Quantitative Module **B**

Linear Programming

Module Outline

LEARNING OBJECTIVES

When you complete this module you should be able to

IDENTIFY OR DEFINE:

Objective function

Constraints

Feasible region

Iso-profit/iso-cost methods

Corner-point solution

Shadow price

DESCRIBE OR EXPLAIN:

How to formulate linear models

Graphical method of linear programming

How to interpret sensitivity analysis

The storm front closed in quickly on Chicago's O'Hare Airport, shutting it down without warning. The heavy thunderstorms, lightning, and poor visibility sent American Airlines passengers and ground crew scurrying. Because American Airlines uses linear programming (LP) to schedule flights, hotels, crews, and refueling, LP has a direct impact on profitability. As the president of AA's Decision Technology Group says, "Finding fast solutions to LP problems is essential. If we get a major weather disruption at one of the hubs, such as Dallas or Chicago, then a lot of flights may get canceled, which means we have a lot of crews and airplanes in the wrong places. What we need is a way to put that whole operation back together again." LP is the tool that helps airlines such as American unsnarl and cope with this weather mess.

Linear Programming (LP)

A mathematical technique designed to help operations managers plan and make decisions relative to the trade-offs necessary to allocate resources.

Many operations management decisions involve trying to make the most effective use of an organization's resources. Resources typically include machinery (such as planes, in the case of an airline), labor (such as pilots), money, time, and raw materials (such as jet fuel). These resources may be used to produce products (such as machines, furniture, food, or clothing) or services (such as airline schedules, advertising policies, or investment decisions). **Linear programming (LP)** is a widely used mathematical technique designed to help operations managers plan and make the decisions necessary to allocate resources.

A few examples of problems in which LP has been successfully applied in operations management are

1. Scheduling school buses to *minimize* the total distance traveled when carrying students.
2. Allocating police patrol units to high crime areas to *minimize* response time to 911 calls.
3. Scheduling tellers at banks so that needs are met during each hour of the day while *minimizing* the total cost of labor.
4. Selecting the product mix in a factory to make best use of machine- and labor-hours available while *maximizing* the firm's profit.
5. Picking blends of raw materials in feed mills to produce finished feed combinations at *minimum* cost.
6. Determining the distribution system that will *minimize* total shipping cost from several warehouses to various market locations.
7. Developing a production schedule that will satisfy future demands for a firm's product and at the same time *minimize* total production and inventory costs.
8. Allocating space for a tenant mix in a new shopping mall so as to *maximize* revenues to the leasing company. (See the *OM in Action* box "Using LP to Select Tenants in a Shopping Mall.")

OM IN ACTION

Using LP to Select Tenants in a Shopping Mall

Homart Development Company is one of the largest shopping-center developers in the U.S. When starting a new center, Homart produces a tentative floor plan, or "footprint," for the mall. This plan outlines sizes, shapes, and spaces for large department stores. Leasing agreements are reached with the two or three major department stores that will become anchor stores in the mall. The anchor stores are able to negotiate highly favorable occupancy agreements. Homart's profits come primarily from the rent paid by the nonanchor tenants—the smaller stores that lease space along the aisles of the mall. The decision as to allocating space to potential tenants is, therefore, crucial to the success of the investment.

The tenant mix describes the desired stores in the mall by their size, general location, and type of merchandise or service provided. For example, the mix might specify two small jewelry stores in a central section of the mall and a medium-size shoe store and a large restaurant in one of the side aisles. In the past, Homart developed a plan for tenant mix using "rules of thumb" developed over years of experience in mall development.

Now, to improve its bottom line in an increasingly competitive marketplace, Homart treats the tenant-mix problem as an LP model. First, the model assumes that tenants can be classified into categories according to the type of merchandise or service they provide. Second, the model assumes that for each store type, store sizes can be estimated by distinct category. For example, a small jewelry store is said to contain about 700 square feet and a large one about 2,200 square feet. The tenant-mix model is a powerful tool for enhancing Homart's mall planning and leasing activities.

Sources: *Chain Store Age* (March 2000): 191–192; *Business World* (March 18, 2002): 1; and *Interfaces* (March–April 1988): 1–9.

REQUIREMENTS OF A LINEAR PROGRAMMING PROBLEM

All LP problems have four properties in common:

Objective function

A mathematical expression in linear programming that maximizes or minimizes some quantity (often profit or cost, but any goal may be used).

1. LP problems seek to *maximize* or *minimize* some quantity (usually profit or cost). We refer to this property as the **objective function** of an LP problem. The major objective of a typical firm is to maximize dollar profits in the long run. In the case of a trucking or airline distribution system, the objective might be to minimize shipping costs.

2. The presence of restrictions, or **constraints**, limits the degree to which we can pursue our objective. For example, deciding how many units of each product in a firm's product line to manufacture is restricted by available labor and machinery. We want, therefore, to maximize or minimize a quantity (the objective function) subject to limited resources (the constraints).

Constraints

Restrictions that limit the degree to which a manager can pursue an objective.

3. There must be *alternative courses of action* to choose from. For example, if a company produces three different products, management may use LP to decide how to allocate among them its limited production resources (of labor, machinery, and so on). If there were no alternatives to select from, we would not need LP.

4. The objective and constraints in linear programming problems must be expressed in terms of *linear equations* or inequalities.

FORMULATING LINEAR PROGRAMMING PROBLEMS

One of the most common linear programming applications is the *product-mix problem*. Two or more products are usually produced using limited resources. The company would like to determine how many units of each product it should produce to maximize overall profit given its limited resources. Let's look at an example.

Active Model B.1

This example is further illustrated in Active model B.1 on the CD-ROM and in the exercise on page 713.

Shader Electronics Example

The Shader Electronics Company produces two products: (1) the Shader Walkman, a portable CD/DVD player, and (2) the Shader Watch-TV, a wristwatch-size internet-connected color television. The production process for each product is similar in that both require a certain number of hours of electronic work and a certain number of labor-hours in the assembly department. Each Walkman takes 4 hours of electronic work and 2 hours in the assembly shop. Each Watch-TV requires 3 hours in electronics and 1 hour in assembly. During the current production period, 240

hours of electronic time are available, and 100 hours of assembly department time are available. Each Walkman sold yields a profit of $7; each Watch-TV produced may be sold for a $5 profit.

Shader's problem is to determine the best possible combination of Walkmans and Watch-TVs to manufacture to reach the maximum profit. This product-mix situation can be formulated as a linear programming problem.

TABLE B.1 ■

Shader Electronics
Company Problem
Data

| | HOURS REQUIRED TO PRODUCE 1 UNIT | | |
| | WALKMANS | WATCH-TVs | |
DEPARTMENT	(X_1)	(X_2)	AVAILABLE HOURS THIS WEEK
Electronic	4	3	240
Assembly	2	1	100
Profit per unit	$7	$5	

We begin by summarizing the information needed to formulate and solve this problem (see Table B.1). Further, let's introduce some simple notation for use in the objective function and constraints. Let

We name the decision variables X_1 and X_2 here but point out that any notation (such as WM and WT) would be fine as well.

$$X_1 = \text{number of Walkmans to be produced}$$
$$X_2 = \text{number of Watch-TVs to be produced}$$

Now we can create the LP *objective function* in terms of X_1 and X_2:

$$\text{Maximize profit} = \$7X_1 + \$5X_2$$

Our next step is to develop mathematical relationships to describe the two constraints in this problem. One general relationship is that the amount of a resource used is to be less than or equal to ($\leq$) the amount of resource *available*.

First constraint: Electronic time used is $\leq$ Electronic time available.

$$4X_1 + 3X_2 \leq 240 \text{ (hours of electronic time)}$$

Second constraint: Assembly time used is $\leq$ Assembly time available.

$$2X_1 + 1X_2 \leq 100 \text{ (hours of assembly time)}$$

Both these constraints represent production capacity restrictions and, of course, affect the total profit. For example, Shader Electronics cannot produce 70 Walkmans during the production period because if $X_1 = 70$, both constraints will be violated. It also cannot make $X_1 = 50$ Walkmans and $X_2 = 10$ Watch-TVs. This constraint brings out another important aspect of linear programming; that is, certain interactions will exist between variables. The more units of one product that a firm produces, the fewer it can make of other products.

GRAPHICAL SOLUTION TO A LINEAR PROGRAMMING PROBLEM

Graphical solution approach
A means of plotting a solution to a two-variable problem on a graph.

The easiest way to solve a small LP problem such as that of the Shader Electronics Company is the **graphical solution approach**. The graphical procedure can be used only when there are two **decision variables** (such as number of Walkmans to produce, X_1, and number of Watch-TVs to produce, X_2). When there are more than two variables, it is *not* possible to plot the solution on a two-dimensional graph; we then must turn to more complex approaches described later in this module.

Graphical Representation of Constraints

Decision variables
Choices available to a decision maker.

To find the optimal solution to a linear programming problem, we must first identify a set, or region, of feasible solutions. The first step in doing so is to plot the problem's constraints on a graph.

The variable X_1 (Walkmans, in our example) is usually plotted as the horizontal axis of the graph, and the variable X_2 (Watch-TVs) is plotted as the vertical axis. The complete problem may be restated as:

$$\text{Maximize profit} = \$7X_1 + \$5X_2$$

Subject to the constraints

$$4X_1 + 3X_2 \leq 240 \text{ (electronics constraint)}$$
$$2X_1 + 1X_2 \leq 100 \text{ (assembly constraint)}$$
$$X_1 \geq 0 \text{ (number of Walkmans produced is greater than or equal to 0)}$$
$$X_2 \geq 0 \text{ (number of Watch-TVs produced is greater than or equal to 0)}$$

These two constraints are also called the nonnegativity constraints.

The first step in graphing the constraints of the problem is to convert the constraint *inequalities* into *equalities* (or equations).

$$\text{Constraint A:} \quad 4X_1 + 3X_2 = 240$$
$$\text{Constraint B:} \quad 2X_1 + 1X_2 = 100$$

The equation for constraint A is plotted in Figure B.1 and for constraint B in Figure B.2.

To plot the line in Figure B.1, all we need to do is to find the points at which the line $4X_1 + 3X_2 = 240$ intersects the X_1 and X_2 axes. When $X_1 = 0$ (the location where the line touches the X_2 axis), it implies that $3X_2 = 240$ and that $X_2 = 80$. Likewise, when $X_2 = 0$, we see that $4X_1 = 240$ and that $X_1 = 60$. Thus, constraint A is bounded by the line running from ($X_1 = 0$, $X_2 = 80$) to ($X_1 = 60$, $X_2 = 0$). The shaded area represents all points that satisfy the original *inequality*.

Constraint B is illustrated similarly in Figure B.2. When $X_1 = 0$, then $X_2 = 100$; and when $X_2 = 0$, then $X_1 = 50$. Constraint B, then, is bounded by the line between ($X_1 = 0$, $X_2 = 100$) and ($X_1 = 50$, $X_2 = 0$). The shaded area represents the original inequality.

