Chapter 5

Linear equations and inequalities in two variables

Vocabulary

- xy-plane
- Plotting ordered pairs
- Graph
- Intercepts (x– and y–intercept of a line in an xy-plane)
- Slope of a line
- Parallel lines
- Perpendicular lines
- Horizontal lines
- Vertical lines
- Slope-intercept form of a linear equation in two variables
- Point-slope form of a linear equation in two variables
- System of linear equations

5.1 Solving linear equations in two variables

We now turn our attention to linear equations with two variables, which we will assume to be called x and y. A linear equation in two variables can always be written in a standard form

$$Ax + By = C,$$

where A and B are constant coefficients and C is a constant. What is "standard" about this form is that the terms involving variables are on one side of the equation, while the constant term (not involving variables) is on the other side of the equation. However, a linear equation may not be written in this standard form. In fact, we will soon see several situations in which it is better to write a linear equation in another form.

As with any algebraic statement, a linear equation in two variables may be true or false, depending on the values for both variables x and y. As we saw earlier in Section 4.1, a solution to a linear equation in two variables consists of a value for each of the two variables, which we indicate by writing them together as an ordered pair.

Let's start by looking at a relatively easy example of a linear equation in two variables:

$$x + y = 5.$$

It's easy to see a few examples of solutions to this equation: $(1, 4)$, $(2, 3)$, and $(3, 2)$, for example. With a little more thought, more exotic solutions come to mind: $(-1, 6)$ and $\left(\frac{1}{2}, 4\frac{1}{2}\right)$, for example. On the other hand, not every ordered pair is a solution to this equation: $(2, 2)$ is not a solution, for example.

5.1.1 A method for producing solutions

In the case that the equation is more complicated, there is still a straightforward method to produce solutions. We illustrate this method in the following example.

Example 5.1.1. *Find three solutions to the equation* $2x - 5y = 10$.

Answer. *Our strategy will be to "eliminate" one of the variables and to solve the remaining linear equation in one variable. We eliminate a variable by choosing a value for that variable, then substituting the value into the original equation. The solution to the original equation will be an ordered pair consisting of the chosen value for the "eliminated" variable and the value obtained by solving the resulting (one-variable) equation.*

For example, let's choose the value 0 *for* x. *Substituting into the given equation for* x *gives* $2(0) - 5y = 10$*; the variable* x *has been "eliminated." We then solve:*

$$\begin{array}{rcrcl} 2(0) & - & 5y & = & 10 \\ 0 & - & 5y & = & 10 \\ & & -5y & = & 10 \\ & & \frac{-5y}{-5} & = & \frac{10}{-5} \\ & & y & = & -2. \end{array}$$

The solution corresponding to our choice of 0 *for* x *is* $(0, -2)$.

For another solution, let's choose the value 0 *for* y*. Substituting this value for* y *gives* $2x - 5(0) = 10$*. Solving:*

$$\begin{array}{rcrcl} 2x & - & 5(0) & = & 10 \\ 2x & - & 0 & = & 10 \\ 2x & & & = & 10 \\ \frac{2x}{2} & & & = & \frac{10}{2} \\ x & & & = & 5. \end{array}$$

The solution corresponding to our choice of 0 *for* y *is* $(5, 0)$*.*

Since we were asked for three solutions, we make one more choice. Let's choose the value 1 *for* y*. Substituting gives* $2x - 5(1) = 10$*. Solving:*

$$\begin{array}{rcrcl} 2x & - & 5(1) & = & 10 \\ 2x & - & 5 & = & 10 \\ & + & 5 & \vdots & +5 \\ \hline 2x & & & = & 15 \\ \frac{2x}{2} & & & = & \frac{15}{2} \\ x & & & = & \frac{15}{2}. \end{array}$$

The solution corresponding to our choice of 1 *for* y *is* $(15/2, 1)$*.*

The three solutions we obtained are $(0, -2)$*,* $(5, 0)$*, and* $(15/2, 1)$*.*

We will organize the data from finding solutions to a linear equation in two variables into a table. For example, we will summarize the three solutions above as:

x	y	*Solution*
$\boxed{0}$	-2	$(0, -2)$
5	$\boxed{0}$	$(5, 0)$
15/2	$\boxed{1}$	$(15/2, 1)$

Notice that we have indicated the value that was chosen with a boxed number, while the value obtained by solving the corresponding equation with an unboxed number.

We can summarize this method for finding solutions.

Finding solutions to an algebraic equation in two variables

To find solutions to an algebraic equation in two variables:

1. Choose a value for one of the variables;
2. Substitute the chosen value into the equation and solve the resulting equation in one variable.

The ordered pair corresponding to the chosen value with the value obtained by solving the resulting equation (in the appropriate order) will be a solution to the original equation in two variables.

One thing should be clear from the method described in the example above: A linear equation in two variables will typically have infinitely many solutions, one for each choice of value for x (or y). This will present some problems from the point of view of solving such equations—finding *all* solutions.

5.1.2 Graphing linear equations in two variables

In Section 4.4 on linear inequalities in one variable, we saw a powerful method for keeping track of solutions of algebraic statements with infinitely many solutions: graphing. However, in the case of algebraic statements in two variables, a number line is not sufficient. To keep track of the values of both variables, we will use the xy–plane (sometimes called the Cartesian plane, after one of the originators of the concept, the French philosopher and mathematician René Descartes).

For the sake of reference, we list here some of the most important properties of an xy-plane (see Figure 5.1):

- It is formed by two number lines placed at right angles and meeting where both are labeled 0. The number lines are called the x-axis (the horizontal number line) and the y-axis (the vertical number line). The point of intersection of the axes is called the origin.
- The positive x-direction is to the right. The positive y-direction is upwards.
- An ordered pair is represented by a point on the xy-plane by means of its *coordinates*. The first number (the x-coordinate) represents the number of units ("in the x-direction") from the y-axis to the point . The second number (the y-coordinate) represents the number of units ("in the y-direction") from the x-axis to the point.

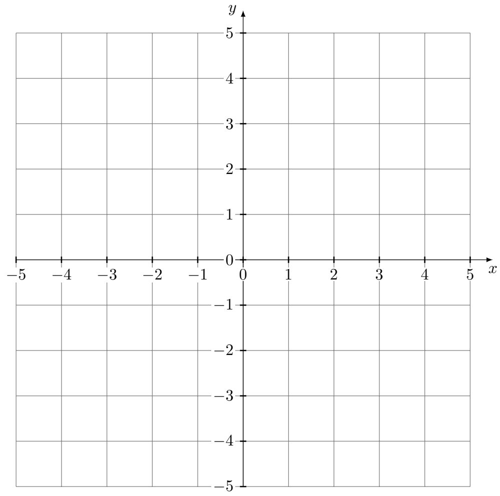
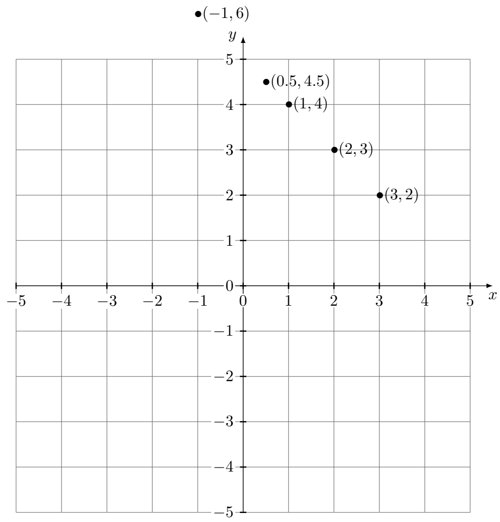


Figure 5.1: An xy-plane

- Points on the x-axis correspond to ordered pairs having 0 as a y-coordinate. Points on the y-axis correspond to ordered pairs having 0 as an x-coordinate.

Let's return to our example $x + y = 5$. Just by inspection, we found several solutions. We will now represent each ordered pair solution with a point in the xy-plane. (This is called *plotting* the ordered pairs.)



Five solutions of $x + y = 5$

This graph, obtained by plotting five solutions of the same linear equation in two variables, points to a crucial fact that will be central to our treatment of linear equations in two variables:

BIG FACT: The geometry of solutions to linear equations in two variables

The points corresponding to plotting all solutions to a linear equation in two variables all lie on a single line. Every point on this line corresponds to a solution to the equation.

This fact, combined with some basic geometry, gives a powerful technique to solve a linear equation in two variables in the form of a graph.

General method to graph linear equations in two variables

To graph all solutions of a linear equation in two variables:

1. Find at least two solutions.
2. Plot the solutions.
3. Draw the line passing through the chosen solutions.

Notice that geometry comes into the picture due to the fact, written down as far back as Euclid, that two (different) points determine a unique line passing through them. This fact is what allows us to "buy two solutions, get infinitely many solutions free."

Combined with our method for producing solutions to linear equations in two variables above, we are hence able to graph any linear equation in two variables.

Example 5.1.2. *Graph the equation* $2x - 5y = 10$.

Answer. *Recall in Example 5.1.1 above, we found three solutions to* $2x - 5y = 10$*, given in the table*

x	y	*Solution*
$\boxed{0}$	-2	$(0, -2)$
5	$\boxed{0}$	$(5, 0)$
$15/2$	$\boxed{1}$	$(15/2, 1)$

We plot these solutions in Figure 5.2.

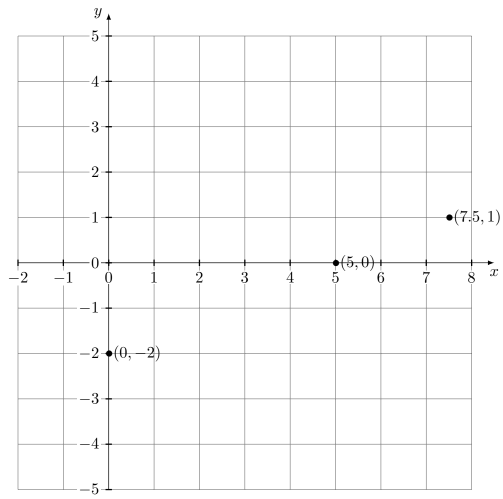


Figure 5.2: Three solutions of $2x - 5y = 10$

Notice that the three solutions appear to lie on the same line, as we expected from our Big Fact. All that remains is to "connect the dots" in Figure 5.3.

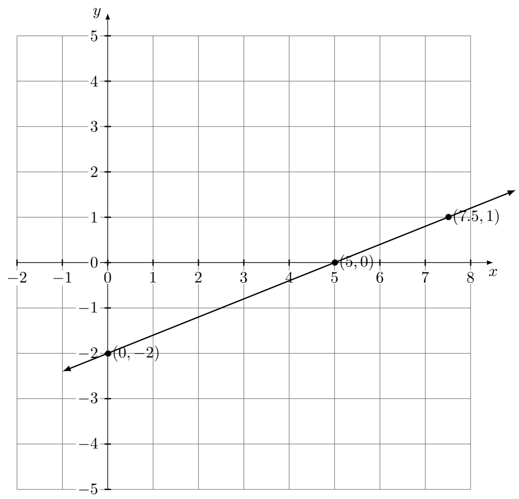


Figure 5.3: All solutions of $2x - 5y = 10$.

It is important to emphasize that the last "connect the dots" step, simplest from the procedural point of view, is also the most significant. We have gone from three solutions to infinitely many solutions—one for each point on the line.

Let's look at two more examples.

Example 5.1.3. *Graph the solutions of* $3x + 4y = 12$.

Answer. *We first find three solutions.*

Choosing 0 *for* x*, we substitute and solve:*

$$\begin{array}{rcrcl} 3(0) & + & 4y & = & 12 \\ 0 & + & 4y & = & 12 \\ & & 4y & = & 12 \\ \frac{4y}{4} & & & = & \frac{12}{4} \\ y & & & = & 3. \end{array}$$

So $(0, 3)$ *is a solution.*

Choosing 0 *for* y*, we substitute and solve:*

$$\begin{array}{rcrcl} 3x & + & 4(0) & = & 12 \\ 3x & + & 0 & = & 12 \\ 3x & & & = & 12 \\ \frac{3x}{3} & & & = & \frac{12}{3} \\ x & & & = & 4. \end{array}$$

So $(4, 0)$ *is a solution.*

Choosing -3 *for* y*, we substitute and solve:*

$$\begin{array}{rcrcl} 3x & + & 4(-3) & = & 12 \\ 3x & - & 12 & = & 12 \\ & + & 12 & \vdots & +12 \\ \hline 3x & & & = & 24 \\ \frac{3x}{3} & & & = & \frac{24}{3} \\ x & & & = & 8. \end{array}$$

So $(8, -3)$ *is a solution.*

Summarizing our results so far, we have the table:

x	y	*Solution*
$\boxed{0}$	3	$(0, 3)$
4	$\boxed{0}$	$(4, 0)$
8	$\boxed{-3}$	$(8, -3)$

We now plot the three solutions and connect them with a line. See Figure 5.4.

