic discrete optimization. Çezik and Iyengar [19] study techniques for generating valid convex constraints for mixed 0–1 conic programs and show that many of the techniques developed for generating linear cuts for linear mixed 0–1 optimization, such as the Gomory cuts, the lift-and-project cuts, and cuts from other hierarchies of tighter relaxations, extend to conic mixed 0–1 optimization. Belotti et al. [12] give conic cuts for conic quadratic integer optimization. Anderson and Jensen [3] give intersection cuts for conic quadratic mixed-integer sets. Kılınç [30] describes minimal inequalities for conic mixed-integer programs. Modaresi et al. [34] give split cuts and extended formulations for conic quadratic mixed-integer programming. Kılınç-Karzan and Yıldız [31] describe two-term disjunction inequalities for the second-order cone. These papers are on general conic quadratic discrete optimization and do not exploit any special structure associated with the problem studied here.

Outline. The rest of the paper is organized as follows: Section 2 describes the main polyhedral results. It includes pack inequalities, their extensions and lifting. The lifting problems of the pack inequalities are themselves optimization problems over supermodular covering knapsack sets. We derive sequence-independent upper bounds and lower bounds on the lifting coefficients. In Section 3 we give a separation algorithm for the pack inequalities for the conic quadratic case. In Section 4 we present a computational study on using the results for solving 0–1 optimization problems with conic quadratic constraints.

2. Polyhedral analysis

In this section we analyze the facial structure of the supermodular knapsack covering polytope. In particular, we introduce the pack inequalities and discuss their extensions and lifting. Throughout the rest of the paper we make the following assumptions:

(A.1) f is non-decreasing,
(A.2) $f(\emptyset) = 0$,
(A.3) $f(N \setminus i) \geq d$ for all $i \in N$.

Because f is supermodular, assumption (A.1) is equivalent to $\rho_i(\emptyset) \geq 0, \ \forall i \in N$, which can be checked easily. Assumption (A.1) holds, for instance, for a function f of the form (3) if $u_i \geq \Omega d_i, \forall i \in N$. Assumption (A.2) can be made without loss of generality as f can be translated otherwise. Finally, if (A.3) does not hold, i.e., $\exists i \in N : f(N \setminus i) < d$, then x_i equals one in every feasible solution. We start with a few basic results that easily follow from the assumptions above.

Proposition 1. Conv(K) *is a full-dimensional polytope.*

Proposition 2. *Inequality $x_i \leq 1, i \in N$, is facet-defining for $\mathrm{conv}(K)$.*

Proposition 3. *Inequality $x_i \geq 0, i \in N$, is facet-defining for $\mathrm{conv}(K)$ if and only if $f(N \setminus \{i,j\}) \geq d, \forall\, j \in N \setminus \{i\}$.*

We refer to the facets defined in Propositions 2–3 as the *trivial facets* of $\mathrm{conv}(K)$.

Proposition 4. *If inequality $\sum_{j \in N} \pi_j x_j \geq \pi_0$ defines a non-trivial facet of $\mathrm{conv}(K)$, then $\pi_0 > 0$ and $0 \leq \pi_j \leq \pi_0, \forall\, j \in N$.*

2.1. Pack inequalities

In this section we define the first class of valid inequalities for K.

Definition 1. A subset P of N is a *pack* for K if $\delta := d - f(P) > 0$. A pack P is maximal if $f(P \cup i) \geq d$, $\forall\, i \in N \setminus P$.

For a pack $P \subset N$ for K, let us define the corresponding *pack inequality* as

$$\mathbf{x}(N \setminus P) \geq 1. \tag{4}$$

The pack inequality simply states that at least one element outside the pack P has to be picked to satisfy the knapsack cover constraint $f(\mathbf{x}) \geq d$. Consider the non-empty restriction $K(P) = \{\mathbf{x} \in K : x_i = 1, \forall\, i \in P\}$ of K.

Proposition 5. *If $P \subset N$ is a pack for K, then the pack inequality (4) is valid for K. Moreover, it defines a facet of $\mathrm{conv}(K(P))$ iff P is a maximal pack.*

Proof. Define $\bar{K} := \{\mathbf{x} \in \{0,1\}^N : \mathbf{x}(N \setminus P) < 1\}$. It is sufficient to show that $f(\mathbf{x}) < d$ for all $\mathbf{x} \in \bar{K}$. Since $\forall\, \mathbf{x} \in \bar{K}$, we have $\mathbf{x}(N \setminus P) = 0$, implying $\mathbf{x} \leq \mathbf{y}$, $\forall\, \mathbf{x} \in \bar{K}$, and $\forall\, \mathbf{y} \in K(P)$; implying

$$f(\mathbf{x}) \leq f(P) < d,$$

where the first inequality follows from assumption (A.1), that f is non-decreasing.