Figure B.3 shows both constraints together. The shaded region is the part that satisfies both restrictions. The shaded region in Figure B.3 is called the *area of feasible solutions*, or simply the **feasible region**. This region must satisfy *all* conditions specified by the program's constraints and is thus the region where all constraints overlap. Any point in the region would be a *feasible solution* to the Shader Electronics Company problem. Any point outside the shaded area would represent an *infeasible solution*. Hence, it would be feasible to manufacture 30 Walkmans

Feasible region
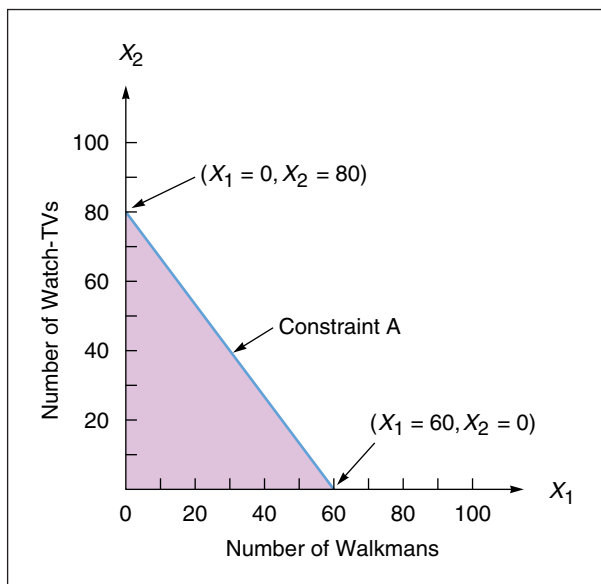The set of all feasible combinations of decision variables.
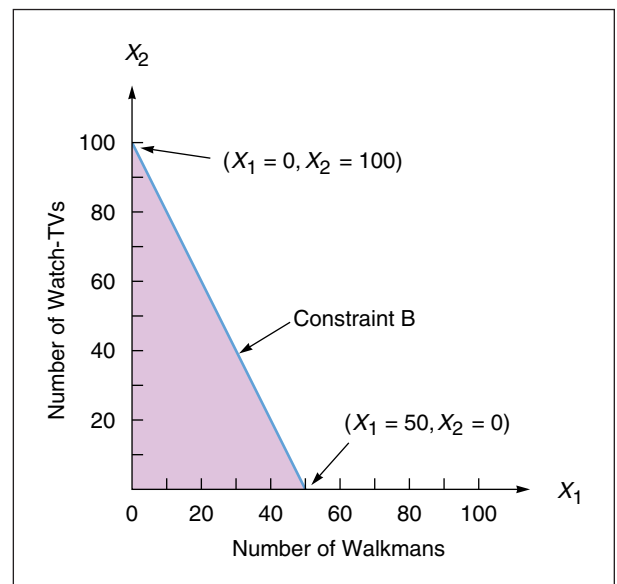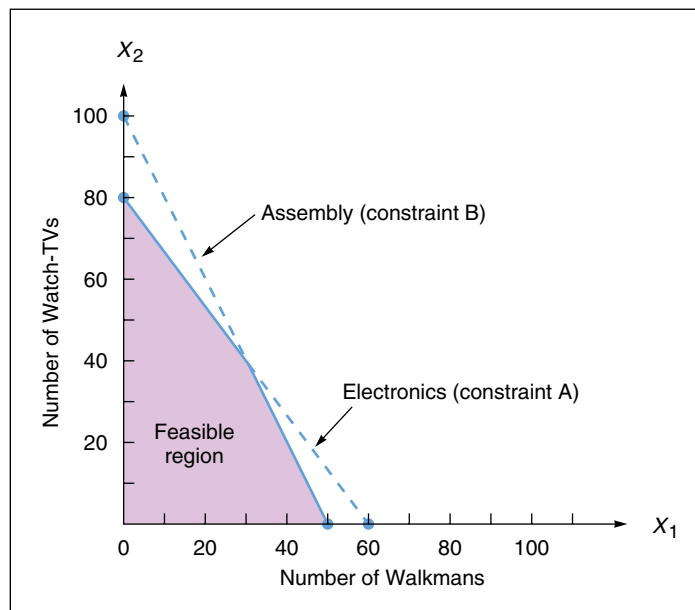

FIGURE B.1 ■ Constraint A

FIGURE B.2 ■ Constraint B

FIGURE B.3 ■

Feasible Solution Region
for the Shader Electronics
Company Problem

and 20 Watch-TVs ($X_1 = 30$, $X_2 = 20$), but it would violate the constraints to produce 70 Walkmans and 40 Watch-TVs. This can be seen by plotting these points on the graph of Figure B.3.

Iso-Profit Line Solution Method

Now that the feasible region has been graphed, we can proceed to find the *optimal* solution to the problem. The optimal solution is the point lying in the feasible region that produces the highest profit.

Once the feasible region has been established, several approaches can be taken in solving for the optimal solution. The speediest one to apply is called the **iso-profit line method**.[1]

Iso-profit line method
An approach to solving a linear programming maximization problem graphically.

We start by letting profits equal some arbitrary but small dollar amount. For the Shader Electronics problem, we may choose a profit of $210. This is a profit level that can easily be obtained without violating either of the two constraints. The objective function can be written as $210 = 7X_1 + 5X_2$.

This expression is just the equation of a line; we call it an *iso-profit line*. It represents all combinations (of X_1, X_2) that will yield a total profit of $210. To plot the profit line, we proceed exactly as we did to plot a constraint line. First, let $X_1 = 0$ and solve for the point at which the line crosses the X_2 axis:

$$\$210 = \$7(0) + \$5X_2$$
$$X_2 = 42 \text{ Watch-TVs}$$

Then let $X_2 = 0$ and solve for X_1:

$$\$210 = \$7X_1 + \$5(0)$$
$$X_1 = 30 \text{ Walkmans}$$

We can now connect these two points with a straight line. This profit line is illustrated in Figure B.4. All points on the line represent feasible solutions that produce a profit of $210.

We see, however, that the iso-profit line for $210 does not produce the highest possible profit to the firm. In Figure B.5, we try graphing two more lines, each yielding a higher profit. The middle equation, $280 = 7X_1 + 5X_2$, was plotted in the same fashion as the lower line. When $X_1 = 0$,

$$\$280 = \$7(0) + \$5X_2$$
$$X_2 = 56 \text{ Watch-TVs}$$

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[1]*Iso* means "equal" or "similar." Thus, an iso-profit line represents a line with all profits the same, in this case $210.
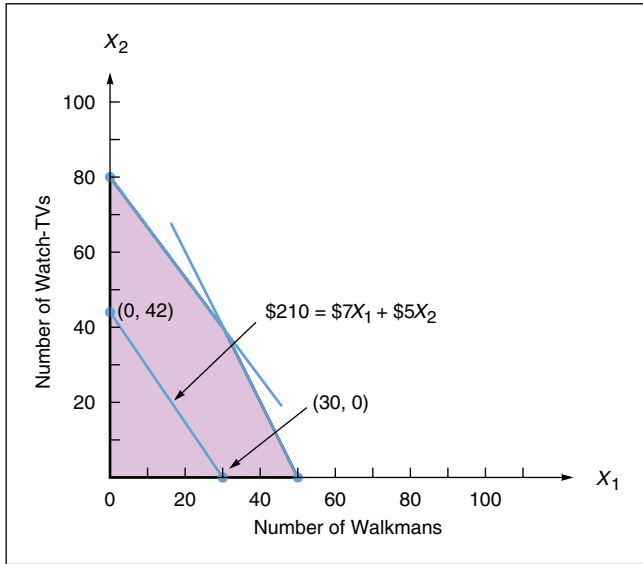
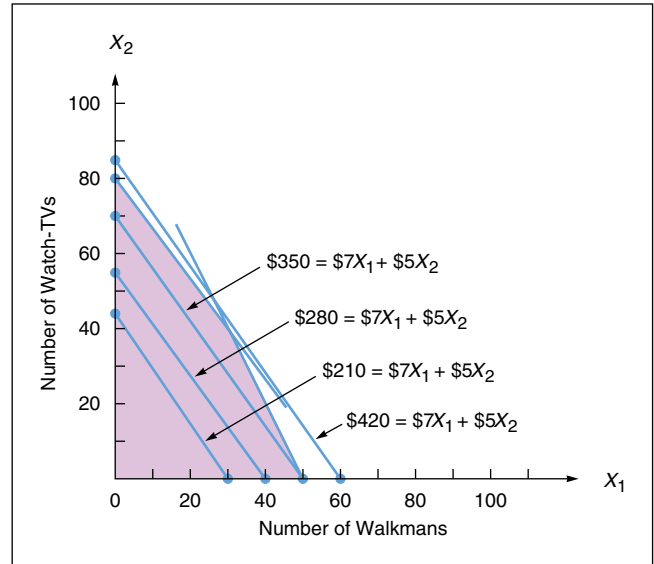**FIGURE B.4** ■ A Profit Line of $210 Plotted for the Shader Electronics Company

FIGURE B.5 ■ Four Iso-Profit Lines Plotted for the Shader Electronics Company

When $X_2 = 0$,

$$\$280 = \$7X_1 + \$5(0)$$
$$X_1 = 40 \text{ Walkmans}$$

Again, any combination of Walkmans (X_1) and Watch-TVs (X_2) on this iso-profit line will produce a total profit of $280.

Note that the third line generates a profit of $350, even more of an improvement. The farther we move from the 0 origin, the higher our profit will be. Another important point to note is that these iso-profit lines are parallel. We now have two clues as to how to find the optimal solution to the original problem. We can draw a series of parallel profit lines (by carefully moving our ruler in a plane parallel to the first profit line). The highest profit line that still touches some point of the feasible region will pinpoint the optimal solution. Notice that the fourth line ($420) is too high to count because it does not touch the feasible region.

The highest possible iso-profit line is illustrated in Figure B.6. It touches the tip of the feasible region at the corner point ($X_1 = 30, X_2 = 40$) and yields a profit of $410.

FIGURE B.6 ■

Optimal Solution for the Shader Electronics Problem
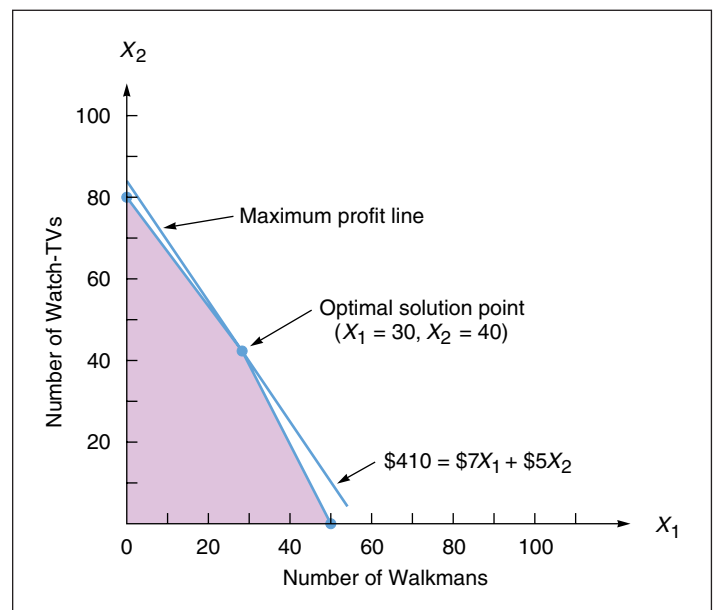
Corner-Point Solution Method

Corner-point method
A method for solving
graphical linear
programming problems.