Notice that choosing 0 first for x and then for y is useful for more than just the ease of working with the number 0. The point whose x-coordinate is 0 (the point $(0, 3)$ in the previous example) is the *y-intercept* of the line: the point

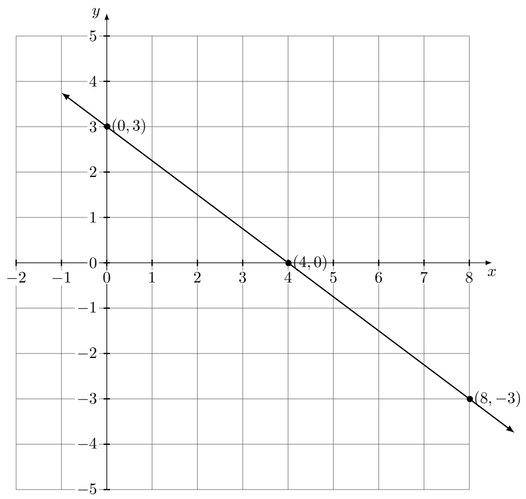


Figure 5.4: All solutions of $3x + 4y = 12$.

where the line intersects the y-axis. Likewise, the point whose y-coordinate is 0 (the point $(4, 0)$ in the previous example) is the x-*intercept* of the line, or the point where the line intersects the x-axis. We will often refer to these two special points on a line in the xy-plane, as they stand out on the graph.

Example 5.1.4. *Graph the solutions of* $y = \frac{1}{4}x - 2$.

Answer. *As usual, we will make three choices to find three solutions. This time, however, we will take advantage of the form in which the equation is written, with the* y *by itself on one side of the equation, and only choose values of* x*.*

Choosing 0 *for* x*, we substitute and solve:*

$$\begin{array}{rcrcr} y & = & \frac{1}{4}(0) & - & 2 \\ y & = & 0 & - & 2 \\ y & = & -2. & & \end{array}$$

So $(0, -2)$ *is a solution.*
Choosing 4 *for* x*, we substitute and solve:*

$$\begin{array}{rcrcr} y & = & \frac{1}{4}(4) & - & 2 \\ y & = & 1 & - & 2 \\ y & = & -1. & & \end{array}$$

So $(4, -1)$ *is a solution.*
Choosing 8 *for* x*, we substitute and solve:*

$$\begin{array}{rcrcr} y & = & \frac{1}{4}(8) & - & 2 \\ y & = & 2 & - & 2 \\ y & = & 0. & & \end{array}$$

So $(8,0)$ is a solution.
Hence we have the table:

x	y	*Solution*
0	-2	$(0,-2)$
4	-1	$(4,-1)$
8	0	$(8,0)$

(Can you see why we chose the values of x that we did?)
Plotting the solutions and connecting them with a line gives Figure 5.5.

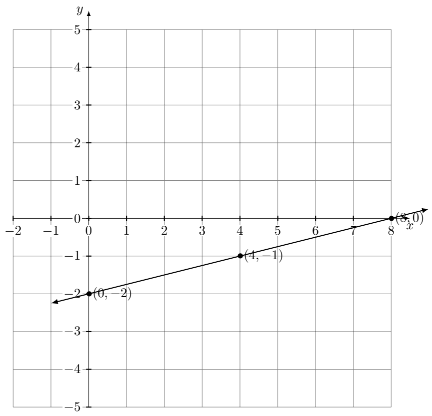


Figure 5.5: All solutions of $y = \frac{1}{4}x - 2$.

5.1.3 Exercises

For each of the linear equations in two variables below, graph the solutions.

1. $x - y = 4$
2. $2x + 3y = -6$
3. $5x - y = 2$
4. $-4x + 3y = 12$
5. $-x + 3y = 9$
6. $y = 2x - 1$
7. $y = \frac{1}{3}x - 2$
8. $y = -\frac{3}{4}x + 1$

5.2 A detour: Slope and the geometry of lines

We saw in the last section how geometry can be helpful in solving a linear equation in two variables. In particular, using the fact that two points determine a line, we were able to find all solutions of a linear equation in two variables (as a graph) just by knowing any two different solutions.

In this section, we continue the theme of how geometry can help us study linear equations in two variables. After defining the slope of a line, we will show how we can use this concept to develop another method for graphing the solutions to such equations. We will also show how this concept allows us to write an equation for a line in the xy-plane.

The *slope* will give a way to measure a line. It will be a single number that is designed to measure the "steepness" of a line.

Consider for example the lines shown in Figure 5.6. Line A is steeper than line B. (Imagine yourself riding a bicycle up two hills represented by the lines. It will be harder to pedal up line A than line B!) So we will want to assign a larger number as the slope of line A than for the slope of line B. Line C is not steep at all; it is "flat." We will want to assign a slope of 0 to this line. Line D appears to be about as steep as line A, but in different "directions." Line A is slanted upwards (from left to right), while line D is slanted downwards. We will assign a positive number as the slopes for lines A and B, but a negative number for the slope of line D. Vertical lines are special in that they do not have a slope. (Don't try to ride your bike down a vertical cliff!)

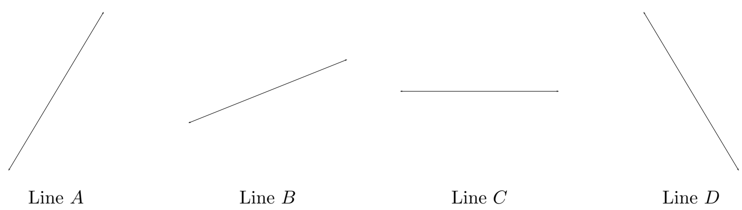


Figure 5.6: Four lines with different slopes.

How do we make this measurement called slope? It turns out than an effective way to assign a number that matches exactly with our expectations from the previous paragraph is to define the slope as the ratio of the vertical change in distance between two points on the line to the horizontal change in distance between the same two points, with the understanding that a change from upper to lower (going from left to right) will be negative[1]. See Figure 5.7.

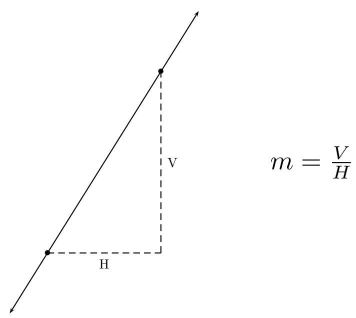


Figure 5.7: The definition of slope m.

Notice that we have defined the slope without reference to a coordinate system, i.e. without an xy-plane. In the case that the line is drawn with reference to a coordinate system, the vertical and horizontal distances in the definition of the slope can be written in terms of the coordinates of two points on the line with coordinates (x_1, y_1) and (x_2, y_2):

The slope of a line in an xy-plane

The slope of a line in an xy-plane passing through the points with coordinates (x_1, y_1) and (x_2, y_2) is given by the ratio

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

(See Figure 5.8.)

It should be pointed out that in this context, the notation Δy and Δx are

[1]This definition in itself is based on an important fact from geometry. Recall that two triangles are *similar* if their corresponding angles have equal measurements. The ratio of corresponding sides of similar triangles are equal. For that reason, the slope does not depend on the two points chosen. Can you see why?

sometimes used to represent the change in x and y respectively, so the slope can be remembered as

$$m = \frac{\Delta y}{\Delta x}.$$

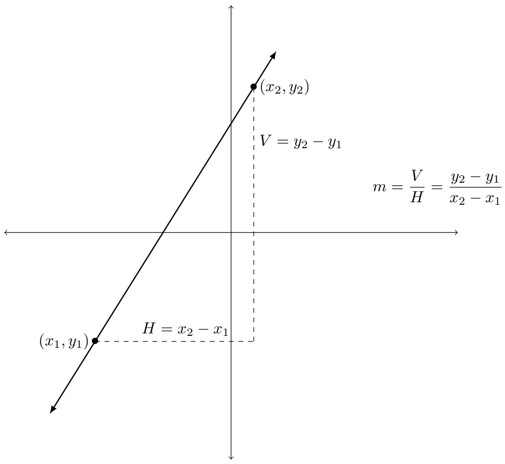


Figure 5.8: The slope defined relative to an xy-plane.

In order to use the formula defining the slope, the coordinates of (any!) two points on the line are needed.

Example 5.2.1. *Find the slope of the line passing through the points with coordinates* $(6, -2)$ *and* $(3, 7)$.

Answer. *Since we are given the coordinates of two points on the line, all that remains to do is to label the coordinates, substitute into the formula defining the slope, and evaluate.*

Labelling,

$$\begin{matrix} & x_1 & & y_1 & & & x_2 & & y_2 & \\ (& 6 & , & -2 &), & (& 3 & , & 7 &) \end{matrix}$$

Substituting and evaluating:

$$\begin{aligned} m &= \frac{(7) - (-2)}{(3) - (6)} \\ &= \frac{7 + 2}{3 + (-6)} \\ &= \frac{9}{-3} \\ &= -3. \end{aligned}$$

The slope is -3.

For the sake of the reader who is seeing the slope formula in action for the first time, let's re-do the previous example, but labeling the coordinates in the opposite way:

$$\begin{array}{cccccccccc} & x_1 & & y_1 & & & x_2 & & y_2 & \\ (& 3 & , & 7 &), & (& 6 & , & -2 &) \end{array}$$

Then substituting,

$$\begin{aligned} m &= \frac{(-2)-(7)}{(6)-(3)} \\ &= \frac{(-2)+(-7)}{3} \\ &= \frac{-9}{3} \\ &= -3. \end{aligned}$$

We obtain the same answer, the slope being -3. This is a special case of the point that we made in the definition: *the slope does not depend on which two points on the line are chosen*, and in particular, does not depend on the order that the points are used.

Although a graph is not necessary for the purpose of computing the slope of a line, the reader might want to plot the two given ordered pairs $(6,-2)$ and $(3,7)$ to visualize the line passing through the corresponding points to verify that the line slants downwards going from left to right, as we would expect from a line with a negative slope.

We next illustrate an example where the required information to compute the slope from the definition is not given directly. We will see shortly that there is another, more effective way to approach this example.

Example 5.2.2. *Use the definition to find the slope of the line given by the equation* $2x + y = 2$.

Answer. *Although we are not given the coordinates of two points on the line, in some ways we have better: we have an equation for the line. We have already seen a method for obtaining as many solutions to this equation as we want—two will be enough.*

Choosing 0 *for* y*, we substitute and solve:*

$$\begin{array}{ccccc} 2x & + & (0) & = & 2 \\ 2x & & & = & 2 \\ \frac{2x}{2} & & & = & \frac{2}{2} \\ x & & & = & 1. \end{array}$$

So $(1,0)$ *is a solution.*

Choosing 0 *for* x*, we substitute and solve:*

$$\begin{array}{ccccc} 2(0) & + & y & = & 2 \\ 0 & + & y & = & 2 \\ & & y & = & 2. \end{array}$$

So $(0, 2)$ *is a solution.*

Summarizing our results so far, we have the table:

x	y	*Solution*
1	$\boxed{0}$	$(1, 0)$
$\boxed{0}$	2	$(0, 2)$

Now, labeling the coordinates of our two solutions,

$$\begin{array}{ccccccccccc} & x_1 & & y_1 & & & x_2 & & y_2 & \\ (& 1 & , & 0 &), & (& 0 & , & 2 &) \end{array}$$

Substituting and evaluating:

$$\begin{aligned} m &= \frac{(2) - (0)}{(0) - (1)} \\ &= \frac{2}{0 + (-1)} \\ &= \frac{2}{-1} \\ &= -2. \end{aligned}$$

The slope is -2.