For the second part, consider the $|N \setminus P|$ points

$$\mathbf{x}^k \in \{0,1\}^N \quad \text{such that } x_j^k = \begin{cases} 1 & \text{if } j \in P \cup k, \\ 0 & \text{if } j \in N \setminus \{P \cup k\}, \end{cases} \quad \forall\, k \in N \setminus P. \tag{5}$$

Since f is non-decreasing and P is a maximal pack, we have $\mathbf{x}^k \in K$, $\forall\, k \in N \setminus P$. The $|N \setminus P|$ points $\mathbf{x}^k$, $k \in N \setminus P$ are in $\mathrm{conv}(K(P))$ and satisfy (4) as equality. It is easily seen that these $|N \setminus P|$ points are linearly independent. Hence for a maximal pack P, (4) defines a facet of $\mathrm{conv}(K(P))$.

Conversely suppose that pack P is not maximal. Thus, $\exists\, i \in N \setminus P$ such that $f(P \cup i) < d$. Then the corresponding valid pack inequality

$$\mathbf{x}(N \setminus (P \cup i)) \geq 1$$

and $x_i \geq 0$ dominate (4). $\quad\square$

Example. Consider the conic-quadratic covering knapsack set

$$K = \left\{ \mathbf{x} \in \{0,1\}^4 : x_1 + 2.5x_2 + 3x_3 + 3x_4 - \sqrt{x_3^2 + x_4^2} \geq 5.5 \right\}.$$

The maximal packs for K and the corresponding pack inequalities are

$$\{1,2\} : x_3 + x_4 \geq 1 \qquad \{1,3\} : x_2 + x_4 \geq 1 \qquad \{1,4\} : x_2 + x_3 \geq 1$$
$$\{2,3\} : x_1 + x_4 \geq 1 \qquad \{2,4\} : x_1 + x_3 \geq 1 \qquad \{3,4\} : x_1 + x_2 \geq 1.$$

2.2. Extended pack inequalities

The pack inequalities (4), typically, do not define facets of $\mathrm{conv}(K)$; however, they can be strengthened by extending them with the elements of the pack. Though unlike in the linear case, for the supermodular covering knapsack set, even simple extensions are sequence-dependent. Proposition 6 describes such an extension of the pack inequalities (4).

Definition 2. Let $P \subset N$ be a pack and $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_{|P|})$ be a permutation of the elements of P. Define $P_i := P \setminus \{\pi_1, \pi_2, \ldots, \pi_i\}$ for $i = 1, \ldots, |P|$ with $P_0 = P$. The *reduction* of P with respect to $\boldsymbol{\pi}$ is defined as $R_{\boldsymbol{\pi}}(P) := P \setminus U_{\boldsymbol{\pi}}(P)$, where

$$U_{\boldsymbol{\pi}}(P) := \left\{ \pi_j \in P : \max_{i \in N \setminus P} \rho_i(N \setminus i) \leq \rho_{\pi_j}(P_j) \right\}. \tag{6}$$

For a given pack P and reduction $R_{\boldsymbol{\pi}}(P) = P \setminus U_{\boldsymbol{\pi}}(P)$, we define the *extended pack inequality* as

$$\mathbf{x}(N \setminus R_{\boldsymbol{\pi}}(P)) \geq |U_{\boldsymbol{\pi}}(P)| + 1. \tag{7}$$

Proposition 6. *If $P \subset N$ is a pack for K and $U_{\boldsymbol{\pi}}(P)$ is defined as in* (6)*, then the extended pack inequality* (7) *is valid for K.*

Proof. Let $L \subseteq N \setminus R_{\boldsymbol{\pi}}(P)$ with $|L| \leq |U_{\boldsymbol{\pi}}(P)|$. To prove the validity of (7) it suffices to show that $f(R_{\boldsymbol{\pi}}(P) \cup L) < d$. Let $J = U_{\boldsymbol{\pi}}(P) \setminus L =: \{j_1, j_2, \ldots, j_{|J|}\}$ be indexed consistently with π. Note that for $Q = U_{\boldsymbol{\pi}}(P) \cap L$, we have $|L \setminus Q| \leq |J|$. Then

$$
\begin{aligned}
f(R_{\boldsymbol{\pi}}(P) \cup L) &= f(R_{\boldsymbol{\pi}}(P) \cup Q) + \rho_{L \setminus Q}(R_{\boldsymbol{\pi}}(P) \cup Q) \\
&\leq f(R_{\boldsymbol{\pi}}(P) \cup Q) + \sum_{\ell \in L \setminus Q} \rho_\ell(N \setminus \ell) \\
&\leq f(R_{\boldsymbol{\pi}}(P) \cup Q) + \sum_{\pi_j \in J} \rho_{\pi_j}(P_j) \\
&\leq f(R_{\boldsymbol{\pi}}(P) \cup Q) + \sum_{j_i \in J} \rho_{j_i}\left(R_{\boldsymbol{\pi}}(P) \cup Q \cup \{j_{i+1}, \ldots, j_{|J|}\}\right) \\
&= f(P) < d,
\end{aligned}
$$

where the first and third inequalities follow from supermodularity of f and the second one from (6), $|L \setminus Q| \leq |J|$, and (A.1). $\square$

We now provide a sufficient condition for the extended pack inequality to be facet-defining for $\mathrm{conv}(K(R_{\boldsymbol{\pi}}(P)))$.