A second approach to solving linear programming problems employs the **corner-point method**. This technique is simpler in concept than the iso-profit line approach, but it involves looking at the profit at every corner point of the feasible region.

The mathematical theory behind linear programming states that an optimal solution to any problem (that is, the values of X_1, X_2 that yield the maximum profit) will lie at a *corner point*, or *extreme point*, of the feasible region. Hence, it is necessary to find only the values of the variables at each corner; the maximum profit or optimal solution will lie at one (or more) of them.

Once again we can see (in Figure B.7) that the feasible region for the Shader Electronics Company problem is a four-sided polygon with four corner, or extreme, points. These points are labeled ①, ②, ③ and ④ on the graph. To find the (X_1, X_2) values producing the maximum profit, we find out what the coordinates of each corner point are, then determine and compare their profit levels.

Point ①:	$(X_1 = 0, X_2 = 0)$	Profit $\$7(0) + \$5(0) = \$0$
Point ②:	$(X_1 = 0, X_2 = 80)$	Profit $\$7(0) + \$5(80) = \$400$
Point ④:	$(X_1 = 50, X_2 = 0)$	Profit $\$7(50) + \$5(0) = \$350$

We skipped corner point ③ momentarily because to find its coordinates *accurately*, we will have to solve for the intersection of the two constraint lines. As you may recall from algebra, we can apply the method of *simultaneous equations* to the two constraint equations:

$$4X_1 + 3X_2 = 240 \quad (\textit{electronics time})$$
$$2X_1 + 1X_2 = 100 \quad (\textit{assembly time})$$

To solve these equations simultaneously, we multiply the second equation by -2:

$$-2(2X_1 + 1X_2 = 100) = -4X_1 - 2X_2 = -200$$

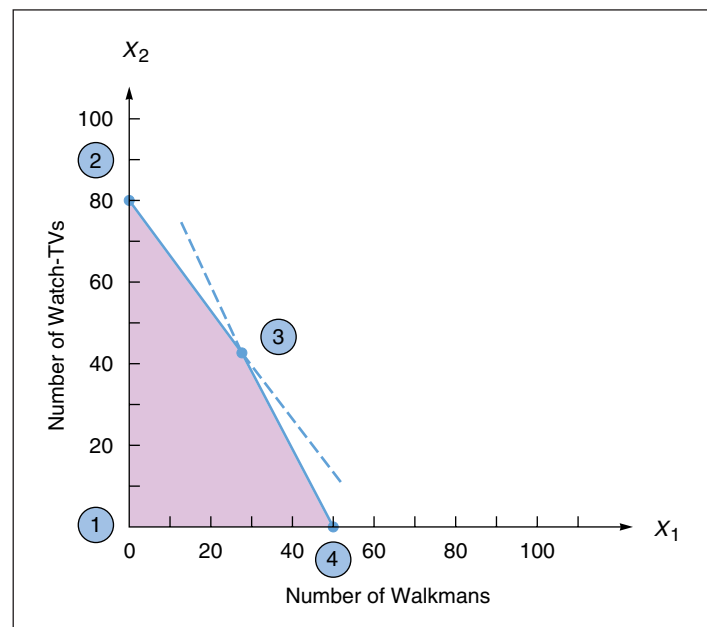and then add it to the first equation:

$$
\begin{aligned}
+4X_1 + 3X_2 &= 240 \\
-4X_1 - 2X_2 &= -200 \\
\hline
+1X_2 &= 40
\end{aligned}
$$

or

$$X_2 = 40$$

FIGURE B.7 ■

The Four Corner Points of
the Feasible Region

Doing this has enabled us to eliminate one variable, X_1, and to solve for X_2. We can now substitute 40 for X_2 in either of the original equations and solve for X_1. Let us use the first equation. When $X_2 = 40$, then

$$4X_1 + 3(40) = 240$$
$$4X_1 + 120 = 240$$

or

$$4X_1 = 120$$
$$X_1 = 30$$

Although the values for X_1 and X_2 are integers for Shader Electronics, this will not always be the case.

Thus, point ③ has the coordinates ($X_1 = 30$, $X_2 = 40$). We can compute its profit level to complete the analysis:

Point ③: ($X_1 = 30$, $X_2 = 40$)   Profit = $7(30) + $5(40) = $410

Because point ③ produces the highest profit of any corner point, the product mix of $X_1 = 30$ Walkmans and $X_2 = 40$ Watch-TVs is the optimal solution to the Shader Electronics problem. This solution will yield a profit of $410 per production period; it is the same solution we obtained using the iso-profit line method.

SENSITIVITY ANALYSIS

Parameter
Numerical value that is given in a model.

Operations managers are usually interested in more than the optimal solution to an LP problem. In addition to knowing the value of each decision variable (the X_is) and the value of the objective function, they want to know how sensitive these answers are to input **parameter** changes. For example, what happens if the coefficients of the objective function are not exact, or if they change by 10% or 15%? What happens if right-hand-side values of the constraints change? Because solutions are based on the assumption that input parameters are constant, the subject of sensitivity analysis comes into play. **Sensitivity analysis**, or postoptimality analysis, is the study of how sensitive solutions are to parameter changes.

Sensitivity analysis
An analysis that projects how much a solution may change if there are changes in the variables or input data.

There are two approaches to determining just how sensitive an optimal solution is to changes. The first is simply a trial-and-error approach. This approach usually involves resolving the entire problem, preferably by computer, each time one input data item or parameter is changed. It can take a long time to test a series of possible changes in this way.

The approach we prefer is the analytic postoptimality method. After an LP problem has been solved, we determine a range of changes in problem parameters that will not affect the optimal solution or change the variables in the solution. This is done without resolving the whole problem. LP software, such as Excel's Solver or POM for Windows, has this capability. Let us examine several scenarios relating to the Shader Electronics example.

Program B.1 is part of the Excel Solver computer-generated output available to help a decision maker know whether a solution is relatively insensitive to reasonable changes in one or more of the parameters of the problem. (The complete computer run for these data, including input and full output, is illustrated in Programs B.2 and B.3 later in this module.)

Sensitivity Report

The *Sensitivity Report* has two parts: Adjustable Cells and Constraints.

We are analyzing only one change at a time.

The Excel *Sensitivity Report* for the Shader Electronics example in Program B.1 has two distinct components: (1) a table titled Adjustable Cells and (2) a table titled Constraints. These tables permit us to answer several what-if questions regarding the problem solution.

It is important to note that while using the information in the sensitivity report to answer what-if questions, we assume that we are considering a change to only a *single* input data value. That is, the sensitivity information does not always apply to simultaneous changes in several input data values.

The *Adjustable Cells* table presents information regarding the impact of changes to the objective function coefficients (i.e., the unit profits of $7 and $5) on the optimal solution. The *Constraints* table presents information related to the impact of changes in constraint right-hand-side (RHS) values (i.e., the 240 hours and 100 hours) on the optimal solution. Although different LP software packages may format and present these tables differently, the programs all provide essentially the same information.

PROGRAM B.1 ■

Sensitivity Analysis for Shader Electronics Using Excel's Solver

The solution values for the variables appear. We should make 30 Walkmans and 40 Watch-TV's.

We will use 240 hours and 100 hours of Electronics and Assembly time, respectively.

If we use one more Electronics hour our profit will increase by $1.50. This is true for up to 60 more hours. The profit will fall by $1.50 for each Electronics hour less than 240 hours, down as low as 200 hours.

Changes in the Resources or Right-Hand-Side Values

The right-hand-side values of the constraints often represent resources available to the firm. The resources could be labor-hours or machine time or perhaps money or production materials available. In the Shader Electronics example, the two resources are hours available of electronics time and hours of assembly time. If additional hours were available, a higher total profit could be realized. How much should the company be willing to pay for additional hours? Is it profitable to have some additional electronics hours? Should we be willing to pay for more assembly time? Sensitivity analysis about these resources will help us answer these questions.

If the size of the feasible region increases, the optimal objective function value could improve.

If the right-hand side of a constraint is changed, the feasible region will change (unless the constraint is redundant), and often the optimal solution will change. In the Shader example, there were 100 hours of assembly time available each week and the maximum possible profit was $410. If the available assembly hours are *increased* to 110 hours, the new optimal solution seen in Figure B.8(a) is (45,20) and the profit is $415. Thus, the extra 10 hours of time resulted in an increase in profit of $5 or $0.50 per hour. If the hours are *decreased* to 90 hours as shown in Figure B.8(b), the new optimal solution is (15,60) and the profit is $405. Thus, reducing the hours by 10 results in a decrease in profit of $5 or $0.50 per hour. This $0.50 per hour change in profit that resulted from a change in the hours available is called the shadow price, or **dual** value. The **shadow price** for a constraint is the improvement in the objective function value that results from a one-unit increase in the right-hand side of the constraint.

Shadow price (or **dual)** The value of one additional unit of a scarce resource in LP.

Validity Range for the Shadow Price   Given that Shader Electronics' profit increases by $0.50 for each additional hour of assembly time, does it mean that Shader can do this indefinitely, essentially earning infinite profit? Clearly, this is illogical. How far can Shader increase its assembly time availability and still earn an extra $0.50 profit per hour? That is, for what level of increase in the RHS value of the assembly time constraint is the shadow price of $0.50 valid?

The shadow price is valid only as long as the change in the RHS is within the Allowable Increase and Allowable Decrease values.

The shadow price of $0.50 is valid as long as the available assembly time stays in a range within which all current corner points continue to exist. The information to compute the upper and lower limits of this range is given by the entries labeled Allowable Increase and Allowable Decrease in the *Sensitivity Report* in Program B.1. In Shader's case, these values show that the shadow price of $0.50 for assembly time availability is valid for an increase of up to 20 hours from the current value and a decrease of up to 20 hours. That is, the available assembly time can range from a low of 80 (= 100 − 20) to a high of 120 (= 100 + 20) for the shadow price of $0.50 to be valid. Note that the allowable decrease implies that for each hour of assembly time that Shader loses (up to 20 hours), its profit decreases by $0.50.

(a)

Changed assembly constraint from $2x_1 + 1x_2 = 100$ to $2x_1 + 1x_2 = 110$

Corner point ③ is still optimal, but values at this point are now $x_1 = 45$, $x_2 = 20$, with a profit = $415.

Electronics constraint is unchanged

(b)

Changed assembly constraint from $2x_1 + 1x_2 = 100$ to $2x_1 + 1x_2 = 90$

Corner point ③ is still optimal, but values at this point are now $x_1 = 15$, $x_2 = 60$, with a profit = $405.