This example also gives us a way to illustrate even more surely that the slope does not depend on the points chosen. Suppose your classmate's choices are different from yours, and they obtain two different solutions $(-1, 4)$ and $(2, -2)$. (Check that these are really solutions to $2x + y = 2$!) In that case, they would label:

$$\begin{array}{ccccccccccc} & x_1 & & y_1 & & & x_2 & & y_2 & \\ (& -1 & , & 4 &), & (& 2 & , & -2 &). \end{array}$$

Substituting and evaluating would give:

$$\begin{aligned} m &= \frac{(-2) - (4)}{(2) - (-1)} \\ &= \frac{(-2) + (-4)}{2 + (1)} \\ &= \frac{-6}{3} \\ &= -2. \end{aligned}$$

The two points were chosen differently, but the slope of the line is the same!

We conclude this subsection with an example that will lead in to the next main use of the slope concept.

Example 5.2.3. *Find the slope of the line given by the equation* $y = \frac{2}{3}x - 4$.

Answer. *As in the last example, we first find any two solutions.*

Choosing 0 *for* x*, we substitute and solve:*

$$\begin{array}{rcrcr} y & = & \frac{2}{3}(0) & - & 4 \\ y & = & 0 & - & 4 \\ y & = & -4. & & \end{array}$$

So $(0, -4)$ *is a solution.*

Choosing 3 *for* x*, we substitute and solve:*

$$\begin{array}{rcrcr} y & = & \frac{2}{3}(3) & - & 4 \\ y & = & 2 & - & 4 \\ y & = & -2. & & \end{array}$$

So $(3, -2)$ *is a solution.*

Summarizing our results so far, we have the table:

x	y	*Solution*
$\boxed{0}$	-4	$(0, -4)$
$\boxed{3}$	-2	$(3, -2)$

Labeling our two solutions,

$$\begin{array}{ccccccccccc} & x_1 & & y_1 & & & & x_2 & & y_2 & \\ (& 0 & , & -4 &), & & (& 3 & , & -2 &) \end{array}$$

Substituting and evaluating:

$$\begin{aligned} m &= \frac{(-2)-(-4)}{(3)-(0)} \\ &= \frac{(-2)+(4)}{3)} \\ &= \frac{2}{3}. \end{aligned}$$

The slope is $2/3$*. We will see very shortly that this answer is no surprise.*

The previous example 5.2.3 is a special case of an important fact relating the slope to linear equations in two variables:

Slope-intercept form of a linear equation in two variables

Suppose that a linear equation is written in the special form

$$y = mx + b,$$

with the variable y by itself on one side of the equation. Then m (the coefficient of x) is the slope of the line, and the y-intercept has coordinates $(0, b)$.

This special form of writing a linear equation in two variables, where the variable y is written by itself on one side of the equation, is known as the *slope-intercept form* of the equation of a line, since both the slope and the y-coordinate of the y-intercept can be read directly from the equation.

Notice that in Example 5.2.3, the equation $y = \frac{2}{3}x - 4$ was written in slope-intercept form. The slope $2/3$ was indeed the coefficient of x. Notice also, although we didn't mention it at the time, that the y-intercept has coordinates $(0, -4)$, a fact that we could also read from the form of the equation. (Keep in mind that the b term in the special slope-intercept form is *added*, so we should think of the equation as being written $y = \frac{2}{3}x + (-4)$.)

If a linear equation in two variables is not written in slope-intercept form, then there is no way to read off the information so easily. However, by changing the form of the equation, we can take advantage of the special slope-intercept form for any equation.

Example 5.2.4. *Find the slope and y-intercept of the line given by the equation $3x - 4y = 12$.*

Answer. *The equation is not written in slope-intercept form, since the variable y is not by itself. However, we can solve for y in terms of x:*

$$\begin{array}{ccccccc} 3x & - & 4y & = & & & 12 \\ -3x & & & \vdots & -3x & & \\ \hline & & -4y & = & -3x & + & 12 \\ & & \frac{-4y}{-4} & = & \frac{-3x+12}{-4} & & \\ & & y & = & \frac{-3x}{-4} & + & \frac{12}{-4} \\ & & y & = & \frac{3}{4}x & - & 3. \end{array}$$

The slope is $3/4$ and the y-intercept has coordinates $(0, -3)$.

We will see several more examples of this procedure in a different context in the following subsection.

5.2.1 Using the slope as an aid in graphing

In this subsection, we show how the slope gives an alternative method to the problem of graphing the solutions to a linear equation with two variables, apart from making a table of values to find solutions. It is based on the following principal:

> The slope, considered as a ratio of the change in the y-coordinates to the change in the x-coordinates of points on the line, gives a way to obtain a new point on the line from a given one.

Specifically, we will think of the slope as a fraction which gives instructions to move "up and to the right" or "down and to the right," depending on whether the slope is positive or negative.[2]

Example 5.2.5. *Find three other points on the line passing through the point with coordinates $(-3,-2)$ and having slope 2.*

Answer. *The slope is $2 = \frac{2}{1}$. So, beginning from the given point's coordinates $(-3,-2)$, we will move our pencil on the graph one unit to the right and two units upwards to obtain our first new point. See Figure 5.9. This new point has coordinates $(-2,0)$, as should be clear from the graph.*

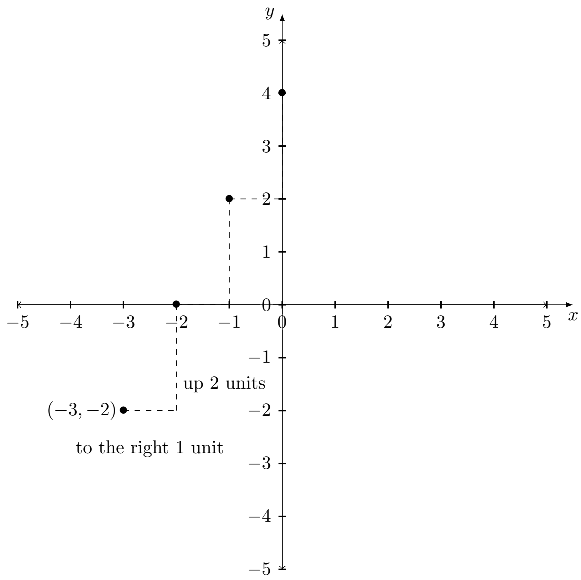


Figure 5.9: Using the slope to find a second point on a line.

Repeating the procedure two more times gives two other new points with coordinates $(-1,2)$ and $(0,4)$. (Even though we could write down a "formula" to obtain the numerical coordinates of one point from the next, it is by far simpler in the cases we will encounter to just read the coordinates from the xy-plane.)

Using the method of the previous example gives us an effective way to graph the solutions of a linear equation in two variables—especially if the equation is written in slope-intercept form.

[2]More properly, we should think of moving "in the same direction" or "in the opposite direction," so that, for example, we can also obtain a second point from a given one on a line with positive slope by moving down and to the left.

Example 5.2.6. *Graph the solutions of* $y = -x + 3$.

Answer. *Notice that the equation is written in slope-intercept form; y is by itself on one side of the equation. The slope is* -1 *(the coefficient of x), while the y-intercept has coordinates* $(0, 3)$.

Using the slope $m = -1 = \frac{-1}{1}$, *we start at the given point with coordinates* $(0, 3)$ *and "move" one unit downward and one unit to the right in order to obtain a second point having coordinates* $(1, 2)$. *This gives us two solutions; the graph will consist of all points on the line passing through these two points.*

The graph is given in Figure 5.10.

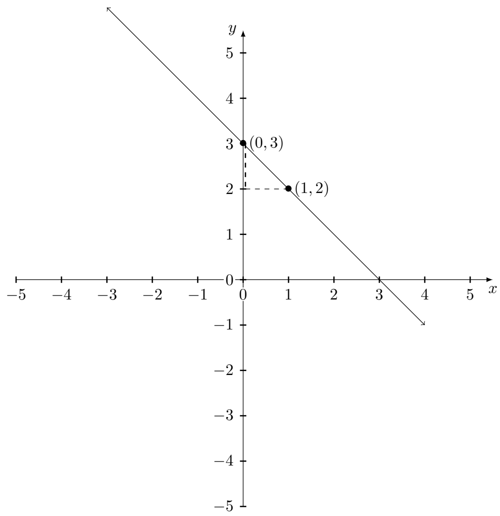


Figure 5.10: All solutions of $y = -x + 3$.

While the previous example was straightforward due to the fact that the equation was written in slope-intercept form to begin with, we have already seen that it doesn't take much effort to rewrite an equation in slope-intercept form if it isn't written that way to begin with, by solving for y.

Example 5.2.7. *Graph the solutions of* $2x - y = 6$.

Answer. *The equation is not written in slope-intercept form, since y is not by itself on one side of the equation. Solving for y in terms of x:*

$$\begin{array}{ccccccc} 2x & - & y & = & & & 6 \\ -2x & & & \vdots & -2x & & \\ \hline & & -y & = & -2x & + & 6 \\ & & \frac{-y}{-1} & = & \frac{-2x+6}{-1} & & \\ & & y & = & \frac{-2x}{-1} & + & \frac{6}{-1} \\ & & y & = & 2x & - & 6. \end{array}$$

We now see that the slope is 2 *and the* y*-intercept has coordinates* $(0, -6)$.

Using the slope $m = 2 = \dfrac{2}{1}$, *we start at the point representing* $(0, -6)$ *and "move" upwards two units and to the right one unit in order to obtain a second solution* $(1, -4)$.

The graph is given in Figure 5.11.

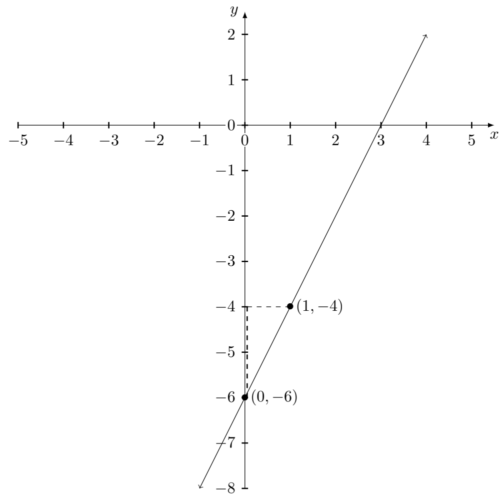


Figure 5.11: All solutions of $2x - y = 6$.

The only possible difficulty in this method of graphing is that when following the method too literally, we will occasionally be forced to plot points with fractional coordinates, as the next example illustrates.

Example 5.2.8. *Graph the solutions of* $3x + 2y = 5$.

Answer. *The equation is not written in slope-intercept form, since* y *is not by itself on one side of the equation. Solving for* y*:*

$$\begin{array}{rcrcrcl} 3x & + & 2y & = & & & 5 \\ -3x & & & \vdots & -3x & & \\ \hline & & 2y & = & -3x & + & 5 \\ & & \frac{2y}{2} & = & \frac{-3x+5}{2} & & \\ & & y & = & \frac{-3x}{2} & + & \frac{5}{2} \\ & & y & = & -\frac{3}{2}x & + & \frac{5}{2}. \end{array}$$

We see that the slope is $-3/2$ *and the* y*-intercept has coordinates* $(0, 5/2)$.

Since $5/2 = 2\ \frac{1}{2}$, *the point representing* $(0, 5/2)$ *is plotted halfway between those representing* $(0, 2)$ *and* $(0, 3)$. *Using the slope* $m = \dfrac{-3}{2}$, *we start at the* *point representing* $(0, 5/2)$ *and "move" downwards three units and to the right two units to obtain a second solution* $(2, -1/2)$.

The graph is given in Figure 5.12.

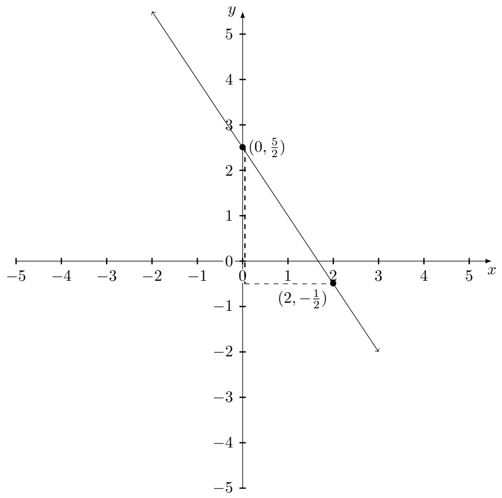


Figure 5.12: All solutions of $3x + 2y = 5$.