Proposition 7. *The extended pack inequality* (7) *is facet-defining for* $\mathrm{conv}(K(R_{\boldsymbol{\pi}}(P)))$ *if P is a maximal pack and for each $i \in U_{\boldsymbol{\pi}}(P)$ there exist distinct $j_i, k_i \in N \setminus P$ such that $f(P \cup \{j_i, k_i\} \setminus i) \geq d$.*

Proof. Consider the points $\chi_{P \cup i}, \forall i \in N \setminus P$ and $\chi_{P \cup \{j_i, k_i\} \setminus i}, \forall i \in U_{\boldsymbol{\pi}}(P)$ and $j_i, k_i \in N \setminus P$, which are on the face defined by (7). The proof will be completed by showing that these $|N \setminus P| + |U_{\boldsymbol{\pi}}(P)|$ points are linearly independent. Let M be the matrix containing these points as rows. Observe that M can be

represented as

$$M = \begin{pmatrix} \mathbb{1}_{n \times m} & \mathrm{Id}_n \\ \mathbb{1}_{m \times m} - \mathrm{Id}_m & \mathrm{H}_{m \times n} \end{pmatrix}$$

where $n = |N \setminus P|$ and $m = |U_{\boldsymbol{\pi}}(P)|$. Here $\mathbb{1}_{n \times m}$ denotes an $n \times m$ matrix all of whose entries are 1. Id_n refers to the $n \times n$ identity matrix and $\mathrm{H}_{m \times n}$ is an $m \times n$ binary matrix ($H_{ij} \in \{0, 1\}$, $1 \leq i \leq m$, $1 \leq j \leq n$) such that all of its rows sum to two $\left(\sum_{j=1}^{n} H_{ij} = 2, \forall 1 \leq i \leq m \right)$. Now, M is non-singular if and only if

$$M \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = \mathbf{0} \quad \Rightarrow \quad \boldsymbol{\alpha}, \boldsymbol{\beta} = \mathbf{0}, \tag{8}$$

where $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are vectors of length of m and n, respectively. The solutions to (8) thus satisfy

$$\mathbb{1}_{n \times m} \boldsymbol{\alpha} + \boldsymbol{\beta} = \mathbf{0}, \tag{9}$$
$$(\mathbb{1}_{m \times m} - \mathrm{Id}_m) \boldsymbol{\alpha} + \mathrm{H}_{m \times n} \boldsymbol{\beta} = \mathbf{0}. \tag{10}$$

Substituting for $\boldsymbol{\beta}$, in (10) from (9) yields

$$(\mathbb{1}_{m \times m} - \mathrm{Id}_m) \boldsymbol{\alpha} - \mathrm{H}_{m \times n} \mathbb{1}_{n \times m} \boldsymbol{\alpha} = \mathbf{0}$$
$$(\mathbb{1}_{m \times m} - \mathrm{Id}_m) \boldsymbol{\alpha} - 2\mathbb{1}_{m \times m} \boldsymbol{\alpha} = \mathbf{0}$$
$$(\mathbb{1}_{m \times m} + \mathrm{Id}_m) \boldsymbol{\alpha} = \mathbf{0}.$$

Thus the problem of proving non-singularity of M boils down to proving non-singularity of $\mathbb{1}_{m \times m} + \mathrm{Id}_m$, which is evident from elementary row operations. □

Example (*Cont.*)**.** Consider the conic-quadratic covering knapsack set in the previous example:

$$K = \left\{ \mathbf{x} \in \{0, 1\}^4 : x_1 + 2.5x_2 + 3x_3 + 3x_4 - \sqrt{x_3^2 + x_4^2} \geq 5.5 \right\}.$$

For the maximal pack $P = \{3, 4\}$, we gave the corresponding pack inequality

$$x_1 + x_2 \geq 1.$$

For permutation $\boldsymbol{\pi} = (3, 4), P_1 = \{4\}$ and $P_2 = \emptyset$. As $\rho_3(P_1) = 4 - \sqrt{2} \approx 2.586$ and $\rho_1(\{2, 3, 4\}) = 1$, $\rho_2(\{1, 3, 4\}) = 2.5$, the corresponding reduction $R_{(3,4)}(P) = \{4\}$ gives the extended pack inequality

$$x_1 + x_2 + x_3 \geq 2. \tag{11}$$

Alternatively, $\boldsymbol{\pi} = (4, 3)$ yields the reduction $R_{(4,3)}(P) = \{3\}$ and the corresponding extended pack inequality

$$x_1 + x_2 + x_4 \geq 2. \tag{12}$$

Observe that inequalities (11) and (12) are the non-trivial facets of $\mathrm{conv}(K(4))$ and $\mathrm{conv}(K(3))$, respectively.