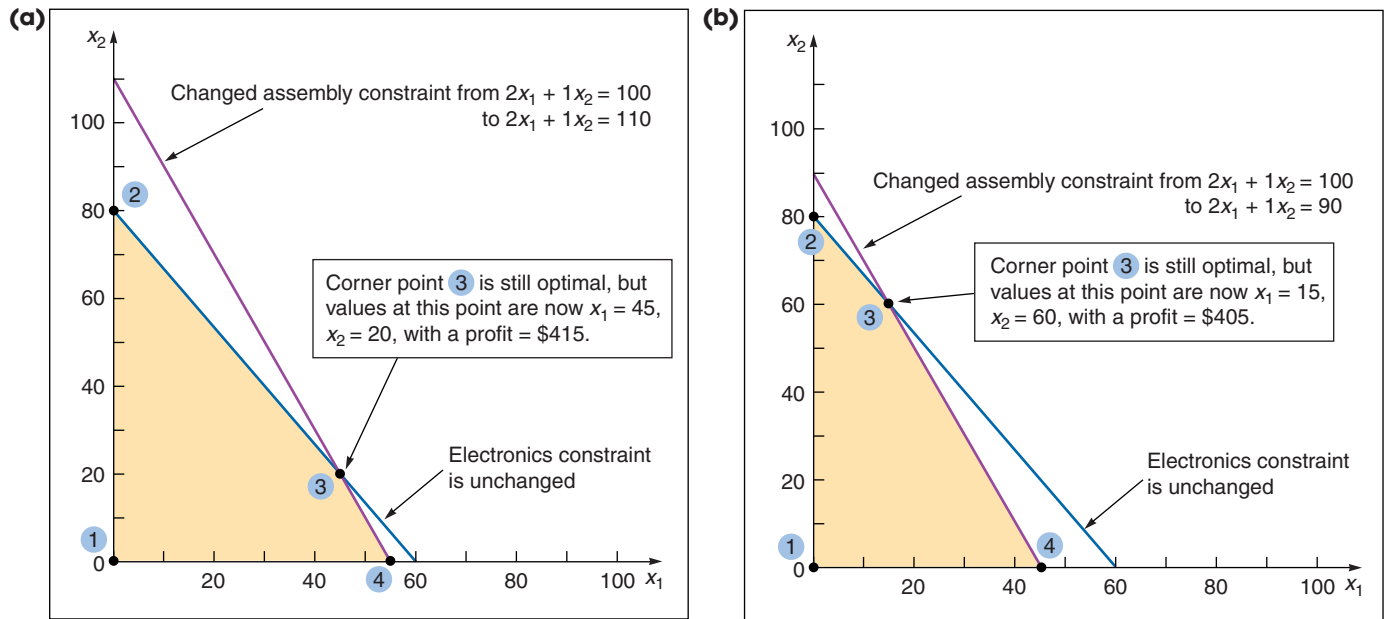Electronics constraint is unchanged

FIGURE B.8 ■ Shader Electronics Sensitivity Analysis on Right-Hand-Side (RHS) Resources

Changes in the Objective Function Coefficient

Let us now focus on the information provided in Program B.1 titled Adjustable Cells. Each row in the Adjustable Cells table contains information regarding a decision variable (i.e., Walkmans or Watch-TVs) in the LP model.

Allowable Ranges for Objective Function Coefficients
As the unit profit contribution of either product changes, the slope of the iso-profit lines we saw earlier in Figure B.5 changes. The size of the feasible region, however, remains the same. That is, the locations of the corner points do not change.

There is an allowable decrease and an allowable increase for each objective function coefficient over which the current optimal solution remains optimal.

The limits to which the profit coefficient of Walkmans or Watch-TVs can be changed without affecting the optimality of the current solution is revealed by the values in the Allowable Increase and Allowable Decrease columns of the *Sensitivity Report* in Program B.1. The allowable increase in the objective function coefficient for Watch-TVs is only $0.25. In contrast, the allowable decrease is $1.50. Hence, if the unit profit of Watch-TVs drops to $4 (i.e., a decrease of $1 from the current value of $5), it is still optimal to produce 30 Walkmans and 40 Watch-TVs. The total profit will drop to $370 (from $410) because each Watch-TV now yields less profit (of $1 per unit). However, if the unit profit drops below $3.50 per Watch-TV (i.e., a decrease of more than $1.50 from the current $5 profit), the current solution is no longer optimal. The LP problem will then have to be resolved using Solver, or other software, to find the new optimal corner point.

A new corner point becomes optimal if an objective function coefficient is decreased or increased too much.

SOLVING MINIMIZATION PROBLEMS

Many linear programming problems involve *minimizing* an objective such as cost instead of maximizing a profit function. A restaurant, for example, may wish to develop a work schedule to meet staffing needs while minimizing the total number of employees. Also, a manufacturer may seek to distribute its products from several factories to its many regional warehouses in such a way as to minimize total shipping costs.

Iso-cost

An approach to solving a linear programming minimization problem graphically.

Minimization problems can be solved graphically by first setting up the feasible solution region and then using either the corner-point method or an **iso-cost** line approach (which is analogous to the iso-profit approach in maximization problems) to find the values of X_1 and X_2 that yield the minimum cost.

Example B1 shows how to solve a minimization problem.

Example B1

A minimization problem with two variables

Cohen Chemicals, Inc., produces two types of photo-developing fluids. The first, a black-and-white picture chemical, costs Cohen $2,500 per ton to produce. The second, a color photo chemical, costs $3,000 per ton.

Based on an analysis of current inventory levels and outstanding orders, Cohen's production manager has specified that at least 30 tons of the black-and-white chemical and at least 20 tons of the color chemical must be produced during the next month. In addition, the manager notes that an existing inventory of a highly perishable raw material needed in both chemicals must be used within 30 days. To avoid wasting the expensive raw material, Cohen must produce a total of at least 60 tons of the photo chemicals in the next month.

We may formulate this information as a minimization LP problem. Let

$$X_1 = \text{number of tons of black-and-white picture chemical produced}$$
$$X_2 = \text{number of tons of color picture chemical produced}$$

Subject to:

$$X_1 \geq 30 \text{ tons of black-and-white chemical}$$
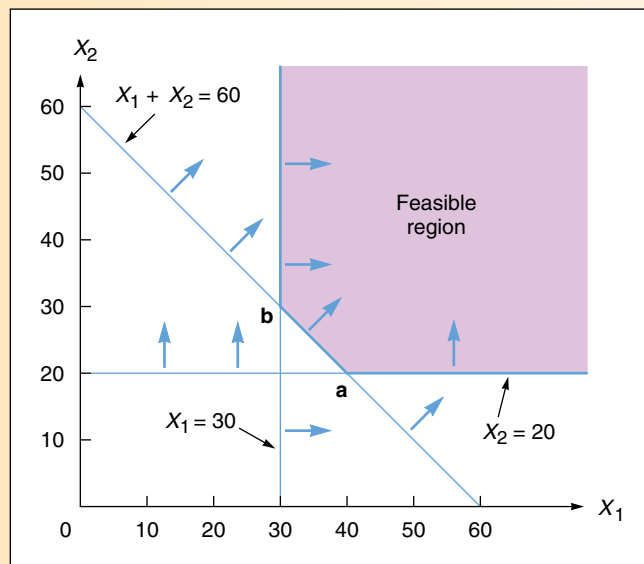$$X_2 \geq 20 \text{ tons of color chemical}$$
$$X_1 + X_2 \geq 60 \text{ tons total}$$
$$X_1, X_2 \geq \$0 \text{ nonnegativity requirements}$$

To solve the Cohen Chemicals problem graphically, we construct the problem's feasible region, shown in Figure B.9.

FIGURE B.9 ■

Cohen Chemicals' Feasible Region

The area is not bounded to the right in a minimization problem as it is in a maximization problem.

Minimization problems are often unbounded outward (that is, on the right side and on the top), but this characteristic causes no problem in solving them. As long as they are bounded inward (on the left side and the bottom), we can establish corner points. The optimal solution will lie at one of the corners.

In this case, there are only two corner points, **a** and **b**, in Figure B.9. It is easy to determine that at point **a**, $X_1 = 40$ and $X_2 = 20$, and that at point **b**, $X_1 = 30$ and $X_2 = 30$. The optimal solution is found at the point yielding the lowest total cost.

Thus

$$\text{Total cost at } \mathbf{a} = 2,500X_1 + 3,000X_2$$
$$= 2,500(40) + 3,000(20)$$
$$= \$160,000$$
$$\text{Total cost at } \mathbf{b} = 2,500X_1 + 3,000X_2$$
$$= 2,500(30) + 3,000(30)$$
$$= \$165,000$$

The lowest cost to Cohen Chemicals is at point **a**. Hence the operations manager should produce 40 tons of the black-and-white chemical and 20 tons of the color chemical.

OM IN ACTION

Scheduling Planes at Delta Airlines with LP

It has been said that an airline seat is the most perishable commodity in the world. Each time an airliner takes off with an empty seat, a revenue opportunity is lost forever. For Delta Airlines, which flies over 2,500 domestic flight legs per day using about 450 aircraft of 10 different models, its schedule is the very heartbeat of the airline.

One flight leg for Delta might consist of a Boeing 757 jet assigned to fly at 6:21 A.M. from Atlanta to arrive in Boston at 8:45 A.M. Delta's problem, like that of every competitor, is to match airplanes such as 747s, 757s, or 767s to flight legs such as Atlanta–Boston and to fill seats with paying passengers. Recent advances in linear programming algorithms and computer hardware have made it possible to solve optimization problems of this scope for the first time. Delta calls its huge LP model "Coldstart" and runs the model every day. Delta is the first airline to solve a problem of this scope.

The typical size of a daily Coldstart model is about 40,000 constraints and 60,000 variables. The constraints include aircraft availability, balancing arrivals and departures at airports, aircraft maintenance needs, and so on. Coldstart's objective is to minimize a combination of operating costs and lost passenger revenue, called "spill costs."

The savings from the model have been phenomenal, at $220,000 per day over Delta's earlier schedule planning tool, which was nicknamed "Warmstart." Delta saves $300 million per year through this use of linear programming.

Sources: Interfaces (September–October, 1999): 123–131; Interfaces (January–February 1994): 104–120; and OR/MS Today (August 1995): 14–15.

LINEAR PROGRAMMING APPLICATIONS

The foregoing examples each contained just two variables (X_1 and X_2). Most real-world problems contain many more variables, however. Let's use the principles already developed to formulate a few more-complex problems. The practice you will get by "paraphrasing" the following LP situations should help develop your skills for applying linear programming to other common operations situations.

Production-Mix Example

Example B2 involves another *production-mix* decision. Limited resources must be allocated among various products that a firm produces. The firm's overall objective is to manufacture the selected products in such quantities as to maximize total profits.