(Notice that we encountered fractional coordinates in this example because the y*-intercept had a fractional* y*-coordinate. If we had used a solution with integer coordinates like* $(1, 1)$, *we could have avoided this inconvenience—but then we would have been on our way to constructing a table.)*

5.2.2 Finding an equation of a given line

So far, we have concentrated on the relationship between the slope and the graph of a linear equation in two variables. The sign of the slope indicates which "direction" the line is slanted. The magnitude of the slope measures the ratio of the vertical change to the horizontal change, and so given one point on the line, the slope indicates how to determine other points on the same line.

However, the slope concept also opens the door to a answering a new kind of question. Suppose we are given a line (in an xy-plane) described by some geometric data. How can we find an equation whose solutions correspond to the given line[3]?

What is meant by describing a line with geometric data? We will consider the following situations:

- A line described by one point on the line and the slope;
- A line described by two points on the line;

[3]Notice that we do not ask for "the" equation of a line. The reader can check, for example, that the equations $x + y = 1$ and $2x + 2y = 2$ have the same solutions, and so describe the same line in an xy-plane.

- A line described by one point on the line, given that it is parallel to another line;
- A line described by one point on the line, given that it is perpendicular to another line.

The simplest example will show that we already have tools to answer this question.

Example 5.2.9. *Find an equation for the line passing through the point with coordinates* $(0, -2)$ *and having slope* 3.

Answer. *Notice that in this case, the point given happens to be the* y*-intercept! (That can be seen even without plotting the point by noticing that the* x*-coordinate is* 0.*) Hence we can treat the slope-intercept form of a line, which we have written as* $y = mx + b$, *as a formula, and substitute the values of* m *and* b.

In this case, $m = 3$ *and* $b = -2$, *so an equation of the line, in slope-intercept form, would be*

$$y = 3x - 2.$$

"That was too good to be true!" Of course, we had been given exactly the data needed to substitute into the slope-intercept "formula" for a line. In the next example, we show that the previous method still applies in a more general context. We also illustrate a second method which is better adapted to the more general setting.

Example 5.2.10. *Find an equation for the line passing through the point with coordinates* $(1, -2)$ *and having slope* -4.

Answer. *This time, the given point is not the* y*-intercept (the* x*-coordinate is not* 0*!), so we cannot proceed as directly as in the previous example.*

Method 1

Even though we do not have all the information needed to substitute into the slope-intercept "formula," we can proceed in two steps.

The first, easy step is to substitute the information we do have, which is the slope ($m = -4$), *into the formula:*

$$y = -4x + b.$$

This time, b *is still unknown.*

In the second step, we will use the coordinates $(1, -2)$ *of the given point to solve for* b, *by substituting the coordinates for* x *and* y *in the equation we have obtained so far:*

$$\begin{array}{rcrcl} y & = & -4x & + & b \\ (-2) & = & -4(1) & + & b \\ -2 & = & -4 & + & b \\ +4 & \vdots & +4 & & \\ \hline 2 & = & & & b. \end{array}$$

The solution for b is 2.

Now, since we have values for m AND b, we can substitute into the slope-intercept "formula" as above. The answer is

$$y = -4x + 2.$$

Method 2

Instead of trying to use the slope-intercept "formula," the second method will use the definition of the slope directly. Namely, we will substitute the coordinates for the given point $(1, -2)$ along with the coordinates of a second unknown point (x, y), *along with the value of the slope, into the formula defining the slope* $m = \dfrac{y_2 - y_1}{x_2 - x_1}$. *Namely, we label:*

$$\begin{array}{ccccccccccc} & x_1 & & y_1 & & & x_2 & & y_2 & \\ (& 1 & , & -2 &), & (& x & , & y &). \end{array}$$

After substituting these values, we solve for y in terms of x:

$$\begin{array}{rrcccr} & -4 & = & \frac{(y)-(-2)}{(x)-(1)} & & \\ & -4 & = & \frac{y+2}{x-1} & & \\ & -4\cdot(x-1) & = & \frac{y+2}{x-1}\cdot(x-1) & & \\ -4x & +4 & = & y & + & 2 \\ & -2 & \vdots & & & -2 \\ \hline -4x & +2 & = & y. & & \end{array}$$

The answer is $y = -4x + 2$.

Notice that in the key step to this method, multiplying both sides by $(x - 1)$ to "cancel" the denominator in the definition of the slope, we assumed that $x - 1 \neq 0$. This is permitted since we were supposing (x, y) to be the coordinates of a second point on the line different from $(1, -2)$.

While Method 1 functions well, it is somewhat artificial in that we are using a "formula" that doesn't match the data we are given. That is why Method 1 is a two-step method.

Method 2, on the other hand, used exactly the information we were given: the slope and the coordinates of *any* one point on the line. Because it applies in the more general setting, we summarize from Method 2 a "formula" for an equation of the line passing through a given point with a given slope.

The point-slope form of a linear equation in two variables

An equation for the line with slope m and passing through the point with coordinates (x_0, y_0) is given by

$$y - y_0 = m(x - x_0).$$

This is known as the *point-slope form of a line*. As indicated in Method 2 of the last example, it derives from the definition of the slope, where we have incorporated the step of "canceling the denominator" into the formula.

Unlike the slope-intercept form of a line, which is useful because we can "read off" geometric data from the equation, the point-slope form of a line is almost exclusively used as a "formula" to find an equation for a line, where values of m, x_0 and y_0 are substituted to obtain an equation involving x and y.

In the remaining examples, we will use the point-slope form of the line to find an equation for the given line.

Example 5.2.11. *Find an equation for the line passing through the points with coordinates* $(4, 1)$ *and* $(-2, 5)$.

Answer. *Unlike the previous examples in this section, this time we are not given the slope. Fortunately, since we have the coordinates of two points on the line, we can use the definition to find the slope.*

Step 1: Find the slope *Labelling*

$$\left(\overset{x_1}{4}, \overset{y_1}{1}\right), \quad \left(\overset{x_2}{-2}, \overset{y_2}{5}\right),$$

we substitute into the definition:

$$\begin{aligned} m &= \frac{(5) - (1)}{(-2) - (4)} \\ &= \frac{4}{-6} \\ &= -\frac{2}{3}. \end{aligned}$$

Step 2: Use the point-slope formula *We now have* $m = -2/3$. *We can choose the coordinates of either of the given points to use in the point-slope formula; let's use the first. Labeling,*

$$\left(\overset{x_0}{4}, \overset{y_0}{1}\right),$$

we can substitute into the point-slope formula and solve for y in terms of x:

$$\begin{array}{ccccccc} y & - & (1) & = & (-\frac{2}{3})(x & - & (4)) \\ y & - & 1 & = & (-\frac{2}{3})(x & - & 4) \\ y & - & 1 & = & (-\frac{2}{3})(x) & - & (-\frac{2}{3})(4) \\ y & - & 1 & = & -\frac{2}{3}x & - & (-\frac{8}{3}) \\ y & - & 1 & = & -\frac{2}{3}x & + & \frac{8}{3} \\ & + & 1 & \vdots & & + & 1 \\ \hline y & & & = & -\frac{2}{3}x & + & \frac{11}{3}. \end{array}$$

(Notice that solving for y in terms of x amounts to writing the answer in slope-intercept form.)

Since this is the first example of its type, let's verify that the result does not depend on which of the two points we choose. If we had instead chosen the second point, we would have obtained

$$\begin{array}{ccccc} & x_0 & & y_0 & \\ (& -2 & , & 5 &). \end{array}$$

We can now substitute into the point-slope formula and solve for y in terms of x:

$$\begin{array}{ccccccc} y & - & (5) & = & (-\frac{2}{3})(x & - & (-2)) \\ y & - & 5 & = & (-\frac{2}{3})(x & + & 2) \\ y & - & 5 & = & \left(-\frac{2}{3}\right)(x) & + & \left(-\frac{2}{3}\right)(2) \\ y & - & 5 & = & -\frac{2}{3}x & + & \left(-\frac{4}{3}\right) \\ y & - & 5 & = & -\frac{2}{3}x & - & \frac{4}{3} \\ & + & 5 & \vdots & & + & 5 \\ \hline y & & & = & -\frac{2}{3}x & + & \frac{11}{3}. \end{array}$$

While the equation looked different immediately after substituting into the point-slope formula, the slope-intercept form of the equation is the same.

The answer, in slope-intercept form, is $y = -\dfrac{2}{3}x + \dfrac{11}{3}$.

The last two examples of this subsection will rely on the the following translation of geometric facts into the language of slopes. Recall that two lines in a plane are **parallel** if they have no point of intersection; two lines in a plane are **perpendicular** if they intersect at right angles. These geometric definitions can be translated (with some work) into algebraic facts by means of the slope.

Parallel and perpendicular lines described by slope

- Two lines are **parallel** if they have *the same slopes.*
- Two lines are **perpendicular** if *the product of their slopes is* -1.

In algebraic terms, suppose two lines have slopes m_1 and m_2. If the lines are parallel, then $m_1 = m_2$. If the lines are perpendicular, then $m_2 = -1/m_1$ (where $m_1 \neq 0$). (It might be helpful to think of $m_2 = -1/m_1$ in words: "m_2 is the opposite of the reciprocal of m_1.")

Example 5.2.12. *Find an equation for the line passing through the point with coordinates* $(-3, 2)$ *and which is parallel to the line* $x + 6y = 1$.

Answer. *We are not given the slope of the line in question. However, we are given the equation of a parallel line. Let's find the slope of the parallel line, then use the same slope for the line in question.*

Step 1: Find the slope of the parallel line.

Since we are given an equation for the parallel line, let's rewrite it in slope-intercept form:

$$\begin{array}{ccccccc} x & + & 6y & = & & & 1 \\ -x & & & \vdots & -x & & \\ \hline & & 6y & = & -x & + & 1 \\ & & \frac{6y}{6} & = & \frac{-x+1}{6} & & \\ & & y & = & \frac{-x}{6} & + & \frac{1}{6} \\ & & y & = & -\frac{1}{6}x & + & \frac{1}{6}. \end{array}$$

The slope of the parallel line is $-1/6$.

Step 2: Use the point-slope formula.

We will substitute $m = -1/6$ *(using the* ***same slope*** *as the parallel line) and the coordinates of the given point*

$$\begin{array}{ccccc} & x_0 & & y_0 & \\ (& -3 & , & 2 &) \end{array}$$

into the point-slope formula, and solve for y in terms of x.

$$\begin{array}{ccccccc}
y & - & (2) & = & (-\frac{1}{6})(x & - & (-3)) \\
y & - & 2 & = & (-\frac{1}{6})(x & + & 3) \\
y & - & 2 & = & \left(-\frac{1}{6}\right)(x) & + & \left(-\frac{1}{6}\right)(3) \\
y & - & 2 & = & -\frac{1}{6}x & + & \left(-\frac{3}{6}\right) \\
y & - & 2 & = & -\frac{1}{6}x & - & \frac{1}{2} \\
 & + & 2 & \vdots & & + & 2 \\
\hline
y & & & = & -\frac{1}{6}x & + & \frac{3}{2}.
\end{array}$$

The answer, in slope-intercept form, is $y = -\dfrac{1}{6}x + \dfrac{3}{2}$.

Example 5.2.13. *Find an equation for the line passing through the point with coordinates $(3, 5)$ which is perpendicular to the line $3x - 2y = 12$.*

Answer. *Again, we are not given the slope of the line in question.*
Step 1: Find the slope of the perpendicular line.

We rewrite the equation of the perpendicular line in slope-intercept form:

$$\begin{array}{cccccc}
3x & - & 2y & = & & 12 \\
-3x & & & \vdots & -3x & \\
\hline
 & & -2y & = & -3x & + \; 12 \\
 & & \frac{-2y}{-2} & = & \frac{-3x+12}{-2} & \\
 & & y & = & \frac{-3x}{-2} & + \; \frac{12}{-2} \\
 & & y & = & \frac{3}{2}x & - \; 6.
\end{array}$$

The slope of the perpendicular line is $3/2$.
Step 2: Use the point-slope formula.