2.3. Lifted pack inequalities

In this section we study the lifting problem of the pack inequalities in order to strengthen them. Lifting has been very effective in strengthening inequalities for the linear 0–1 knapsack set [9–11,24,25,39]. The lifting problem for the pack inequalities for K is itself an optimization problem over the supermodular covering knapsack set.

Precisely, we lift the pack inequality (4) to a valid inequality of the form

$$x(N \setminus P) - \sum_{i \in P} \alpha_i (1 - x_i) \geq 1. \tag{13}$$

The lifting coefficients α_i, $i \in P$ can be computed iteratively in some sequence: Suppose the pack inequality (4) is lifted with variables x_i, $i \in J \subseteq P$ to obtain the intermediate valid inequality

$$x(N \setminus P) - \sum_{i \in J} \alpha_i (1 - x_i) \geq 1 \tag{14}$$

in some sequence of J, then x_k, $k \in P \setminus J$, can be introduced to (14) by computing

$$\alpha_k = \varphi(I, k) - 1 - \boldsymbol{\alpha}(J), \tag{15}$$

where $\varphi(I, k)$ is the optimal objective value of the following lifting problem, $L(I, k)$:

$$\varphi(I, k) := \min_{T \subseteq I} \left\{ |(N \setminus P) \cap T| + \sum_{i \in J \cap T} \alpha_i : f(T \cup P \setminus (J \cup k)) \geq d \right\} \tag{16}$$

and

$$I = (N \setminus P) \cup J.$$

The lifting coefficients are typically a function of the sequence used for lifting. The extension given in Proposition 6 may be seen as a simple approximation of the lifted inequalities (13).

Proposition 8. *If $P \subset N$ is a pack for K, and α_i, $\forall i \in P$ are defined as in (15), then the lifted pack inequality (13) is valid for K. Moreover, inequality (13) defines a facet of $\mathrm{conv}(K)$ if P is a maximal pack.*

Corollary 9. *The lifted pack inequality*

$$x(N \setminus P) - \sum_{i \in P} \widehat{\alpha}_i (1 - x_i) \geq 1, \tag{17}$$

where $\widehat{\alpha}_k = \lceil \widehat{\varphi}(I, k) \rceil - 1 - \widehat{\boldsymbol{\alpha}}(J)$, $k \in P \setminus J$ and $\widehat{\varphi}(I, k)$ is any lower bound on $\varphi(I, k)$, is valid for K.

Computing the lifting coefficients α_k, $k \in P$, exactly may be computationally prohibitive in general as the feasible set of the lifting problem (16) is defined over a supermodular covering knapsack. For a deeper understanding of the structure of the lifted inequalities, it is of interest to identify bounds on the lifting coefficients that are independent of a chosen lifting sequence. As we shall see later, these bounds may help to generate approximate lifting coefficients quickly. We start with the following lemma.

Lemma 10. *Let $P \subset N$ be a maximal pack with $\delta := d - f(P)(> 0)$ and for $h = 0, 1, 2, 3, \ldots, |N \setminus P|$, define*

$$\mu_h := \max \{ f(T \cup P) : |T| = h, \ T \subseteq N \setminus P \} \tag{18}$$

$$\nu_h := \min \{ f(T \cup P) : |T| = h, \ T \subseteq N \setminus P \}. \tag{19}$$

Then, for all $h = 0, 1, 2, 3, \ldots, |N \setminus P| - 1$, the following inequalities hold:

(i) $\nu_{h+1} \geq \nu_h + \delta$,
(ii) $\mu_{h+1} \geq \mu_h + \delta$.

Proof. Since P is a maximal pack, $\rho_k(P) \geq \delta$, $\forall k \in N \setminus P$.