Example B2

A production mix problem

Failsafe Electronics Corporation primarily manufactures four highly technical products, which it supplies to aerospace firms that hold NASA contracts. Each of the products must pass through the following departments before they are shipped: wiring, drilling, assembly, and inspection. The time requirements in each department (in hours) for each unit produced and its corresponding profit value are summarized in this table:

PRODUCT	DEPARTMENT				UNIT PROFIT
	WIRING	DRILLING	ASSEMBLY	INSPECTION	
XJ201	.5	3	2	.5	$ 9
XM897	1.5	1	4	1.0	$12
TR29	1.5	2	1	.5	$15
BR788	1.0	3	2	.5	$11

The production time available in each department each month and the minimum monthly production requirement to fulfill contracts are as follows:

DEPARTMENT	CAPACITY (IN HOURS)	PRODUCT	MINIMUM PRODUCTION LEVEL
Wiring	1,500	XJ201	150
Drilling	2,350	XM897	100
Assembly	2,600	TR29	300
Inspection	1,200	BR788	400

The production manager has the responsibility of specifying production levels for each product for the coming month. Let

$$X_1 = \text{number of units of XJ201 produced}$$
$$X_2 = \text{number of units of XM897 produced}$$
$$X_3 = \text{number of units of TR29 produced}$$
$$X_4 = \text{number of units of BR788 produced}$$

Maximize profit $= 9X_1 + 12X_2 + 15X_3 + 11X_4$

subject to
$$.5X_1 + 1.5X_2 + 1.5X_3 + 1X_4 \leq 1,500 \text{ hours of wiring available}$$
$$3X_1 + 1X_2 + 2X_3 + 3X_4 \leq 2,350 \text{ hours of drilling available}$$
$$2X_1 + 4X_2 + 1X_3 + 2X_4 \leq 2,600 \text{ hours of assembly available}$$
$$.5X_1 + 1X_2 + .5X_3 + .5X_4 \leq 1,200 \text{ hours of inspection}$$
$$X_1 \geq 150 \text{ units of XJ201}$$
$$X_2 \geq 100 \text{ units of XM897}$$
$$X_3 \geq 300 \text{ units of TR29}$$
$$X_4 \geq 400 \text{ units of BR788}$$
$$X_1, X_2, X_3, X_4 \geq 0$$

Diet Problem Example

Example B3 illustrates the *diet problem*, which was originally used by hospitals to determine the most economical diet for patients. Known in agricultural applications as the *feed-mix problem*, the diet problem involves specifying a food or feed ingredient combination that will satisfy stated nutritional requirements at a minimum cost level.

Example B3

A diet problem

The Feed 'N Ship feedlot fattens cattle for local farmers and ships them to meat markets in Kansas City and Omaha. The owners of the feedlot seek to determine the amounts of cattle feed to buy to satisfy minimum nutritional standards and, at the same time, minimize total feed costs.

Each grain stock contains different amounts of four nutritional ingredients: A, B, C, and D. Here are the ingredient contents of each grain, in *ounces per pound of grain*.

	FEED		
INGREDIENT	STOCK X	STOCK Y	STOCK Z
A	3 oz.	2 oz.	4 oz.
B	2 oz.	3 oz.	1 oz.
C	1 oz.	0 oz.	2 oz.
D	6 oz.	8 oz.	4 oz.

The cost per pound of grains X, Y, and Z is $0.02, $0.04, and $0.025, respectively. The minimum requirement per cow per month is 64 ounces of ingredient A, 80 ounces of ingredient B, 16 ounces of ingredient C, and 128 ounces of ingredient D.

The feedlot faces one additional restriction—it can obtain only 500 pounds of stock Z per month from the feed supplier, regardless of its need. Because there are usually 100 cows at the Feed 'N Ship feedlot at any given time, this constraint limits the amount of stock Z for use in the feed of each cow to no more than 5 pounds, or 80 ounces, per month. Let

$$X_1 = \text{number of pounds of stock X purchased per cow each month}$$
$$X_2 = \text{number of pounds of stock Y purchased per cow each month}$$
$$X_3 = \text{number of pounds of stock Z purchased per cow each month}$$

Minimum cost = $.02X_1 + .04X_2 + .025X_3$ subject to

$$\text{Ingredient A requirement: } 3X_1 + 2X_2 + 4X_3 \geq 64$$
$$\text{Ingredient B requirement: } 2X_1 + 3X_2 + 1X_3 \geq 80$$
$$\text{Ingredient C requirement: } 1X_1 + 0X_2 + 2X_3 \geq 16$$
$$\text{Ingredient D requirement: } 6X_1 + 8X_2 + 4X_3 \geq 128$$
$$\text{Stock Z limitation: } \qquad X_3 \leq 80$$
$$X_1, X_2, X_3 \geq 0$$

The cheapest solution is to purchase 40 pounds of grain X_1, at a cost of $0.80 per cow.

Production Scheduling Example

One of the most important areas of linear programming application is *production scheduling*. Solving a production scheduling problem allows the production manager to set an efficient, low-cost production schedule for a product over several production periods. Basically, the problem resembles the common product-mix model for each period in the future. Production levels must allow the firm to meet demand for its product within manpower and inventory limitations. The objective is either to maximize profit or to minimize the total cost (of production plus inventory).

Example B4

A production scheduling problem

The T. E. Callarman Appliance Company is thinking of manufacturing and selling trash compactors on an experimental basis over the next 6 months. The manufacturing costs and selling prices of the compactors are projected to vary from month to month. Table B.2 gives these forecast costs and prices.

TABLE B.2 ■ Manufacturing Costs and Selling Prices

MONTH	MANUFACTURING COST	SELLING PRICE (DURING MONTH)
July	$60	—
August	$60	$80
September	$50	$60
October	$60	$70
November	$70	$80
December	—	$90

All compactors manufactured during any month are shipped out in one large load at the end of that month. The firm can sell as many as 300 units per month, but its operation is limited by the size of its warehouse, which holds a maximum of 100 compactors.

Callarman's operations manager, Richard Deckro, needs to determine the number of compactors to manufacture and sell each month to maximize the firm's profit. Callarman has no compactors on hand at the beginning of July and wishes to have no compactors on hand at the end of the test period in December.

To formulate this LP problem, Deckro lets

$X_1, X_2, X_3, X_4, X_5, X_6 =$ number of units *manufactured* during July (first month),
August (second month), etc.

$Y_1, Y_2, Y_3, Y_4, Y_5, Y_6 =$ number of units *sold* during July, August, etc.

He notes that because the company starts with no compactors (and because it takes 1 month to gear up and ship out the first batch), it cannot sell any units in July (that is, $Y_1 = 0$). Also, because it wants zero inventory at the end of the year, manufacture during the month of December must be zero (that is, $X_6 = 0$).

Profit for Callarman Appliances is sales minus manufacture cost. Hence, Deckro's objective function is

$$\text{Maximize profit} = 80Y_2 + 60Y_3 + 70Y_4 + 80Y_5 + 90Y_6$$
$$- (60X_1 + 60X_2 + 50X_3 + 60X_4 + 70X_5)$$

The first part of this expression is the sales price times the units sold each month. The second part is the manufacture cost, namely, the costs from Table B.2 times the units manufactured.

To set up the constraints, Deckro needs to introduce a new set of variables: $I_1, I_2, I_3, I_4, I_5, I_6$. These represent the inventory at the end of a month (after all sales have been made and after the amount produced during the month has been stocked in the warehouse). Thus

$$\begin{matrix} \text{Inventory at} \\ \text{end of this} \\ \text{month} \end{matrix} = \begin{matrix} \text{Inventory at} \\ \text{end of} \\ \text{previous month} \end{matrix} + \begin{matrix} \text{Current} \\ \text{month's} \\ \text{production} \end{matrix} - \begin{matrix} \text{This month's} \\ \text{sales} \end{matrix}$$

For July, this is $I_1 = X_1$ because there is neither previous inventory nor sales. For August,

$$I_2 = I_1 + X_2 - Y_2$$

Constraints for the remaining months are as follows:

$$
\begin{aligned}
\text{September:} \quad & I_3 = I_2 + X_3 - Y_3 \\
\text{October:} \quad & I_4 = I_3 + X_4 - Y_4 \\
\text{November:} \quad & I_5 = I_4 + X_5 - Y_5 \\
\text{December:} \quad & I_6 = I_5 - Y_6
\end{aligned}
$$

Constraints for the storage capacity are

$$I_1 \le 100, \quad I_2 \le 100, \quad I_3 \le 100, \quad I_4 \le 100, \quad I_5 \le 100$$

and $I_6 = 0$ (to end up with zero inventory at the end of December).
Constraints for demand are

$$\text{all} \quad Y_i \le 300$$

The final solution yields a profit of $19,000 with

$$
\begin{aligned}
& X_1 = 100, X_2 = 200, X_3 = 400, X_4 = 300, X_5 = 300, X_6 = 0, \\
& Y_1 = 100, Y_2 = 300, Y_3 = 300, Y_4 = 300, Y_5 = 300, Y_6 = 100, \\
& I_1 = 100, \quad I_2 = 0, \quad I_3 = 100, \quad I_4 = 100, \quad I_5 = 100, \quad I_6 = 0
\end{aligned}
$$

Alternative optimal solutions exist.

Labor Scheduling Example

Labor scheduling problems address staffing needs over a specific time period. They are especially useful when managers have some flexibility in assigning workers to jobs that require overlapping or interchangeable talents. Large banks and hospitals frequently use LP to tackle their labor scheduling. Example B5 describes how one bank uses LP to schedule tellers.

Example B5

Scheduling bank tellers

Arlington Bank of Commerce and Industry is a busy bank that has requirements for between 10 and 18 tellers depending on the time of day. Lunchtime, from noon to 2 P.M., is usually heaviest. The table below indicates the workers needed at various hours that the bank is open.

TIME PERIOD	NUMBER OF TELLERS REQUIRED	TIME PERIOD	NUMBER OF TELLERS REQUIRED
9 A.M.–10 A.M.	10	1 P.M.–2 P.M.	18
10 A.M.–11 A.M.	12	2 P.M.–3 P.M.	17
11 A.M.–Noon	14	3 P.M.–4 P.M.	15
Noon–1 P.M.	16	4 P.M.–5 P.M.	10

The bank now employs 12 full-time tellers, but many people are on its roster of available part-time employ-ees. A part-time employee must put in exactly 4 hours per day but can start anytime between 9 A.M. and 1 P.M. Part-timers are a fairly inexpensive labor pool because no retirement or lunch benefits are provided them. Full-timers, on the other hand, work from 9 A.M. to 5 P.M. but are allowed 1 hour for lunch. (Half the full-timers eat at 11 A.M., the other half at noon.) Full-timers thus provide 35 hours per week of productive labor time.

By corporate policy, the bank limits part-time hours to a maximum of 50% of the day's total requirement.

Part-timers earn $6 per hour (or $24 per day) on average, whereas full-timers earn $75 per day in salary and benefits on average. The bank would like to set a schedule that would minimize its total manpower costs. It will release 1 or more of its full-time tellers if it is profitable to do so.