For the slope of the line in question, we will use the ***opposite of the reciprocal*** *of the slope of the perpendicular line: we will substitute $m = -2/3$ along with and the coordinates of the given point*

$$\begin{array}{ccccc}
 & x_0 & & y_0 & \\
(& 3 & , & 5 &)
\end{array}$$

into the point-slope formula:

$$\begin{array}{rrrcrr}
y & - & (5) & = & (-\frac{2}{3})(x & - & (3)) \\
y & - & 5 & = & (-\frac{2}{3})(x & - & 3) \\
y & - & 5 & = & \left(-\frac{2}{3}\right)(x) & - & \left(-\frac{2}{3}\right)(3) \\
y & - & 5 & = & -\frac{2}{3}x & - & \left(-\frac{6}{3}\right) \\
y & - & 5 & = & -\frac{2}{3}x & + & 2 \\
 & + & 5 & \vdots & & + & 5 \\
\hline
y & & & = & -\frac{2}{3}x & + & 7.
\end{array}$$

The answer, in slope-intercept form, is $y = -\dfrac{2}{3}x + 7$.

Notice that in none of the examples in this section were we asked to graph the lines in question. Having done the work of writing their equations in slope-intercept form, however, doing so would have not been much extra effort.

5.2.3 Special cases: Horizontal and vertical lines

In the beginning of our discussion of linear equations in two variables, we mentioned that such an equation (in variables x and y) could always be written in the form $Ax + By = C$, where A, B, and C are constants. We did not specify that these constants were not 0 (although if they are *both* 0, the equation is no longer a linear equation!). In the case that either A or B is zero, the corresponding term is "missing," and it appears that the equation has only one variable. However, the context determines whether we should consider the equation in a one-variable setting or a two-variable setting.

Let's start with the case of horizontal lines. When we first introduced the slope concept, we specified that horizontal lines should have slope 0.

Example 5.2.14. *Find an equation of the horizontal line with slope* 0 *and passing through the point with coordinates* $(-3, -7)$.

Answer. *We have been given exactly the information needed to use the point-slope formula. So we will substitute and solve for* y *in terms of* x.

$$\begin{array}{rrrcrrr}
y & - & y_0 & = & m(x & - & x_0) \\
y & - & (-7) & = & (0)(x & - & (-3)) \\
y & + & 7 & = & (0)(x & + & 3) \\
y & + & 7 & = & 0 & & \\
 & - & 7 & \vdots & -7 & & \\
\hline
y & & & = & -7. & &
\end{array}$$

The answer is $y = -7$. Notice that although the example was clearly stated in the setting of two variables (an ordered pair was given!), only one variable appears in the equation describing the line.

Let's consider the equation $y = -7$ from the previous example more carefully. A solution to this equation, which in this context will be an ordered pair (x, y), must make the equation $y = -7$ after substituting its coordinates into the equation. However, there is no place to substitute x-values. In other words, the equation $y = -7$ imposes no restrictions at all on x! A table might look like:

x	y	Solution
$\boxed{0}$	-7	$(1, -7)$
$\boxed{-4}$	-7	$(-4, -7)$
$\boxed{29}$	-7	$(29, -7)$
$\boxed{-0.717}$	-7	$(-0.717, -7)$
$\boxed{3}$	-7	$(3, -7)$

Whatever x value we choose, the equation requires that the y-coordinate be -7.

Turning our attention to vertical lines, we immediately run into the problem that a vertical line does not have a slope (roughly speaking, the slope of a vertical line is "infinite"). Because of this, our strategy of relying on the point-slope formula would lead nowhere.

However, our discussion of a table of values for solutions to an equation in two variables with one variable missing still applies.

Example 5.2.15. *Graph the equation $x = -1$ in an xy-plane.*

Answer. *We will make a table to find solutions. Since the equation $x = -1$ does not involve the variable y, we will be free to choose any value of for y. However, the only x-value that will make the equation true will be -1. One possible table might be:*

x	y	*Solution*
-1	$\boxed{0}$	$(-1, 0)$
-1	$\boxed{-2}$	$(-1, -2)$
-1	$\boxed{1}$	$(-1, 1)$

Plotting these three solutions and drawing the line through them, we obtain Figure 5.13.

Notice that the line given by the equation $x = -1$ is a vertical line.

There are some obvious patterns in the previous two examples, which we can summarize as follows:

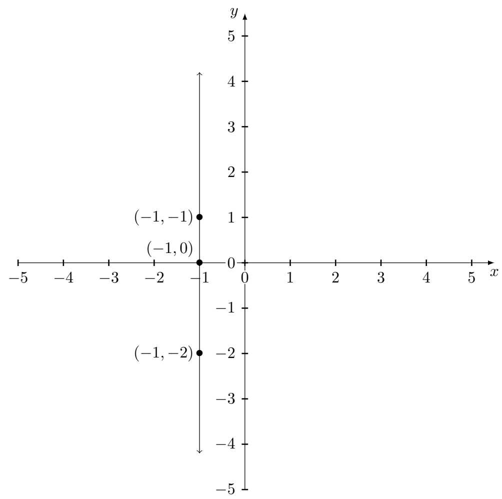


Figure 5.13: All solutions to $x = -1$.

Horizontal and vertical lines

- An equation for a *horizontal* line passing through the point with coordinates (a, b) is $y = b$.
- An equation for a *vertical* line passing through the point with coordinates (a, b) is $x = a$.

For future reference, it is worth remembering two special cases of this pattern in an xy-plane:

- An equation for the x-axis is $y = 0$.
- An equation for the y-axis is $x = 0$.

5.2.4 Exercises

1. Find the slope of the lines in an xy-plane described by the following information:

   (a) Passing through the points $(-2, 4)$ and $(1, -2)$.

   (b) Passing through the points $(0, -3)$ and $(4, 5)$.

   (c) Passing through the points $(-1, 4)$ and $(-1, -2)$.

   (d) Having equation $x - 3y = 4$.

(e) Having equation $2x + 3y = -6$.

(f) Having equation $5x - y = 2$.

(g) Having equation $y = 2x - 1$.

(h) Having equation $y = \frac{1}{3}x - 2$.

(i) Having equation $y = -\frac{3}{4}x + 1$.

(j) Having equation $y = 4$.

(k) Having equation $y = -x$

2. Find the slope and y-intercept of the line given by the equation $y = -\frac{3}{4}x + 1$.

3. Find the slope and y-intercept of the line given by the equation $5x - y = 2$.

4. Find an equation of the line having slope $3/4$ and passing through the point $(3, -2)$.

5. Find an equation of the line passing through the points $(2, -1)$ and $(5, 1)$.

6. Find an equation of the line passing through the point $(4, -2)$ and parallel to the line given by $3x - 4y = 6$.

7. Find an equation of the line passing through the point $(1, 0)$ and perpendicular to the line given by $x + 4y = 2$.

The following exercises give an alternate method to approach problems of the type in Examples 5.2.12 and 5.2.13.

8. (*) Show that for any values of A, B, C_1, C_2, $(A \neq 0)$ the line described by the equations $Ax + By = C_1$ is parallel to the line described by $Ax + By = C_2$.

9. Use the result of the previous exercise to find an equation of the line parallel to $3x + 5y = 8$ and passing through the point with coordinates $(2, -3)$.

10. (*) Show that for any values of A, B, C_1, C_2, $(A, B \neq 0)$ the line described by the equations $Ax + By = C_1$ is perpendicular to the line described by $-Bx + Ay = C_2$.

11. Use the result of the previous exercise to find an equation of the line perpendicular to $-x + 5y = 7$ and passing through the point with coordinates $(-1, 5)$.

5.3 Solving linear inequalities in two variables

We will approach linear inequalities in two variables in the same way as we approached linear inequalities in one variable. The reader should review Section 4.4.2 on one-variable inequalities briefly before proceeding; just like in that section, we will outline two approaches to solving two-variable inequalities.

As we have seen, a solution to a linear inequality in two variables is a value for each of the two variables which, when substituted into the inequality, make the inequality true. As in the case of linear equations in two variables, we will represent a solution with an ordered pair.

Let's look at an example: $x + y < 3$. Given any ordered pair, we can test to see whether or not it is a solution by substituting and evaluating. For example, $(3, 4)$ is not a solution since $(3) + (4) < 3$ is false. On the other hand, $(0, 1)$ is a solution, since $(0) + (1) < 3$ is true. You should check that $(-1, 1)$, $(2, -3)$ and $(0, 0)$ are also solutions to $x + y < 3$, while $(3, 3)$ and $(1, 2)$ are not solutions. After checking these ordered pairs, it is not hard to believe that the inequality has infinitely many solutions—as well as infinitely many ordered pairs which are not solutions.

As usual in the case when we have infinitely many solutions, we will attempt to draw a graph to represent all the solutions. However, plotting the solutions (and non-solutions) to the inequality $x + y < 3$ shows that coming up with a "pattern" will take a little more thought, see Figure 5.14.

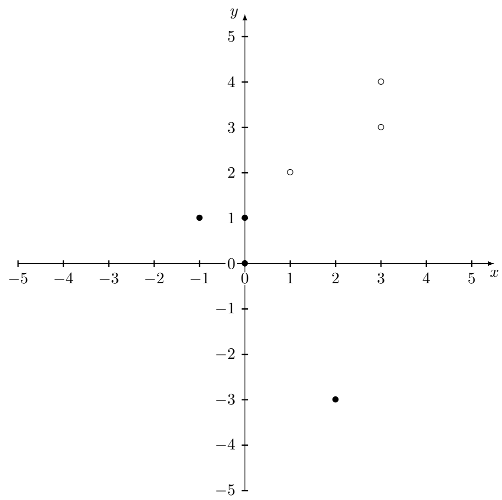


Figure 5.14: Four solutions (•) and three non-solutions (∘) to $x + y < 3$.

The key to seeing a pattern here is to take a step back and remember that solutions to a linear *equation* all lie on a line. Points not on the line do not represent solutions to the linear equation—or, equivalently, represent solutions to a linear *inequality*. In other words, if an ordered pair (a, b) is not a solution to the equation $Ax + By = C$ (and so the corresponding point is not on the line given by $Ax+By = C$), then the ordered pair (a, b) is a solution to $Ax+By \neq C$.

Now there are two ways that the inequality $Ax+By \neq C$ can be true: either $Ax + By < C$ is true, or $Ax + By > C$ is true. It is an important fact about an xy-plane that *all points representing solutions to* $Ax + By < C$ *lie on the same side of the line* $Ax + By = C$ *in an* xy*-plane, and all points representing solutions to* $Ax + By > C$ *lie on the other side of the same line.* Figure 5.15 is the same as Figure 5.14, except with the "border" line $x + y = 3$ indicated. (Notice that the ordered pair $(1, 2)$, which is not a solution to $x + y < 3$, is represented by a point on the border line described by $x + y = 3$.)

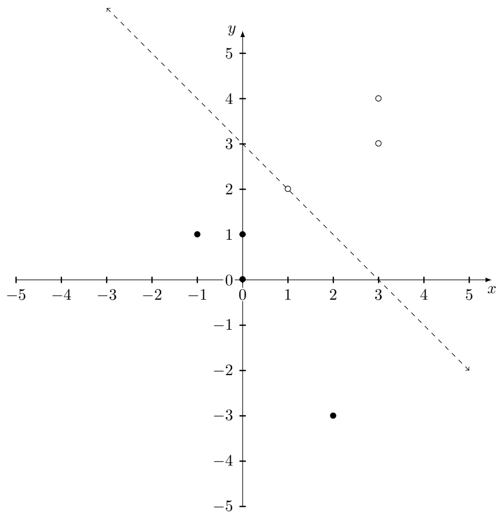


Figure 5.15: Solutions (•) and non-solutions (◦) to $x + y < 3$, with border line $x + y = 3$.

We can summarize the above discussion as follows: *The graph of all solutions to a typical linear inequality in two variables will consist of all points on one side of a line in an* xy*-plane. The border line will not (or will) be included depending on whether the inequality is strict (or not).*

Our strategy to solve a linear inequality in two variables will then be the following:

General strategy to solve linear inequalities in two variables

To solve a linear inequality in two variables:

1. Draw the border line. Use a dotted line for strict inequalities (so that points on the border line do not represent solutions) or a solid line for non-strict inequalities (so that the border points do represent solutions).

2. Shade the side of the border line that consists of solutions.

As in Section 4.4.2, we will discuss two methods to decide which side of the border line to shade.

Method 1: Test point method

The idea of this method is to choose any point in the xy-plane *not on the border line*. Test whether the chosen point represents a solution to the inequality. If it does represent a solution, shade all points on the *same side* of the border line as the test point. If it does not, shade all points on the *opposite side* of the border line.

We will give three examples using this method.

Example 5.3.1. *Graph the solutions of* $x - 3y < 6$.