(i) Let T_{h+1}^* be an optimal solution corresponding to (19) and let $k \in T_{h+1}^*$. Then by supermodularity of f and maximality of P, we have

$$\delta \leq \rho_k(P) \leq \rho_k \left((T_{h+1}^* \setminus k) \cup P \right)$$
$$= f(T_{h+1}^* \cup P) - f \left((T_{h+1}^* \setminus k) \cup P \right).$$

Adding ν_h to both sides yields,

$$\delta + \nu_h \leq f(T^*_{h+1} \cup P) = \nu_{h+1}.$$

(ii) Let T^*_h be an optimal solution to (18) and let $k \in N \setminus \{P \cup T^*_h\}$. It follows from supermodularity of f and maximality of P that

$$\delta \leq \rho_k(P) \leq \rho_k(T^*_h \cup P)$$
$$\leq f(T^*_h \cup k \cup P) - f(T^*_h \cup P).$$

Adding μ_h to both sides yields,

$$\delta + \mu_h \leq f(T^*_h \cup k \cup P) \leq \mu_{h+1}.$$

In summary, for a maximal pack $P, \nu_h \leq f(T \cup P), \forall T \subseteq N \setminus P$ with $|T| \geq h$, and $\mu_h \geq f(T \cup P), \forall T \subseteq N \setminus P$ with $|T| \leq h$. $\square$

Proposition 11 is inspired by a similar result by Balas [9] for the linear 0–1 knapsack problem.

Proposition 11. *Let $P \subset N$ be a pack with $\delta := d - f(P) > 0$ and μ_h and $\nu_h, h = 0, 1, 2, 3, \ldots, |N \setminus P|$ be defined as in (18) and (19). Suppose that the lifted pack inequality*

$$x(N \setminus P) - \sum_{i \in P} \alpha_i(1 - x_i) \geq 1 \tag{20}$$

defines a facet of conv(K). For any $i \in P$, the following statements hold:

(i) *if $\rho_i(\emptyset) \geq f(N) - \nu_{|N \setminus P| - h}$, then $\alpha_i \geq h$;*
(ii) *if $\rho_i(N \setminus i) \leq \mu_{1+h} - d$, then $\alpha_i \leq h$.*

Proof. (i) The lifting coefficient of $x_i, i \in P$, is the smallest if x_i is the last variable introduced to (20) in a lifting sequence. Let $\alpha_i = \varphi(N \setminus i, i) - 1 - \boldsymbol{\alpha}(P \setminus i)$. Also, because the intermediate lifting inequality before introducing x_i is valid for K, we have $\varphi(N \setminus i, \emptyset) \geq 1 + \boldsymbol{\alpha}(P \setminus i)$. Thus, it is sufficient to show that $\varphi(N \setminus i, i) - \varphi(N \setminus i, \emptyset) \geq h$.

We claim that in any feasible solution S to the lifting problem $L(N \setminus i, i)$ (when x_i is lifted last), at least $h + 1$ variables in $N \setminus P$ are positive. For contradiction, suppose that at most h variables in $N \setminus P$ are positive. Let $J \subseteq N \setminus P$ and $\tilde{P} \subseteq P \setminus i$ be such that $S = J \cup \tilde{P}$. We have

$$f(J \cup \tilde{P}) \leq f(J \cup P \setminus i)$$
$$= f(J \cup P) - \rho_i(J \cup P \setminus i)$$
$$\leq f(J \cup P) - \rho_i(\emptyset)$$
$$\leq f(J \cup P) - f(N) + \nu_{|N \setminus P| - h}$$
$$= f(P) + \rho_J(P) - f(N) + \nu_{|N \setminus P| - h}$$
$$\leq f(P) + \rho_J(N \setminus J) - f(N) + \nu_{|N \setminus P| - h}$$
$$= f(P) - f(N \setminus J) + \nu_{|N \setminus P| - h}$$
$$\leq f(P) < d,$$

where the penultimate inequality follows from the fact that $f(N \setminus J) \geq \nu_{|N \setminus P| - h}, \forall J \subseteq N \setminus P, |J| \leq h$. Thus, χ_S is infeasible for $L(N \setminus i, i)$.

Now let $S^* = J^* \cup P^*$ with $J^* \subseteq N \setminus P, P^* \subseteq P \setminus i$, be an optimal solution to $L(N \setminus i, i)$. Let $J \subset J^*$ be such that $|J| = h$. The existence of such a J is guaranteed by the argument in previous paragraph. We claim that $S^* \setminus J$ is a feasible solution to $L(N \setminus i, \emptyset)$. To see this, observe that

$$f((S^* \setminus T) \cup i) \geq f(S^* \setminus T) + f(i)$$
$$= f(S^* \setminus T) + \rho_i(\emptyset)$$
$$\geq f(S^*) - f(N) + f(N \setminus T) + \rho_i(\emptyset)$$
$$\geq f(S^*) - f(N) + \nu_{|N \setminus P| - h} + \rho_i(\emptyset)$$
$$\geq f(S^*) \geq d,$$

where the third inequality follows from the supermodularity of f and the penultimate inequality follows from our assumption $\rho_i(\emptyset) \geq f(N) - \nu_{|N \setminus P| - h}$. Thus, we see that $\varphi(N \setminus i, i) - \varphi(N \setminus i, \emptyset) \geq |T| = h$.