We can let

$$F = \text{full-time tellers}$$
$$P_1 = \text{part-timers starting at 9 A.M. (leaving at 1 P.M.)}$$
$$P_2 = \text{part-timers starting at 10 A.M. (leaving at 2 P.M.)}$$
$$P_3 = \text{part-timers starting at 11 A.M. (leaving at 3 P.M.)}$$
$$P_4 = \text{part-timers starting at noon (leaving at 4 P.M.)}$$
$$P_5 = \text{part-timers starting at 1 P.M. (leaving at 5 P.M.)}$$

Objective function:

$$\begin{array}{c}\text{Minimize total daily}\\ \text{manpower cost}\end{array} = \$75F + \$24(P_1 + P_2 + P_3 + P_4 + P_5)$$

Constraints: For each hour, the available labor-hours must be at least equal to the required labor-hours.

$$\begin{array}{lll}
F + P_1 & \geq 10 & \text{(9 A.M. to 10 A.M. needs)} \\
F + P_1 + P_2 & \geq 12 & \text{(10 A.M. to 11 A.M. needs)} \\
\tfrac{1}{2}F + P_1 + P_2 + P_3 & \geq 14 & \text{(11 A.M. to noon needs)} \\
\tfrac{1}{2}F + P_1 + P_2 + P_3 + P_4 & \geq 16 & \text{(noon to 1 P.M. needs)} \\
F \quad\quad + P_2 + P_3 + P_4 + P_5 & \geq 18 & \text{(1 P.M. to 2 P.M. needs)} \\
F \quad\quad\quad\quad + P_3 + P_4 + P_5 & \geq 17 & \text{(2 P.M. to 3 P.M. needs)} \\
F \quad\quad\quad\quad\quad\quad + P_4 + P_5 & \geq 15 & \text{(3 P.M. to 4 P.M. needs)} \\
F \quad\quad\quad\quad\quad\quad\quad\quad + P_5 & \geq 10 & \text{(4 P.M. to 5 P.M. needs)}
\end{array}$$

Only 12 full-time tellers are available, so

$$F \leq 12$$

Part-time worker-hours cannot exceed 50% of total hours required each day, which is the sum of the tellers needed each hour.

$$4(P_1 + P_2 + P_3 + P_4 + P_5) \leq .50(10 + 12 + 14 + 16 + 18 + 17 + 15 + 10)$$

or

$$4P_1 + 4P_2 + 4P_3 + 4P_4 + 4P_5 \leq 0.50(112)$$
$$F, P_1, P_2, P_3, P_4, P_5 \geq 0$$

There are two alternative optimal schedules that Arlington Bank can follow. The first is to employ only 10 full-time tellers ($F = 10$) and to start 7 part-timers at 10 A.M. ($P_2 = 7$), 2 part-timers at 11 A.M. and noon ($P_3 = 2$ and $P_4 = 2$), and 3 part-timers at 1 P.M. ($P_5 = 3$). No part-timers would begin at 9 A.M.

The second solution also employs 10 full-time tellers, but starts 6 part-timers at 9 A.M. ($P_1 = 6$), 1 part-timer at 10 A.M. ($P_2 = 1$), 2 part-timers at 11 A.M. and noon ($P_3 = 2$ and $P_4 = 2$), and 3 part-timers at 1 P.M. ($P_5 = 3$). The cost of either of these two policies is $1,086 per day.

THE SIMPLEX METHOD OF LP

Simplex method
An algorithm developed by Dantzig for solving linear programming problems of all sizes.

Most real-world linear programming problems have more than two variables and thus are too complex for graphical solution. A procedure called the **simplex method** may be used to find the optimal solution to such problems. The simplex method is actually an algorithm (or a set of instructions) with which we examine corner points in a methodical fashion until we arrive at the best solution—highest profit or lowest cost. Computer programs (such as POM for Windows) and Excel spreadsheets are available to solve linear programming problems via the simplex method.

For details regarding the algebraic steps of the simplex algorithm, see Tutorial 3 on the CD-ROM that accompanies this book, or refer to a management science textbook.[2]

SUMMARY

This module introduces a special kind of model, linear programming. LP has proven to be especially useful when trying to make the most effective use of an organization's resources.

The first step in dealing with LP models is problem formulation, which involves identifying and creating an objective function and constraints. The second step is to solve the problem. If there are only two decision variables, the problem can be solved graphically, using the corner-point method or the iso-profit/iso-cost line method. With either approach, we first identify the feasible region, then find the corner point yielding the greatest profit or least cost. LP is used in a wide variety of business applications, as the examples and homework problems in this module reveal.

KEY TERMS

Linear programming (LP) *(p. 692)*
Objective function *(p. 693)*
Constraints *(p. 693)*
Graphical solution approach *(p. 694)*
Decision variables *(p. 694)*
Feasible region *(p. 695)*
Iso-profit line method *(p. 696)*

Corner-point method *(p. 698)*
Parameter *(p. 699)*
Sensitivity analysis *(p. 699)*
Shadow price (or dual) *(p. 700)*
Iso-cost *(p. 701)*
Simplex method *(p. 708)*

USING SOFTWARE TO SOLVE LP PROBLEMS

All LP problems can also be solved with the simplex method, using software such as POM for Windows or Excel. This approach produces valuable economic information such as the shadow price, or dual, and provides complete sensitivity analysis on other inputs to the problems. Excel uses Solver, which requires that you enter your own constraints. Excel OM does not have an LP module. POM for Windows requires only that demand data, supply data, and shipping costs be entered. In the following section we illustrate how to create an Excel spreadsheet for LP problems.

Using Excel Spreadsheets

Excel offers the ability to analyze linear programming problems using built-in problem-solving tools. Excel's tool is named Solver. Solver is limited to 200 changing cells (variables), each with 2 boundary constraints and up to 100 additional constraints. These capabilities make Solver suitable for the solution of complex, real-world problems.

We use Excel to set up the Shader Electronics problem in Program B.2. The objective and constraints are repeated here:

Objective function: Maximize profit =

$$\$7(\text{No. of Walkmans}) + \$5(\text{No. of Watch-TVs})$$

$$\text{Subject to:} \quad 4(\text{Walkmans}) + 3(\text{Watch-TVs}) \leq 240$$

$$2(\text{Walkmans}) + 1(\text{Watch-TV}) \leq 100$$

[2]See, for example, Barry Render, Ralph M. Stair, and Michael Hanna, *Quantitative Analysis for Management*, 9th ed. (Upper Saddle River, NJ: Prentice Hall, 2006): Chapters 7–9 or Barry Render, Ralph M. Stair, and Raju Balakrishnan, *Managerial Decision Modeling with Spreadsheets*, 2nd ed. (Upper Saddle River, NJ: Prentice Hall, 2006): Chapters 2–4.

PROGRAM B.2 ■ Using Excel to Formulate the Shader Electronics Problem
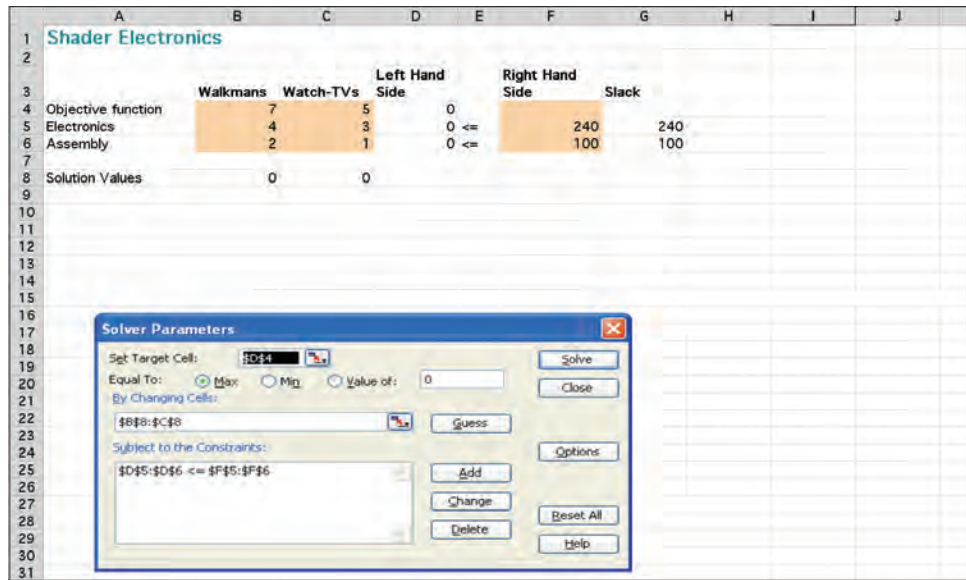
		COMPUTATIONS	
VALUE	CELL	EXCEL FORMULA	ACTION
Left Hand Side	D4	=SUMPRODUCT($B$8:$C$8,B4:C4)	Copy to D5:D6
Slack	G5	=F5-D5	Copy to G6
			Select Tools, Solver
			Set solver parameters as displayed
			Press Solve

The Excel screen in Program B.3 shows Solver's solution to the Shader Electronics Company problem. Note that the optimal solution is now shown in the *changing cells* (cells B8 and C8, which served as the variables). The Reports selection performs more extensive analysis of the solution and its environment. Excel's sensitivity analysis capability was illustrated earlier in Program B.1.

PROGRAM B.3 ■ Excel Solution to Shader Electronics LP Problem
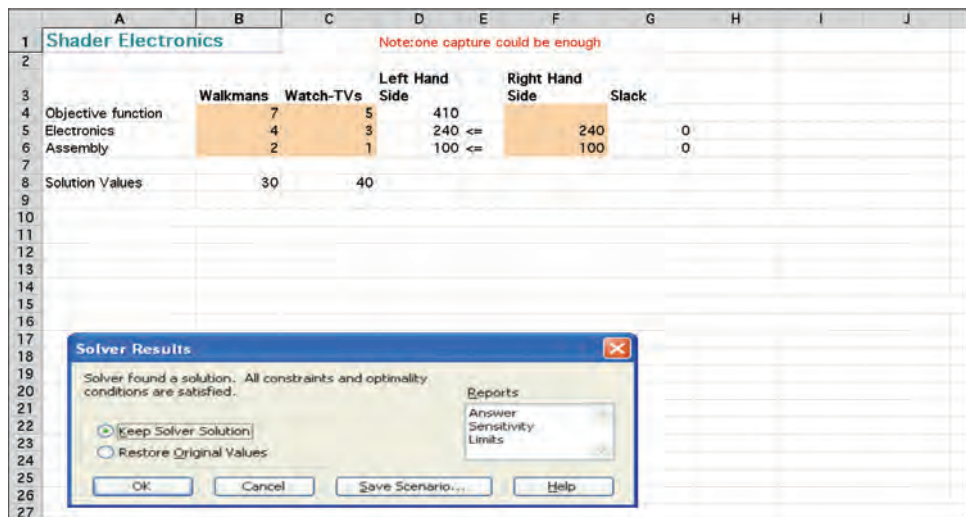
Using POM for Windows

POM for Windows can handle LP problems with up to 22 constraints and 99 variables. As output, the software provides optimal values for the variables, optimal profit or cost, and sensitivity analysis. In addition, POM for Windows provides graphical output for problems with only two variables. Please see Appendix IV for further details.

SOLVED PROBLEMS

Solved Problem B.1

Smitty's, a clothing manufacturer that produces men's shirts and pajamas, has two primary resources available: sewing-machine time (in the sewing department) and cutting-machine time (in the cutting department). Over the next month, Smitty can schedule up to 280 hours of work on sewing machines and up to 450 hours of work on cutting machines. Each shirt produced requires 1.00 hour of sewing time and 1.50 hours of cutting time. Producing each pair of pajamas requires .75 hour of sewing time and 2 hours of cutting time.

To express the LP constraints for this problem mathematically, we let

$$X_1 = \text{number of shirts produced}$$
$$X_2 = \text{number of pajamas produced}$$

SOLUTION

First constraint: $1X_1 + .75X_2 \le 280$ hours of sewing-machine time available—our first scarce resource

Second constraint: $1.5X_1 + ②X_2 \le 450$ hours of cutting-machine time available—our second scarce resource

Note: This means that each pair of pajamas takes 2 hours of the cutting resource.

Smitty's accounting department analyzes cost and sales figures and states that each shirt produced will yield a $4 contribution to profit and that each pair of pajamas will yield a $3 contribution to profit.