Answer. *The first step is to graph the border line represented by* $x - 3y = 6$*; notice that we will draw the border as a* dotted *line since the inequality is strict (and the points on the border do not represent solutions to the inequality). To do that, we can use either of our methods for graphing linear equations. We list here a possible table of values to find two solutions:*

x	y	*Solution*
$\boxed{0}$	-2	$(0, -2)$
6	$\boxed{0}$	$(6, 0)$

Now we choose a test point to determine which side of the border line to shade. Let's choose one with coordinates $(1, 1)$*. To test it, we substitute these coordinates into the original inequality* $x - 3y < 6$*:*

$$\begin{array}{rcrcl} (1) & - & 3(1) & < & 6 \\ 1 & - & 3 & < & 6 \\ & & -2 & < & 6. \end{array}$$

The inequality is true, and so $(1, 1)$ *is a solution. We shade all points on the* same side *of the border line as the one representing* $(1, 1)$ *to represent all solutions of* $x - 3y < 6$*. See Figure 5.16.*

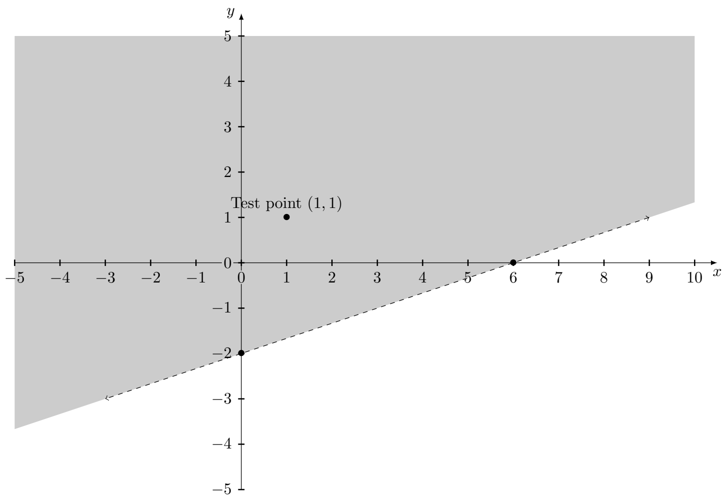


Figure 5.16: All solutions of $x - 3y < 6$.

Example 5.3.2. *Graph the solutions of* $2x + 5y \geq 10$.

Answer. *We first graph the border line represented by* $2x + 5y = 10$*; we will draw the line as a* solid *line since the inequality is non-strict, and so points on the border do represent solutions to the inequality. In order to graph the border line, we might use the following table of values:*

x	y	*Solution*
$\boxed{0}$	2	$(0, 2)$
5	$\boxed{0}$	$(5, 0)$

Now we choose a test point; this time let's choose the origin, with coordinates $(0, 0)$. *Substituting these coordinates into the original inequality* $2x + 5y \geq 10$,

$$\begin{array}{rcrcl} 2(0) & + & 5(0) & \geq & 10 \\ 0 & + & 0 & \geq & 10 \\ & & 0 & \geq & 10. \end{array}$$

The inequality is false; $(0, 0)$ *is not a solution to* $2x + 5y \geq 10$. *We shade all points on the* opposite side *of the border line as the origin to represent all solutions of* $2x + 5y \geq 10$. *See Figure 5.17.*

Example 5.3.3. *Graph all solutions of* $y < -\frac{1}{3}x + 1$.

Answer. *First, as always, we graph the border line represented by* $y = -\frac{1}{3}x + 1$. *We will draw it using a dashed line since the inequality is strict. This time, since the equation is written in slope-intercept form, we see that the* y*-intercept has*

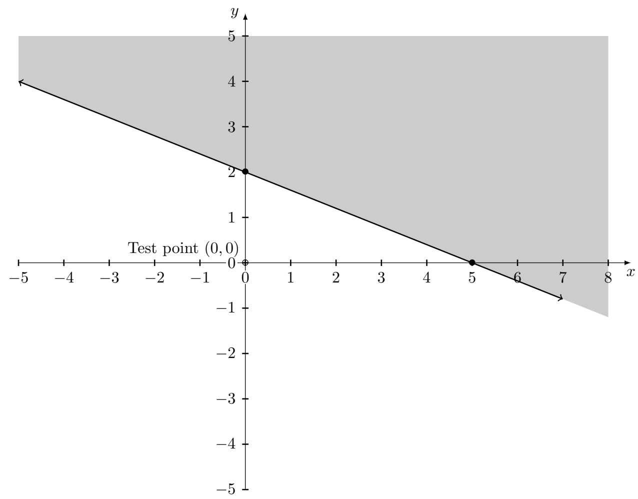


Figure 5.17: All solutions of $2x + 5y \geq 10$.

coordinates $(0, 1)$. *A second solution can be obtained by "moving" down one unit and to the right three units to give* $(3, 0)$.

Now we choose a test point; since zero is a nice number to work with let's choose the origin with coordinates $(0, 0)$ *again. To decide whether it is a solution, we substitute into* $y < -\dfrac{1}{3}x + 1$:

$$\begin{array}{cccc} (0) & < -\frac{1}{3}(0) & + & 1 \\ 0 & < & 0 & + \ 1 \\ 0 & < & 1. & \end{array}$$

The inequality is true; $(0, 0)$ *is a solution of* $y < -\dfrac{1}{3}x + 1$. *We will shade all points on the* same side *of the border line as the origin* $(0, 0)$. *See Figure 5.18.*

Method 2: Standard form method

Many students look at a few examples of linear inequalities and try to find patterns, or "shortcuts," to the test point method. "Wouldn't it be great," someone might say, "if every 'less than' inequality had a graph shaded below the border line! Then I don't have to waste my time with test points." However, look back at Examples 5.3.1 and 5.3.3; if there is a pattern, it is not so simple. In fact, since the an inequality can be written in so many equivalent forms, there is really no hope for an easy "shortcut."

However, if we make the effort of writing the inequality in a standard form, it is possible to make "rules" for which side of the border line to shade. Here is an example of such rules:

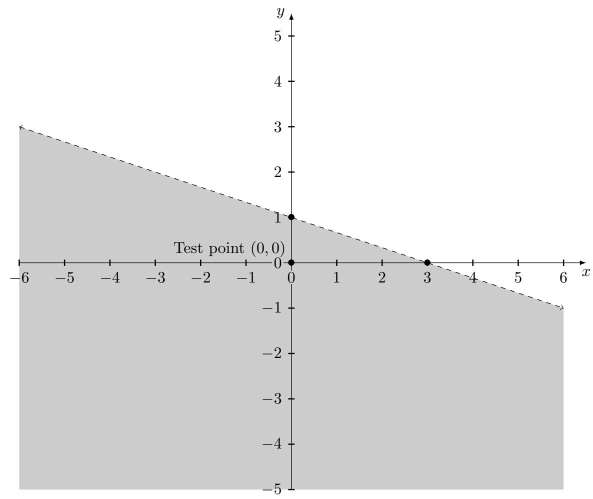


Figure 5.18: All solutions of $y < -\frac{1}{3}x + 1$.

1. The points representing solutions to a linear inequality of the form $y < mx + b$ (or $y \leq mx + b$) lie *below* the border line given by $y = mx + b$.
2. The points representing solutions to a linear inequality of the form $y > mx + b$ (or $y \geq mx + b$) lie *above* the border line given by $y = mx + b$.

Notice what is "standard" about this standard form: The y variable is by itself on the *left* side of the inequality. As with linear equalities in one variable, where the standard form consisted of having the x variable by itself on the left side of the inequality, if the inequality is not in standard form, we can use our basic addition and multiplication principals to rewrite the inequality in standard form. Keep in mind that as always, multiplying or dividing both sides of an inequality by a negative quantity requires changing the sense of the inequality.

Example 5.3.3 already gave an example of these standard form rules, since in that example the y variable was already by itself on the left side. Notice in Figure 5.18 that the shaded region is *below* the border line, as the rules for standard form dictate for the inequality $<$.

Here are two more examples illustrating the standard form method for graphing linear inequalities in two variables.

Example 5.3.4. *Graph the solutions of* $3x + 4y \geq 12$.

Answer. *In this case, the inequality is not in our standard form. We solve for* y*:*

$$\begin{array}{ccccccc} 3x & + & 4y & \geq & & & 12 \\ -3x & & & \vdots & -3x & & \\ \hline & & 4y & \geq & -3x & + & 12 \\ & & \frac{4y}{4} & \geq & \frac{-3x+12}{4} & & \\ & & y & \geq & \frac{-3x}{4} & + & \frac{12}{4} \\ & & y & \geq & -\frac{3}{4}x & + & 3. \end{array}$$

Notice that at no point did we divide by a negative number; the sense of the inequality $\geq$ *does not change.*

One advantage of the standard form we have chosen is that the equation of the border line $y = -\dfrac{3}{4}x + 3$ *is in slope-intercept form. The* y*-intercept has coordinates* $(0, 3)$ *and the slope is* $-\dfrac{3}{4}$*, so the coordinates of a second point on the line is obtained by "moving" down three units and to the right four units from* $(0, 3)$*, giving* $(4, 0)$*. We draw the border line with a solid line since the inequality* $\geq$ *is not strict.*

Since the inequality in standard form is $\geq$*, we will shade* above *the border line. See Figure 5.19.*

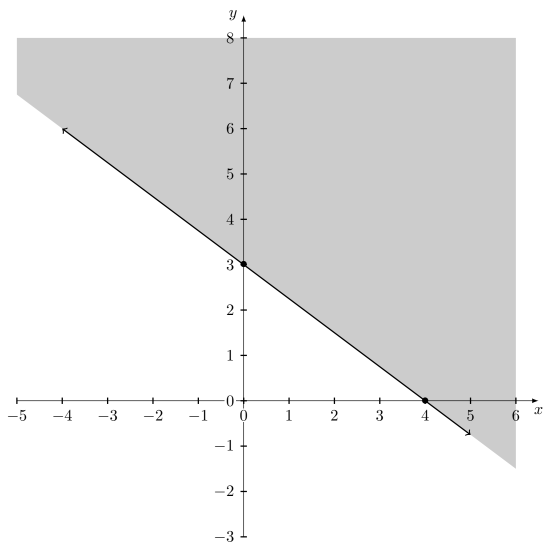


Figure 5.19: All solutions of $3x + 4y \geq 12$.

Example 5.3.5. *Graph the solutions of* $4x - 2y > 5$.

Answer. *We will first write the inequality in standard form:*

$$\begin{array}{rcrcccc} 4x & - & 2y & > & & & 5 \\ -4x & & & \vdots & -4x & & \\ \hline & & -2y & > & -4x & + & 5 \\ & & \frac{-2y}{-2} & < & \frac{-4x+5}{-2} & & \\ & & y & < & \frac{-4x}{-2} & + & \frac{5}{-2} \\ & & y & < & 2x & - & \frac{5}{2}. \end{array}$$

This time, when we divided by -2 on the fourth line, the sense of the inequality changes from $>$ to $<$.

The border line, represented by $y = 2x - \dfrac{5}{2}$, has y-intercept $\left(0, -\dfrac{5}{2}\right)$ and slope $m = 2 = \dfrac{2}{1}$. We can obtain a second point by starting from $\left(0, -\dfrac{5}{2}\right)$ and "moving" up 2 units and to the right 1 unit to give $\left(1, -\dfrac{1}{2}\right)$. (Notice that $-5/2 = -2.5$) We will draw the border line with a dashed line since the original inequality $>$ is strict.

Even though the original inequality was $>$, in standard form the inequality changed to $<$ (when we divided by a negative number). For that reason, we will shade below *the border line. See Figure 5.20.*

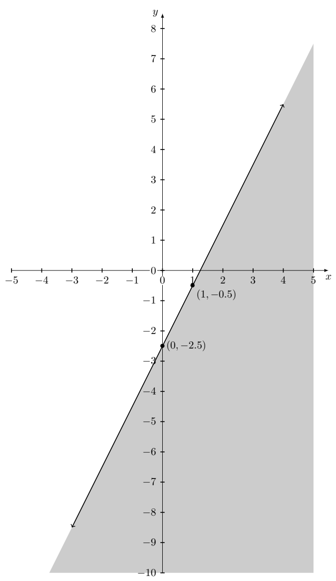


Figure 5.20: All solutions of $4x - 2y > 5$.

5.3.1 Exercises

Solve the following linear inequalities in two variables. In each case, graph all solutions and list five individual solutions.