(ii) For this part, it is sufficient to show that if the pack inequality (4) is lifted first with x_i, then $\alpha_i \leq h$. Consider the lifting problem, $L_i(N \setminus P)$. Let $T \subseteq N \setminus P, |T| = h + 1$, such that $f(T \cup P) = \mu_{h+1}$. We claim that T is feasible for $L_i(N \setminus P)$. Consider the following

$$f(T \cup P \setminus i) = f(T \cup P) - \rho_i(T \cup P \setminus i)$$
$$\geq \mu_{h+1} - \rho_i(N \setminus i)$$
$$\geq d.$$

Hence an optimal solution to $L_i(N \setminus P)$ has at most $h + 1$ variables positive, i.e., $\varphi(N \setminus P, i) \leq h + 1$. Thus we have $\alpha_i = \varphi(N \setminus P, i) - 1 \leq h$. $\square$

Computing the bounds μ_h and $\nu_h, h = 1, \ldots, |N \setminus P|$ is $\mathcal{NP}$-hard as they require minimizing and maximizing supermodular functions over a cardinality restriction. Nevertheless, Lemma 10 and Proposition 11 can be utilized together in order to derive approximate lifted inequalities efficiently as μ_1, ν_1 and $\mu_{|N \setminus P| - 1}, \nu_{|N \setminus P| - 1}$ can be computed in linear time by enumeration.

Proposition 11 yields that for a maximal pack P if for any $i \in P, \rho_i(N \setminus i) \leq \mu_1 - d$, then the corresponding lifting coefficient α_i for x_i is zero and thus x_i can be dropped from consideration for extensions and lifting of the pack inequality. Similarly, if for any $i \in P, \rho_i(\emptyset) \geq f(N) - \nu_{|N \setminus P| - 1}$, then the lifting coefficient α_i of x_i is at least one and thus x_i can be included in every extension or lifting of the pack inequality. Also, if $\rho_i(\emptyset) \geq f(N) - \nu_1, i \in P$, then the corresponding lifting coefficient is set to $|N \setminus P| - 1$. Furthermore, Proposition 11 and a repeated application of Lemma 10 suggest the following corollary.

Corollary 12. *For $h = 1, \ldots, |N \setminus P| - 1$*

(1) *if $\rho_i(\emptyset) \geq f(N) - \nu_1 - \delta(|N \setminus P| - h - 1)$, then $\alpha_i \geq h$.*
(2) *if $\rho_i(N \setminus i) \leq \mu_1 + h\delta - d$, then $\alpha_i \leq h$.*

3. Separation

In this section, we give a separation algorithm for the pack inequalities for the supermodular covering knapsack set K defined with respect to any supermodular set function f assuming the functional form (2) with g concave and increasing on $\mathbb{R}_+$ and $\mathbf{c} \geq \mathbf{0}$. Observe that conic quadratic supermodular function defining K_{CQ} assumes this functional form.

Given $\bar{\mathbf{x}} \in \mathbb{R}^N$ such that $0 \leq \bar{\mathbf{x}} \leq 1$, we are interested in finding a pack P with $\sum_{i \in N \setminus P} \bar{x}_i < 1$, if there exists any. Then, the separation problem with respect to the pack inequalities can be formulated as

$$\zeta = \min \left\{ \bar{\mathbf{x}}'(\mathbb{1} - \mathbf{z}) : \mathbf{u}'\mathbf{z} - g(\mathbf{c}'\mathbf{z}) < d, \, z \in \{0,1\}^N \right\}, \tag{21}$$

where the constraint $\mathbf{u}'\mathbf{z} - g(\mathbf{c}'\mathbf{z}) < d$ ensures that a feasible $\mathbf{z}$ corresponds to a pack. Thus, there is a violated pack inequality if and only if $\zeta < 1$.

In order to find violated pack inequalities quickly, we employ a heuristic that rounds off fractional solutions to the continuous relaxation of (21):

$$\max \left\{ \bar{\mathbf{x}}'\mathbf{z} : \mathbf{u}'\mathbf{z} - y \leq d, \, \mathbf{c}'\mathbf{z} \geq h(y), \, \mathbf{0} \leq \mathbf{z} \leq \mathbf{1}, \, y \in \mathbb{R} \right\}, \tag{22}$$

where h is the inverse of g (h exists as g is increasing). Because g is increasing concave, h is increasing convex; hence (22) is a convex optimization problem. Also, observe that, for a fixed value of $y \in \mathbb{R}$, there can be at most two fractional z_i, $i \in N$ in any extreme point solution to (22).