This information can be used to create the LP *objective function* for this problem:

Objective function: maximize total contribution to profit = $4X_1 + $3X_2

Solved Problem B.2

We want to solve the following LP problem for Failsafe Computers using the corner-point method:

Maximize profit = $9X_1 + $7X_2
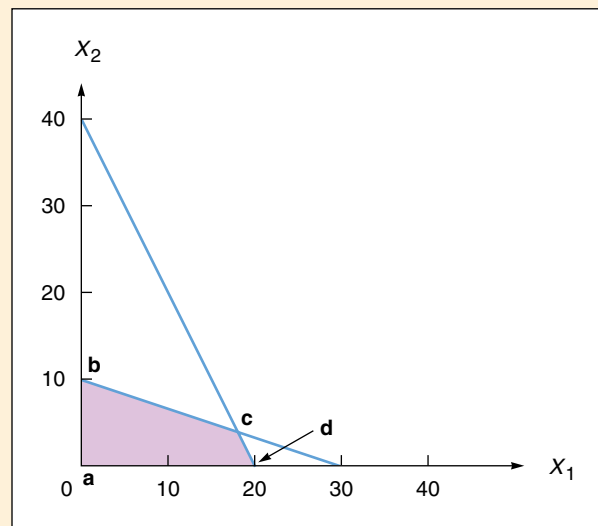
$$2X_1 + 1X_2 \le 40$$

$$X_1 + 3X_2 \le 30$$

FIGURE B.10 ■ Failsafe Computers' Feasible Region

SOLUTION

Figure B.10 illustrates these constraints:

Corner-point **a**: $(X_1 = 0, X_2 = 0)$ Profit = 0

Corner-point **b**: $(X_1 = 0, X_2 = 10)$ Profit = 9(0) + 7(10) = $70

Corner-point **d**: $(X_1 = 20, X_2 = 0)$ Profit = 9(20) + 7(0) = $180

Corner-point **c** is obtained by solving equations $2X_1 + 1X_2 = 40$ and $X_1 + 3X_2 = 30$ simultaneously. Multiply the second equation by -2 and add it to the first.

$$2X_1 + 1X_2 = 40$$
$$\underline{-2X_1 - 6X_2 = -60}$$
$$- 5X_2 = -20$$

Thus $X_2 = 4$.

$$X_1 + 3(4) = 30 \quad \text{or} \quad X_1 + 12 = 30 \quad \text{or} \quad X_1 = 18$$

Corner-point **c**: $(X_1 = 18, X_2 = 4)$ Profit $= 9(18) + 7(4) = \$190$

Hence the optimal solution is

$$(x_1 = 18, x_2 = 4) \qquad \text{Profit} = \$190$$

Solved Problem B.3

Holiday Meal Turkey Ranch is considering buying two different types of turkey feed. Each feed contains, in varying proportions, some or all of the three nutritional ingredients essential for fattening turkeys. Brand Y feed costs the ranch $.02 per pound. Brand Z costs $.03 per pound. The rancher would like to determine the lowest-cost diet that meets the minimum monthly intake requirement for each nutritional ingredient.

The following table contains relevant information about the composition of brand Y and brand Z feeds, as well as the minimum monthly requirement for each nutritional ingredient per turkey.

COMPOSITION OF EACH POUND OF FEED

INGREDIENT	BRAND Y FEED	BRAND Z FEED	MINIMUM MONTHLY REQUIREMENT
A	5 oz.	10 oz.	90 oz.
B	4 oz.	3 oz.	48 oz.
C	.5 oz.	0	1.5 oz.
Cost/lb	$.02	$.03	

SOLUTION

If we let

X_1 = number of pounds of brand Y feed purchased

X_2 = number of pounds of brand Z feed purchased

then we may proceed to formulate this linear programming problem as follows:

$$\text{Minimize cost (in cents)} = 2X_1 + 3X_2$$

subject to these constraints:

$$5X_1 + 10X_2 \geq 90 \text{ oz.} \quad (\textit{ingredient A constraint})$$
$$4X_1 + 3X_2 \geq 48 \text{ oz.} \quad (\textit{ingredient B constraint})$$
$$\tfrac{1}{2} X_1 \geq 1\tfrac{1}{2} \text{ oz.} \quad (\textit{ingredient C constraint})$$

Figure B.11 illustrates these constraints.

The iso-cost line approach may be used to solve LP minimization problems such as that of the Holiday Meal Turkey Ranch. As with iso-profit lines, we need not compute the cost at each corner point, but instead draw a series of parallel cost lines. The lowest cost line (that is, the one closest in toward the origin) to touch the feasible region provides us with the optimal solution corner.

For example, we start in Figure B.12 by drawing a 54¢ cost line, namely, $54 = 2X_1 + 3X_2$. Obviously, there are many points in the feasible region that would yield a lower total cost. We proceed to
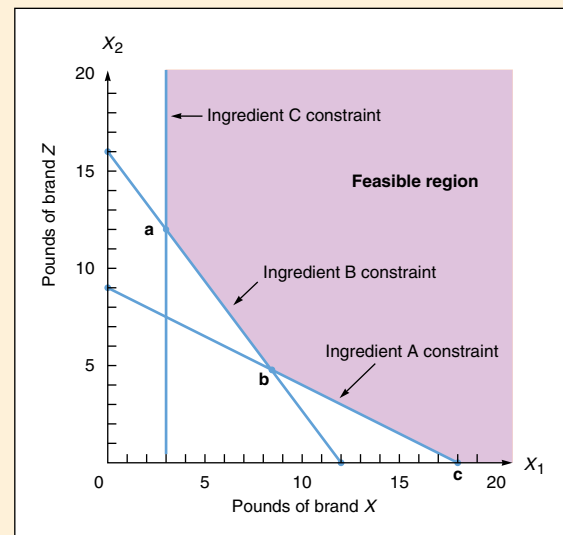

FIGURE B.11 ■ Feasible Region for the Holiday Meal Turkey Ranch Problem

move our iso-cost line toward the lower left, in a plane parallel to the 54¢ solution line. The last point we touch while still in contact with the feasible region is the same as corner point **b** of Figure B.11. It has the coordinates $(X_1 = 8.4, X_2 = 4.8)$ and an associated cost of 31.2 cents.

FIGURE B.12 ■

Graphical Solution to the Holiday
Meal Turkey Ranch Problem Using
the Iso-Cost Line

*Note that the last line parallel to the 54¢
iso-cost line that touches the feasible
region indicates the optimal corner point.*
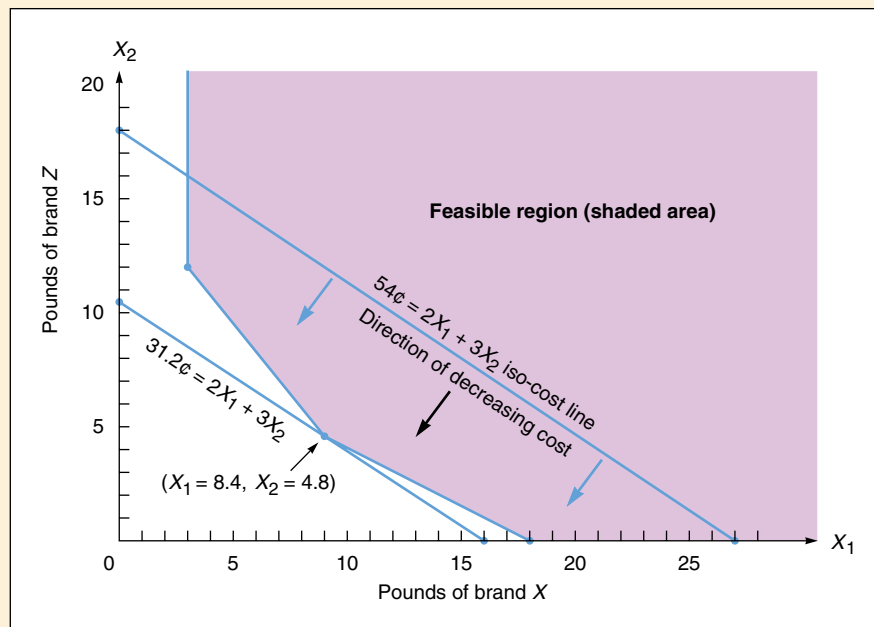
INTERNET AND STUDENT CD-ROM EXERCISES

Visit our Companion Web site or use your student CD-ROM to help with material in this module.

On Our Companion Web site, www.prenhall.com/heizer

• Self-Study Quizzes
• Practice Problems
• Internet Homework Problems
• Internet Cases

On Your Student CD-ROM

• PowerPoint Lecture
• Practice Problems
• Active Model Exercise
• POM for Windows

DISCUSSION QUESTIONS

1. List at least four applications of linear programming problems.
2. What is a "corner point"? Explain why solutions to linear programming problems focus on corner points.
3. Define the feasible region of a graphical LP problem. What is a feasible solution?
4. Each linear programming problem that has a feasible region has an infinite number of solutions. Explain.
5. Under what circumstances is the objective function more important than the constraints in a linear programming model?
6. Under what circumstances are the constraints more important than the objective function in a linear programming model?
7. Why is the diet problem, in practice, applicable for animals but not for people?
8. How many feasible solutions are there in a linear program? Which ones do we need to examine to find the optimal solution?

9. Define shadow price (or dual).
10. Explain how to use the iso-cost line in a graphical minimization problem.
11. Compare how the corner-point and iso-profit line methods work for solving graphical problems.
12. Where a constraint crosses the vertical or horizontal axis, the quantity is fairly obvious. How does one go about finding the quantity coordinates where two constraints cross, not at an axis?
13. Suppose a linear programming (maximation) problem has been solved and that the optimal value of the objective function is $300. Suppose an additonal constraint is added to this problem. Explain how this might affect each of the following:
    (a) The feasible region.
    (b) The optimal value of the objective function.

ACTIVE MODEL EXERCISE

The Active Model describes the Shader Electronics Example maximization problem with two less-than-or-equal-to constraints. You can use the scrollbars to change any of the eight numbers in the example or to move the iso-profit line.

ACTIVE MODEL B.1 ■

Analysis of the Shader Electronics Example Data
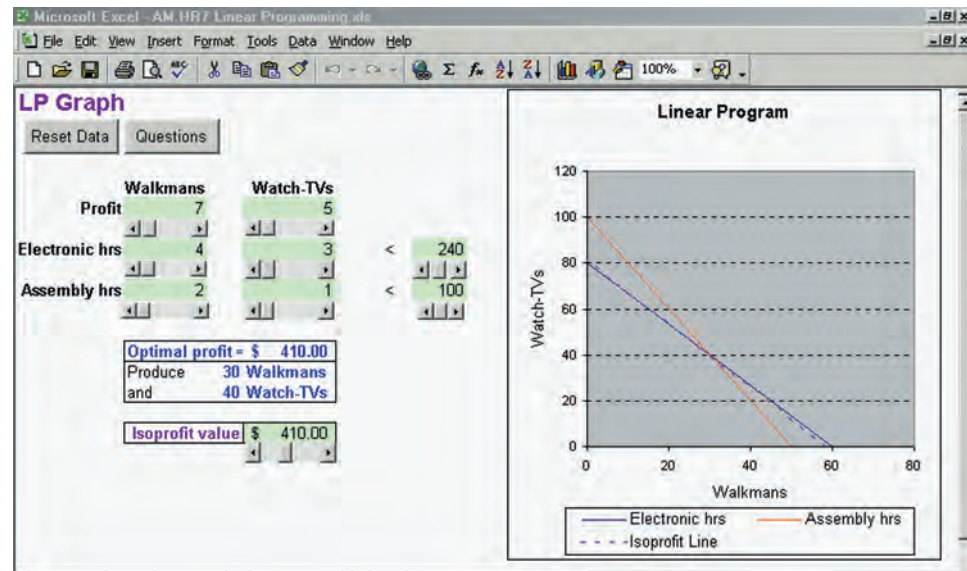
Questions

1. By how much does the profit on Walkmans need to rise to make it the only product manufactured?
2. By how much does the profit on Walkmans need to fall to stop manufacturing it?
3. What happens to the profit as the number of assembly hours increases by 1 hour at a time? For how many hours is this true?
4. What happens if we can reduce the electronics time for Watch-TVs to 2.5 hours?