1. $-x - y > 6$
2. $2x + 5y \leq 10$
3. $3x - 2y \geq 12$
4. $-4x + y > 4$
5. $y \geq -\frac{1}{2}x + 4$
6. $y < 1$

5.4 Solving systems of linear equations

A *system* of equations represents a situation where a solution must make all of several equations true, as opposed to just one equation. In this section we will consider only systems of two linear equations in two unknowns. A solution to such a system will be an ordered pair which, when substituted, makes both equations true.

For example, the following is a typical system of linear equations:

$$\begin{cases} 2x & + & 5y & = & 13 & \circledcirc \\ x & - & 2y & = & 2 & \circledast \end{cases} \tag{5.1}$$

There are a few things to notice about our notation in writing systems of linear equations:

- The system is indicated by the symbol $\{$. This indicates that a solution must make both equations true. (Caution: Not every text uses this symbol.)
- We write both equations is the general form $Ax + By = C$, where all variable terms are on the left side of the equations and all constant terms are on the right side. (If an equation is not written in this form, it can be rewritten as an equivalent one in the general form by using the addition principle.)
- We have written the equations so that like terms are in the same "column," with x-terms written above x-terms and y-terms written above y-terms.
- We use the symbols $\circledcirc$ and $\circledast$ to represent the two equations. For example, in this case, "Equation $\circledcirc$," or just $\circledcirc$, will refer to the equation $2x + 5y = 13$.

Let's look at some potential solutions for System (5.1). The reader should check the validity of the statements below:

- $(9, -1)$ is a solution to Equation $\circledcirc$, but $(9, -1)$ is not a solution to Equation $\circledast$. So $(9, -1)$ IS NOT a solution to System (5.1).
- $(6, 2)$ is a NOT solution to Equation $\circledcirc$, but $(6, 2)$ is a solution to Equation $\circledast$. So $(6, 2)$ IS NOT a solution to System (5.1).
- $(4, 1)$ is a solution to Equation $\circledcirc$, and $(4, 1)$ is not a solution to Equation $\circledast$. So $(4, 1)$ IS a solution to System (5.1).
- $(0, 0)$ is a not solution to Equation $\circledcirc$, and $(0, 0)$ is not a solution to Equation $\circledast$. So $(0, 0)$ IS NOT a solution to System (5.1).

The preceding paragraph should convince the reader that to find solutions to a system of equations, it is not enough to solve the two equations separately. At this point, we found one solution to System (5.1), but we can't be sure that it is the *only* solution. To do that, we need a method to solve systems of linear equations.

Before discussing a general method, we can again let geometry give us a guide as to what to expect. We know, for example, that Equation ⊚ has infinitely many solutions, which form a line when plotted in an xy-plane. We also know that Equation ⊛ also has infinitely many solutions, which form a different line when plotted in an xy-plane. So, if we graph both equations in the same xy-plane, a point will represent a solution to *both* equations if it lies on both lines—in other words, if it is a point of intersection of the two lines. But we know from elementary geometry that two non-parallel lines have exactly one point of intersection. So we have the following conclusion:

> A typical system of two linear equations in two variables will have exactly one solution. The solution, when plotted on an xy-plane, represents the point of intersection of the lines represented by the two equations.

In fact, this discussion already gives one method to solve a system of linear equations: Graph both equations on the same xy-plane, and the solution will be the coordinates of the point of intersection. However, this method requires a high degree of accuracy in plotting, and we will not generally rely on this method to solve systems of linear equations.

There is another, more algebraic way to solve systems of linear equations. Beginning with one of the equations, we could solve for one of the variables, say y, in terms of the other variable x. Then we could substitute this expression for y in terms of x into the second equation to obtain a new equation in just one variable x. This new linear equation (in one variable) will typically have one solution. Substituting this solution into the first equation (for y in terms of x) will give a corresponding value of y. The solution will be an ordered pair consisting of the solutions for x and y.

The method in the preceding paragraph is sometimes known (for obvious reasons) as the *substitution method*. Despite the fact this method applies to a wide variety of systems beyond those that we are considering here, we will not pursue this method any further. In many situations it requires detailed calculations with fractions, which as it turns out can be avoided in most cases we will encounter.

Solving systems of linear equations: Elimination method

We are going to outline a method for solving systems of two linear equations in two variables x and y, both of which have *integer* coefficients for both variables (this can always be arranged using the method of Section 4.2.4). We will arrive at the method by considering several examples, from simpler to more general.

Example 5.4.1. *Solve:*

$$\begin{cases} x + y = 4 & \circledcirc \\ x - y = 1 & \circledast \end{cases} \tag{5.2}$$

Answer. *Looking at System (5.2), we can notice that the* y*-terms have a special form in Equations* $\circledcirc$ *and* $\circledast$*: they are "opposites," in the sense that their coefficients (*1 *and* -1*) have the same magnitude but opposite sign. Let's apply the addition principle, which as a reminder states that we can add the same quantity to both sides of an equation without changing the solutions. For a solution to Equation* $\circledast$*, both sides are equal, so we will "add Equation* $\circledast$ *to Equation* $\circledcirc$*," meaning add the left sides and right sides of the equations.*

Eliminate y*:*

$$\begin{array}{cccccc} x & + & y & = & 4 & \circledcirc \\ x & - & y & = & 1 & \circledast \\ \hline 2x & & & = & 5 & \circledcirc + \circledast \end{array}$$

Notice that the new equation, which we denote $\circledcirc + \circledast$*, is an equation in one variable, with solution* $5/2$*. We have learned so far that if* (x, y) *is a solution to System (5.2), then* x *must have the value* $5/2$*.*

What is the corresponding y*-value for the solution? One way to find this would be to substitute the* x*-value* $5/2$ *into either Equation* $\circledcirc$ *or* $\circledast$ *to obtain a new equation in one variable* y *and then solve. However, let's stay in the spirit of "elimination."*

The preceding step of eliminating y *worked so well because the original coefficients of* y *were so nice. If we wanted to eliminate* x*, adding the equations directly does not work, as we just saw. While the coefficients of* x*, which are both* 1*, do have the same magnitude, they have the same sign, and so are not "opposites."*

But why not, instead of adding the two equations, subtract them—or, what is the same, add the opposite of Equation $\circledast$ *to Equation* $\circledcirc$*?*

Eliminate x*:*

$$\begin{array}{cccccccc} x & + & y & = & 4 & \circledcirc & & \\ -x & + & y & = & -1 & \circledast & \times & (-1) \\ \hline & & 2y & = & 3 & \circledcirc & - & \circledast \end{array}$$

Notice that we multiplied every term on both sides *of Equation* $\circledast$ *by* -1*. We represent this with the notation* $\circledast \times (-1)$*.*

So after adding to obtain $\circledcirc - \circledast$ *(which is the same as* $\circledcirc + (-1) \times \circledast$*), we obtain the equation in one variable* $2y = 3$*, which has solution* $3/2$*. This tells us that if* (x, y) *is a solution to System (5.2), then* y *must have the value* $3/2$*.*

From our preceding discussion, we expect that System (5.2) has one solution. We conclude:

The solution to System (5.2) is $\left(\dfrac{5}{2}, \dfrac{3}{2}\right)$.

It is worth pointing out from this first example that we did encounter fractions in our solution, even though the system involved equations without fractional coefficients. This is completely normal. However, we did not encounter fractions until the very last step of each elimination, and in fact, we never had to perform operations with these fractions. As we will see, this is typical for systems with integer coefficients and a major advantage of the elimination method.

From our first example, we can already see the outlines of the elimination method: Combine the two equations in such a way that one of the variables is "eliminated" in order to find a value for the other variable. Then repeat the process, eliminating the other variable to find the value for the remaining unknown. The solution is the ordered pair formed by the two values obtained in this way.

What remains to investigate is exactly how to combine the equations in such a way that one variable is always eliminated. The next example is a step in that direction.

Example 5.4.2. *Solve:*

$$\begin{cases} 3x & + & y & = & 9 & \circledcirc \\ x & - & 2y & = & -6 & \circledast \end{cases} \tag{5.3}$$

Answer. *In this example, unfortunately, the coefficients of neither variable are "opposites." In fact, neither adding nor subtracting the equations will eliminate either of the variables this time.*

However, we don't give up hope. Notice that even though the coefficients of y *are not opposites, at least they have opposite signs! If there was only a way to change the equations in such a way that the magnitudes were equal...*

Actually, the notation we used in the first example already had the clue to a way around this problem. If we multiply both sides of Equation $\circledcirc$ *by* 2*, then the new* y *term will be opposite that of the* y*-term in Equation* $\circledast$*; adding the resulting equations will eliminate* y*!*

Eliminate y*:*

$$\begin{array}{ccccccccc} 6x & + & 2y & = & 18 & \circledcirc & \times & 2 \\ x & - & 2y & = & -6 & \circledast & & \\ \hline 7x & & & = & 12 & 2\times\circledcirc & + & \circledast \end{array}$$

After eliminating y*, we obtain an equation in just one variable* (x) *whose solution is* $12/7$*. The conclusion is that if* (x, y) *is a solution to System (5.3), then* x *must be* $12/7$*.*

Turning now to the y*-coordinate of the solution, we want to eliminate* x*. This time, the coefficients of* x *not only have different magnitudes* (1 *and* 3)*, but they have the same sign. They are far from being opposites. A little thought, though, can convince us that again, we already have the idea of how to cope with this: why not multiply Equation* $\circledast$ *by the* negative *number* -3*. That way, the resulting coefficients of* x *will have the same magnitudes but opposite signs:*

Eliminate x:

$$\begin{array}{rcrcrcc} 3x & + & y & = & 9 & & \circledcirc \\ -3x & + & 6y & = & 18 & & \circledast \times (-3) \\ \hline & & 7y & = & 27 & & \circledcirc \quad + \quad (-3) \times \circledast \end{array}$$

In the resulting equation $\circledcirc + (-3) \times \circledast$*, we have eliminated* x *to obtain an equation in one variable* y *with solution* $27/7$*.*

The solution to System (5.3) is

$$\left(\frac{12}{7}, \frac{27}{7}\right).$$

One thing should be clear from these examples so far: Be careful of signs when multiplying both sides by a negative number!

Example 5.4.3. *Solve:*

$$\begin{cases} 2x & + & 5y & = & 13 & \circledcirc \\ x & - & 2y & = & 2 & \circledast \end{cases} \tag{5.4}$$

Answer. *This is the example that we used in the opening of the section (System (5.1). We already saw the solution at that time, when we were checking if various ordered pairs were solutions. Now we will apply the elimination method to actually find the solution "from scratch."*

In looking at the system, we see that we can eliminate x *exactly as in the previous example. We will multiply Equation* $\circledast$ *by* -2 *(notice that the coefficients of* x *initially have the same sign, so we need to multiply by a negative number in order to make the resulting coefficients "opposite."*

Eliminate x:

$$\begin{array}{rcrcrcc} 2x & + & 5y & = & 13 & & \circledcirc \\ -2x & + & 4y & = & -4 & & \circledast \times (-2) \\ \hline & & 9y & = & 9 & & \circledcirc \quad + \quad (-2) \times \circledast \end{array}$$

The equation $9y = 9$ *has* 1 *as a solution, so the* y*-coordinate of the solution to System (5.4) is* 1*.*

When we turn to eliminating y*, we encounter a new problem. The good news is that the coefficients of* y *(*5 *and* -2*) have opposite signs. But there is no way to multiply just one of the equations by an integer to make the coefficients of* y *"opposites," as we need to eliminate* y*.*

It turns out that the way around this difficulty is not hard: we will use the multiplication principle on both *equations. First, we find a common multiple of the magnitudes* 2 *and* 5*. That is, we find an integer that both* 2 *and* 5 *divide evenly. The least common multiple of* 2 *and* 5 *is* 10*. (The reader might notice that finding a common multiple of* 2 *and* 5 *is exactly the same mental process as finding a common denominator*[4] *for two fractions with denominators* 2 *and* 5*.)*

[4] In fact, a common denominator is just a common multiple of the denominators.