For the convex relation (22) let $\lambda \geq 0, \nu \leq 0, \boldsymbol{\alpha} \leq \mathbf{0}, \boldsymbol{\beta} \leq \mathbf{0}$ be the dual variables for the constraints in the order listed. From the first order optimality conditions

$$\bar{x}_i - \lambda u_i - \nu c_i - \alpha_i + \beta_i = 0, \quad \forall \, i \in N,$$
$$\lambda + \nu h'(y) = 0,$$

and the complementary slackness conditions

$$\alpha_i z_i = 0, \quad \forall \, i \in N,$$
$$\beta_i (z_i - 1) = 0, \quad \forall \, i \in N,$$

we see that optimal solutions satisfy

$$\bar{x}_i \begin{cases} \leq \lambda u_i + \nu c_i, & z_i = 0 \\ = \lambda u_i + \nu c_i, & 0 < z_i < 1 \\ \geq \lambda u_i + \nu c_i, & z_i = 1. \end{cases}$$

Since in an extreme point of (22) there are at most two variables with $0 < z_i, z_j < 1$, we compute $\binom{|N|}{2}$ candidate values for λ and ν, which are solutions of

$$\bar{x}_i = \lambda u_i + \nu c_i, \qquad \bar{x}_j = \lambda u_j + \nu c_j, \quad i, j \in N, \, i < j.$$

For candidate values (λ, ν) satisfying $\lambda \geq 0, \, \nu \leq 0$, we assign variables $z_i, i \in N$ equal to one, in the non-increasing order of $\bar{x}_i / (\lambda u_i + \nu c_i)$, until $\mathbf{z}$ defines a pack and check for the violation of the corresponding pack inequality.

4. Computational experiments

In this section we present our computational experiments on testing the effectiveness of the pack inequalities and their extensions for solving 0–1 optimization problems with conic quadratic covering knapsack constraints. For the computational experiments we use the MIP solver of CPLEX Version 12.5 that solves conic quadratic relaxations at the nodes of the branch-and-bound tree. CPLEX heuristics are turned off and a single thread is used. The search strategy is set to traditional branch-and-bound, rather than the default dynamic search as it is not possible to add user cuts in CPLEX while retaining the dynamic search strategy. In addition, the solver time limit and memory limit have been set to 3600 s and 1 GB, respectively. All experiments are performed on a 2.93 GHz Pentium Linux workstation with 8 GB main memory.

In Tables 1 and 2 we report the results of the experiments for varying number of variables (n), constraints (m), and values for Ω. For each combination, five random instances are generated with u_i from uniform

Table 1
Effect of cuts with barrier algorithm.

m	n	Ω	igap	cplex barrier					packs						extended packs					
				rgap	egap	nodes	time	#	cuts	rgap	egap	nodes	time	#	cuts	rgap	egap	nodes	time	#
		1	22.8	22.8	0	4,087	27	5	31	7.0	0	367	4	5	40	3.9	0	188	2	5
	50	3	22.4	22.4	0	4,673	29	5	28	8.6	0	338	4	5	31	5.5	0	120	2	5
		5	25.1	25.1	0	15,854	92	5	31	8.7	0	503	5	5	33	6.0	0	249	3	5
10		1	11.4	11.4	0	153,652	2673	2	25	7.6	0	27,394	599	5	30	6.6	0	15,214	345	5
	100	3	10.4	10.4	0	95,758	1688	5	24	7.1	0	16,167	324	5	28	5.9	0	11,737	254	5
		5	11.1	11.1	0	160,024	2726	2	22	8.1	0	50,734	1021	5	27	7.4	0	26,763	581	5
		1	18.5	18.5	0	74,833	1209	5	51	8.2	0	4,104	98	5	59	5.8	0	2,115	55	5
	50	3	21.1	21.1	0.4	129,208	1943	3	49	8.0	0	9,028	208	5	63	5.6	0	4,764	126	5
		5	21.5	21.5	0	68,563	1043	5	48	8.1	0	1,028	26	5	55	5.2	0	433	12	5
20		1	11.3	11.3	5.6	71,460	3589	0	27	9.1	4	58,659	3329	1	34	8.6	3.2	54,293	3146	1
	100	3	11.3	11.3	5.2	78,661	3589	0	35	9.1	4	66,359	3589	0	42	8.4	3.4	61,112	3589	0
		5	11.3	11.3	4.0	77,952	3589	0	31	9.0	3	61,461	3159	1	39	8.3	2.3	56,462	3085	1
Average				**16.5**	**1.27**	**77,894**	**1850**			**8.2**	**0.92**	**24,679**	**1031**			**6.4**	**0.74**	**19,454**	**933**	
Stdev				**5.82**						**0.72**						**1.46**				

Table 2
Effect of cuts with outer linear approximation.