Once the common multiple 10 *is found, we will multiply both equations by a number so that the magnitude of the coefficient of* y *is* 10*. That is, in this case, we will multiply Equation* ⊚ *by* 2 *and Equation* ⊛ *by* 5*.*

Eliminate y*:*

$$\begin{array}{rcrcrccc} 4x & + & 10y & = & 26 & \circledcirc & \times & 2 \\ 5x & - & 10y & = & 10 & \circledast & \times & 5 \\ \hline 9x & & & = & 36 & 2\times\circledcirc & + & 5\times\circledast \end{array}$$

The resulting equation $9x = 36$ *has* 4 *as a solution, so the* x*-coordinate of the solution to System (5.4) is* 4*.*

Putting this together with the result of the previous elimination step, we find that the solution to System (5.4) is $(4, 1)$*.*

With the three preceding examples as guides, we can write down a general method that describes the "elimination" that is at the heart of the elimination method.

In order to eliminate a variable from a system of linear equations:

- Find a common multiple of the magnitudes of the coefficients of the variable to be eliminated;
- If the coefficients of the desired variable originally had different signs, multiply each equation by a positive number so that the magnitude of the coefficients of the desired variable in the resulting equations is the common multiple;
- If the coefficients of the desired variable originally had the same sign, multiply one equation by a positive number and one equation by a negative number so that the magnitude of the coefficients of the desired variable is the common multiple.

After these preparations, adding the resulting equations will result in a new equation that does not involve the variable to be eliminated.

The next example illustrates the general method.

Example 5.4.4. *Solve:*

$$\begin{cases} 6x & + & 4y & = & 16 & \circledcirc \\ 9x & - & 5y & = & 7 & \circledast \end{cases} \tag{5.5}$$

Answer. *To eliminate* x*, we see that the least common multiple of the coefficients of* x *(*6 *and* 9*) is* 18*. Since the signs of the coefficients are the same, we will multiply one equation (say Equation* ⊛*) by a negative number. Specifically, we will multiply Equation* ⊚ *by* 3 *and Equation* ⊛ *by* -2*:*

Eliminate x:

$$\begin{array}{rcrcrl} 18x & + & 12y & = & 48 & \circledcirc \times 3 \\ -18x & + & 10y & = & -14 & \circledast \times (-2) \\ \hline & & 22y & = & 34 & 3 \times \circledcirc \quad + \quad (-2) \times \circledast \end{array}$$

The solution to the resulting equation $22y = 34$ *is* $17/11$ *(after reducing), so the* y*-coordinate of the solution to System (5.5) is* $17/11$

Now to eliminate y*, we see that the least common multiple of the magnitudes of the coefficients of* y *in System (5.5) (*4 *and* -5*) is* 20*. Since the signs of the coefficients are already different, we will multiply both equations by positive numbers to achieve the common multiple. Specifically, we will multiply Equation* $\circledcirc$ *by* 5 *and Equation* $\circledast$ *by* 4*:*

Eliminate y:

$$\begin{array}{rcrcrl} 30x & + & 20y & = & 80 & \circledcirc \times 5 \\ 36x & - & 20y & = & 28 & \circledast \times 4 \\ \hline 66x & & & = & 108 & 5 \times \circledcirc \quad + \quad 4 \times \circledast \end{array}$$

The solution to the equation $66x = 108$ *is* $18/11$ *(after reducing), so the* x*-coordinate of the solution to System (5.5) is* $18/11$*.*

Together with the first elimination step, we see that the solution to System (5.5) is

$$\left(\frac{18}{11}, \frac{17}{11}\right).$$

5.4.1 Systems that do not have exactly one solution

By thinking of a system of two linear equations in two unknowns graphically, we came to the conclusion that a "typical" such system will have exactly one solution, just like a "typical" linear equation in one variable will have exactly one solution. However, keeping in mind Section 4.2.3, we might expect that not every system is "typical."

To see what might go wrong, consider following example.

Example 5.4.5. *Solve:*

$$\begin{cases} x + 2y = 9 & \circledcirc \\ 3x + 6y = 10 & \circledast \end{cases} \tag{5.6}$$

Answer. *We will apply our elimination method, as usual.*

To eliminate x*, we see that the least common multiple of the coefficients of* x *(*1 *and* 3*) is* 3*. The signs of the coefficients are the same, so we will multiply one equation (say Equation* $\circledcirc$*) by a negative number. Specifically, we will multiply Equation* $\circledcirc$ *by* -3 *and Equation* $\circledast$ *by* 1*:*

Eliminate x:

$$\begin{array}{rcrcrl} -3x & - & 6y & = & -27 & \circledcirc \times (-3) \\ 3x & + & 6y & = & 10 & \circledast \times 1 \\ \hline & & 0 & = & -17 & (-3) \times \circledcirc \ + \ \circledast \end{array}$$

Although we were aiming to eliminate x, both variables were eliminated in the resulting equation!

As in Section 4.2.3, the question in such cases is whether the new equation is true or false.

Since the equation $0 = -17$ is false, System (5.6) has no solution.

What went "wrong" in the previous example? Why did our elimination procedure end up eliminating both variables, instead of the "typical" one variable at a time?

To investigate Example 5.2.12 more closely, let's rewrite both equations in slope-intercept form. Solving Equation $\circledcirc$ for y gives:

$$\begin{array}{rcrcrcr} x & + & 2y & = & & & 9 \\ -x & & & \vdots & -x & & \\ \hline & & 2y & = & -x & + & 9 \\ & & \frac{2y}{2} & = & \frac{-x+9}{2} & & \\ & & y & = & \frac{-x}{2} & + & \frac{9}{2} \\ & & y & = & -\frac{1}{2}x & + & \frac{9}{2}. \end{array}$$

We see that the slope of the line given by Equation $\circledcirc$ is $-1/2$ and the y-intercept is $(0, 9/2)$.

Turning to Equation $\circledast$, we solve for y:

$$\begin{array}{rcrcrcr} 3x & + & 6y & = & & & 10 \\ -3x & & & \vdots & -3x & & \\ \hline & & 6y & = & -3x & + & 10 \\ & & \frac{6y}{6} & = & \frac{-3x+10}{6} & & \\ & & y & = & \frac{-3x}{6} & + & \frac{10}{6} \\ & & y & = & -\frac{1}{2}x & + & \frac{5}{3}. \end{array}$$

The slope of the line given by Equation $\circledast$ is $-1/2$ and the y-intercept is $(0, 5/3)$.

Comparing, we see that the two lines represented by Equations $\circledcirc$ and $\circledast$ have the same slope, but different y-intercepts. In other words, the Equations represent parallel lines!

In fact, this shouldn't be a big surprise. Our geometric thinking that led us to the conclusion that the typical system of two linear equations in two variables had a single solution was that two different lines in a plane typically intersect in one point—except when the two lines are parallel, in which case they have no point of intersection.

Keeping in mind that we saw two "unusual" situations in Section 4.2.3, let's look at one last example.

Example 5.4.6. *Solve:*

$$\begin{cases} 2x - y = 5 & \circledcirc \\ 4x - 2y = 10 & \circledast \end{cases} \tag{5.7}$$

Answer. *Let's eliminate* x *first.The least common multiple of the coefficients of* x *(*2 *and* 4*) is* 4*. The signs of the coefficients are the same, so we will multiply one equation (say Equation* $\circledcirc$*) by a negative number. Specifically, we will multiply Equation* $\circledcirc$ *by* -2 *and Equation* $\circledast$ *by* 1*:*

Eliminate x*:*

$$\begin{array}{rcrcrl} -4x & + & 2y & = & -10 & \circledcirc \times (-2) \\ 4x & + & 2y & = & 10 & \circledast \times 1 \\ \hline & & 0 & = & 0 & (-2) \times \circledcirc + \circledast \end{array}$$

Again, we have eliminated both variables. This time, though, the resulting equation is true.

In the one-variable situation in Section 4.2.3, this would have led us to conclude that all real numbers were solutions. However, in this case, not every ordered pair is a solution. For example, the reader can check that $(0, 0)$ *is not a solution to System (5.7).*

To understand what the resulting true equation is telling us, let's again rewrite the equations in slope-intercept form to see what some geometry can tell us.

Solving Equation $\circledcirc$ *for* y *gives:*

$$\begin{array}{rcrcrcr} 2x & - & y & = & & & 5 \\ -2x & & & \vdots & -2x & & \\ \hline & & -y & = & -2x & + & 5 \\ & & \frac{-y}{-1} & = & \frac{-2x+5}{-1} & & \\ & & y & = & \frac{-2x}{-1} & + & \frac{5}{-1} \\ & & y & = & 2x & - & 5. \end{array}$$

The slope of the line given by Equation $\circledcirc$ *is* 2 *and the* y*-intercept is* $(0, -5)$*.*

Solving Equation $\circledast$ for y gives:

$$\begin{array}{rcrcrcrcr}
4x & - & 2y & = & & & 10 \\
-4x & & & \vdots & -4x & & \\ \hline
& & -2y & = & -4x & + & 10 \\
& & \frac{-2y}{-2} & = & \frac{-4x+10}{-2} & & \\
& & y & = & \frac{-4x}{-2} & + & \frac{10}{-2} \\
& & y & = & 2x & - & 5.
\end{array}$$

The slope of the line given by Equation $\circledast$ is 2 and the y-intercept is $(0, -5)$.

Notice that the two equations represent lines with the same slope *and the* same y-intercept*—they actually represent the same line.*

In other words, System (5.7) has infinitely many solutions*, all of which are represented by the points on the line given by either Equation $\circledcirc$ or Equation $\circledast$. We could graph the solutions: See Figure 5.21.*

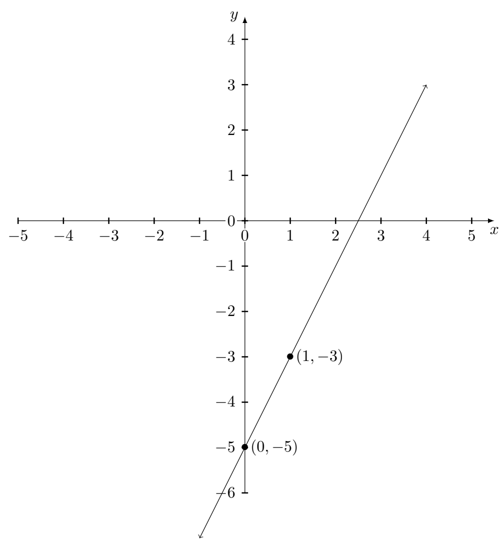


Figure 5.21: All solutions of

$$\begin{cases} 2x - y = 5 \\ 4x - 2y = 10 \end{cases}$$

The unusual cases in this section are detected using the elimination method when both variables are eliminated. We can summarize our results as follows:

Suppose that, in using the elimination method to solve a system of two linear equations in two unknowns, an equation results which involves neither of the two variables.

- If the resulting equation is false, the system has no solution. The lines represented by the two equations are parallel lines.
- If the resulting equation is true, the system has infinitely many solutions, represented by the points on the graph of either equation. The lines represented by the two equations are the same.

5.4.2 Exercises

Solve the following systems of linear equations.

1. $\begin{cases} x - y = 8 \\ 2x + y = 1 \end{cases}$

2. $\begin{cases} 3x - 2y = -1 \\ 2x + y = -3 \end{cases}$

3. $\begin{cases} x - 2y = 4 \\ 5x + 6y = 3 \end{cases}$

4. $\begin{cases} 2x - 3y = 6 \\ x + 4y = 8 \end{cases}$

5. $\begin{cases} 3x + 2y = 4 \\ -x + 5y = 10 \end{cases}$

6. $\begin{cases} 4x - y = 6 \\ 2x + 3y = 8 \end{cases}$

7. $\begin{cases} -2x + y = 5 \\ 4x - 2y = 8 \end{cases}$

8. $\begin{cases} y = 2x - 5 \\ y = \dfrac{x+4}{2} \end{cases}$

5.5 Chapter summary

- A typical linear equation in two variables has infinitely many solutions. When graphed on an xy-plane, the points corresponding to solutions of a linear equation in two variables form a line.
- The most basic strategy to graph all solutions of a linear equation in two variables is to plot two solutions, then draw the line passing through these two.
- An alternate method to graph all solutions to a linear equation is to use the slope of the line and the coordinates of one point on the line. This is most useful when the equation is written in slope-intercept form.
- Given the slope of a line and the coordinates of one point on the line, the point-slope form of a line gives a "formula" to write an equation of the line.
- Linear inequalities in two variables typically have infinitely many solutions. The points corresponding to these solutions in an xy-plane all lie on the same half the xy-plane with border line given by the corresponding linear equation.
- A typical system of two linear equations in two variables has one solution.