m	n	Ω	igap	cplex outer approx.					packs						extended packs					
				rgap	egap	nodes	time	#	cuts	rgap	egap	nodes	time	#	cuts	rgap	egap	nodes	time	#
		1	22	8.9	0	1,942	1	5	64	4.8	0	587	1	5	71	3.6	0	260	1	5
	50	3	24.7	10.7	0	2,792	1	5	71	6.2	0	744	1	5	82	4.3	0	396	1	5
20		5	22.9	9.7	0	1,834	1	5	64	4.8	0	217	0	5	72	3.1	0	134	0	5
		1	11.7	7.0	0	517,570	575	5	57	6.8	0	195,698	229	5	62	6.8	0	151,307	157	5
	100	3	13.1	7.7	0	249,849	234	5	52	7.4	0	127,090	138	5	57	6.9	0	113,878	124	5
		5	12.4	7.0	0	242,291	211	5	50	6.7	0	106,121	126	5	61	6.4	0	94,831	105	5
		1	24.2	12.4	0	3,393	3	5	67	9.7	0	1,108	1	5	84	8.4	0	572	1	5
	50	3	23.6	10.2	0	1,335	1	5	66	7.2	0	481	1	5	81	5.9	0	121	1	5
30		5	23.9	10.8	0	2,241	2	5	67	7.3	0	368	1	5	78	6.0	0	274	1	5
		1	13.6	9.4	0.3	1,160,006	2159	4	61	8.9	0.3	755,204	1458	4	73	8.6	0.2	790,050	1585	4
	100	3	12.9	8.6	0.3	1,094,672	2238	2	67	8.2	0	686,178	1451	4	78	8.0	0	596,803	1195	5
		5	12.9	8.3	0	1,047,374	1903	5	64	8.2	0	636,519	1319	5	77	7.7	0	387,649	698	5
Average				**9.2**	**0.05**	**360,442**	**611**			**7.2**	**0.03**	**209,193**	**394**			**6.3**	**0.02**	**178.023**	**322**	
Stdev				**1.64**						**1.48**						**1.83**				

$[0, 100]$ and σ_i from uniform $[0, u_i/5]$. The covering knapsack right-hand-side constant d is set to $0.5\,\kappa$, where $\kappa = \max_{i \in N} f(N \setminus i)$. So that constraints are not completely dense, we set the density of the constraints to $20/\sqrt{n}$.

In Table 1 we compare the initial relaxation gap ($igap$), the root relaxation gap ($rgap$), the end gap ($egap$), the gap between best upper bound and lower bound at termination, the number of cuts generated ($cuts$), the number of nodes explored ($nodes$), the CPU time in seconds ($time$), and the number of instances solved to optimality (#) using the barrier algorithm and several cut generation options. The initial relaxation gap ($igap$) is computed as $\frac{(f_u - f_i)}{f_u}$, where f_i denotes the objective value of the initial relaxation and f_u denotes the objective of the best feasible solution found across all versions. The root gap ($rgap$) and the end gap ($egap$) are computed as $\frac{(f_u - f_r)}{f_u}$ and $\frac{(f_u - f_l)}{f_u}$, where f_r is the objective value of the relaxation at the root node and f_l is the best lower bound for the optimal objective at termination. The columns under heading cplex show the performance of CPLEX with no user cuts added. The other columns show the performance of the algorithm using maximal pack cuts and extended maximal pack cuts with preprocessing as described in Corollary 12. The pack inequalities and their extensions are added only at the root node of the search tree using the separation algorithm discussed in Section 3.

We observe in Table 1 that the addition of the pack cuts reduces the root gap and the number of nodes and leads to faster solution times. As expected, the extended pack cuts are more effective than the simpler pack cuts. On average, the root gap is reduced from 16.5% to 6.43% for all instances with the extended pack cuts. Using extended packs leads to a reduction of 49.5% in the solution times and 75% in the number of branch and bound nodes explored. For problems that could be solved by CPLEX alone, the average solution time is reduced from 769 s to mere 97 s. For problems that could not be solved by either of the three versions, the average end gap is reduced from 4.8% to 2.8% using the extended packs. Over all instances, the average number of nodes are 77,894, 24,679 and 19,454 for CPLEX with barrier algorithm without user cuts, with packs and extended packs, respectively. On the other hand, the average CPU times are 1850, 1031 and 933 s for CPLEX without user cuts, with packs and extended packs, respectively.

In Table 2 we present similar comparisons, but this time using the CPLEX linear outer approximation for solving conic quadratic problems at the nodes instead of the barrier algorithm. We observe, in this case, that CPLEX adds its own cuts from the linear constraints. Therefore, compared to Table 1, in general the root gaps are smaller and the solution times are faster. Adding extended pack cuts reduces the average root gap from 9.23% to 6.31%. This leads to 50.6% reduction in the number of search nodes and 47.2% reduction in the solution times. For larger instances that are not solved to optimality, the average end gap is reduced from 0.9% to 0.3%.

In conclusion, we find the pack inequalities and their extensions to be quite effective in strengthening the convex relaxations of the conic quadratic covering 0–1 knapsacks and reducing the solution times of optimization problems with such constraints.

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