PROBLEMS*

• $P_{\swarrow}$ B.1   Solve the following linear programming problem graphically:

$$\text{Maximize} \quad Z = 4X + 6Y$$
$$\text{Subject to:} \quad X + 2Y \le 8$$
$$5X + 4Y \le 20$$
$$X, Y \ge 0$$

$P_{\swarrow}$ • B.2   Solve the following linear programming problem graphically.

$$\text{Maximize} \quad Z = X + 10Y$$
$$\text{Subject to:} \quad 4X + 3Y \le 36$$
$$2X + 4Y \le 40$$
$$Y \ge 3$$
$$X, Y \ge 0$$

*Note: **P** means the problem may be solved with POM for Windows; $\swarrow$ means the problem may be solved with Excel; and $P_{\swarrow}$ means the problem may be solved with POM for Windows and/or Excel. Recall that Excel OM does not have an LP module because the *LP Solver* is built into Excel.

: $P_{\times}$ B.3   Graphically analyze the following problem:

$$\text{Maximize profit} = \$4X_1 + \$6X_2$$
$$\text{Subject to:}\quad 1X_1 + 2X_2 \le 8$$
$$6X_1 + 4X_2 \le 24$$

   a) What is the optimal solution?
   b) If the first constraint is altered to $1X_1 + 3X_2 \le 8$, does the feasible region or optimal solution change?

: $P_{\times}$ B.4   Consider the following linear programming problem:

$$\text{Maximize}\quad Z = 30X_1 + 10X_2$$
$$\text{Subject to:}\quad 3X_1 + X_2 \le 300$$
$$X_1 + X_2 \le 200$$
$$X_1 \quad \le 100$$
$$X_2 \ge 50$$
$$X_1 - X_2 \le 0$$
$$X_1, X_2 \ge 0$$

   a) Solve the problem graphically.
   b) Is there more than one optimal solution? Explain.

• $P_{\times}$ B.5   Solve the following LP problem graphically:

$$\text{Minimize}\quad Z = 24X + 15Y$$
$$\text{Subject to:}\quad 7X + 11Y \ge 77$$
$$16X + 4Y \ge 80$$
$$X, Y \ge 0$$

: $P_{\times}$ B.6   Ed Silver Dog Food Company wishes to introduce a new brand of dog biscuits composed of chicken- and liver-flavored biscuits that meet certain nutritional requirements. The liver-flavored biscuits contain 1 unit of nutrient A and 2 units of nutrient B; the chicken-flavored biscuits contain 1 unit of nutrient A and 4 units of nutrient B. According to federal requirements, there must be at least 40 units of nutrient A and 60 units of nutrient B in a package of the new mix. In addition, the company has decided that there can be no more than 15 liver-flavored biscuits in a package. If it costs 1¢ to make 1 liver-flavored biscuit and 2¢ to make 1 chicken-flavored, what is the optimal product mix for a package of the biscuits to minimize the firm's cost?
   a) Formulate this as a linear programming problem.
   b) Solve this problem graphically, giving the optimal values of all variables.
   c) What is the total cost of a package of dog biscuits using the optimal mix?

• $P_{\times}$ B.7   The Electrocomp Corporation manufactures two electrical products: air conditioners and large fans. The assembly process for each is similar in that both require a certain amount of wiring and drilling. Each air conditioner takes 3 hours of wiring and 2 hours of drilling. Each fan must go through 2 hours of wiring and 1 hour of drilling. During the next production period, 240 hours of wiring time are available and up to 140 hours of drilling time may be used. Each air conditioner sold yields a profit of $25. Each fan assembled may be sold for a $15 profit.
       Formulate and solve this LP production-mix situation, and find the best combination of air conditioners and fans that yields the highest profit.

• $P_{\times}$ B.8   The Lauren Shur Tub Company manufactures two lines of bathtubs, called model A and model B. Every tub requires blending a certain amount of steel and zinc; the company has available a total of 25,000 lb of steel and 6,000 lb of zinc. Each model A bathtub requires a mixture of 125 lb of steel and 20 lb of zinc, and each yields a profit of $90. Each model B tub requires 100 lb of steel and 30 lb of zinc and can be sold for a profit of $70. Find by graphical linear programming the best production mix of bathtubs.

• $P_{\times}$ B.9   The Grand Valley Company, run by the J. Motwani family, produces two products: bed mattresses and box springs. A prior contract requires that the firm produce at least 30 mattresses or box springs, in any combination, per week. In addition, union labor agreements demand that stitching machines be kept running at least 40 hours per week, which is one production period. Each box spring takes 2 hours of stitching time, and each mattress takes 1 hour on the machine. Each mattress produced costs $20; each box spring costs $24.
   a) Formulate this problem so as to minimize total production costs.
   b) Solve graphically.

P   **B.10**   MSA Computer Corporation manufactures two models of minicomputers, the Alpha 4 and the Beta 5. The firm employs 5 technicians, working 160 hours each per month, on its assembly line. Management insists that full employment (that is, *all* 160 hours of time) be maintained for each worker during next month's operations. It requires 20 labor-hours to assemble each Alpha 4 computer and 25 labor-hours to assemble each Beta 5 model. MSA wants to see at least 10 Alpha 4s and at least 15 Beta 5s produced during the production period. Alpha 4s generate a $1,200 profit per unit, and Betas yield $1,800 each.

      Determine the most profitable number of each model of minicomputer to produce during the coming month.

P   **B.11**   The Sweet Smell Fertilizer Company markets bags of manure labeled "not less than 60 lb dry weight." The packaged manure is a combination of compost and sewage wastes. To provide good-quality fertilizer, each bag should contain at least 30 lb of compost but no more than 40 lb of sewage. Each pound of compost costs Sweet Smell 5¢ and each pound of sewage costs 4¢. Use a graphical LP method to determine the least-cost blend of compost and sewage in each bag.

P   **B.12**   Consider Faud Shatara's following linear programming formulation:

$$\text{Minimize cost} = \$1X_1 + \$2X_2$$
$$\text{Subject to:} \quad X_1 + 3X_2 \geq 90$$
$$8X_1 + 2X_2 \geq 160$$
$$3X_1 + 2X_2 \geq 120$$
$$X_2 \leq 70$$

a)   Graphically illustrate the feasible region and apply the iso-cost line procedure to indicate to Krista which corner point produces the optimal solution.

b)   What is the cost of this solution?

·   **B.13**   The LP relationships that follow were formulated by Jeffrey Rummel at the Connecticut Chemical Company. Which ones are invalid for use in a linear programming problem, and why?

$$\text{Maximize} = 6X_1 + \tfrac{1}{2} X_1 X_2 + 5X_3$$
$$\text{Subject to:} \quad 4X_1 X_2 + 2X_3 \quad \leq 70$$
$$7.9X_1 - 4X_2 \quad \geq 15.6$$
$$3X_1 + 3X_2 + 3X_3 \geq 21$$
$$19X_2 - \tfrac{1}{3} X_3 = 17$$
$$-X_1 - X_2 + 4X_3 = 5$$
$$4X_1 + 2X_2 + 3\sqrt{X_3} \leq 80$$

P   **B.14**   Kalyan Singhal Corp. makes three products, and it has three machines available as resources as given in the following LP problem:

$$\text{Maximize contribution} = 4X_1 + 4X_2 + 7X_3$$
$$\text{Subject to:} \quad 1X_1 + 7X_2 + 4X_3 \leq 100 \text{ (hours on machine 1)}$$
$$2X_1 + 1X_2 + 7X_3 \leq 110 \text{ (hours on machine 2)}$$
$$8X_1 + 4X_2 + 1X_3 \leq 100 \text{ (hours on machine 3)}$$

a)   Determine the optimal solution using LP software.

b)   Is there unused time available on any of the machines with the optimal solution?

c)   What would it be worth to the firm to make an additional hour of time available on the third machine?

d)   How much would the firm's profit increase if an extra 10 hours of time were made available on the second machine at no extra cost?

P   **B.15**   Consider the following LP problem developed at Jeff Spencer's San Antonio optical scanning firm:

$$\text{Maximize profit} = \$1X_1 + \$1X_2$$
$$\text{Subject to:} \quad 2X_1 + 1X_2 \leq 100$$
$$1X_1 + 2X_2 \leq 100$$

a)   What is the optimal solution to this problem? Solve it graphically.

b)   If a technical breakthrough occurred that raised the profit per unit of X_1 to $3, would this affect the optimal solution?

c)   Instead of an increase in the profit coefficient X_1, to $3, suppose that profit was overestimated and should only have been $1.25. Does this change the optimal solution?

: P   **B.16**   The Arden County, Maryland, superintendent of education is responsible for assigning students to the three high schools in his county. He recognizes the need to bus a certain number of students, for several sectors of the county are beyond walking distance to a school. The superintendent partitions the county into five geographic sectors as he attempts to establish a plan that will minimize the total number of student miles traveled by bus. He also recognizes that if a student happens to live in a certain sector and is assigned to the high school in that sector, there is no need to bus him because he can walk to school. The three schools are located in sectors B, C, and E.

The accompanying table reflects the number of high-school-age students living in each sector and the distance in miles from each sector to each school.

| | DISTANCE TO SCHOOL | | | |
SECTOR	SCHOOL IN SECTOR B	SCHOOL IN SECTOR C	SCHOOL IN SECTOR E	NUMBER OF STUDENTS
A	5	8	6	700
B	0	4	12	500
C	4	0	7	100
D	7	2	5	800
E	12	7	0	400
				2,500

Each high school has a capacity of 900 students.
a)   Set up the objective function and constraints of this problem using linear programming so that the total number of student miles traveled by bus is minimized.
b)   Solve the problem.

: P   **B.17**   The National Credit Union has $250,000 available to invest in a 12-month commitment. The money can be placed in Treasury notes yielding an 8% return or in municipal bonds at an average rate of return of 9%. Credit union regulations require diversification to the extent that at least 50% of the investment be placed in Treasury notes. Because of defaults in such municipalities as Cleveland and New York, it is decided that no more than 40% of the investment be placed in bonds. How much should the National Credit Union invest in each security so as to maximize its return on investment?

: P   **B.18**   Boston's famous Limoges Restaurant is open 24 hours a day. Servers report for duty at 3 A.M., 7 A.M., 11 A.M., 3 P.M., 7 P.M., or 11 P.M., and each works an 8-hour shift. The following table shows the minimum number of workers needed during the 6 periods into which the day is divided.

PERIOD	TIME	NUMBER OF SERVERS REQUIRED
1	3 A.M.–7 A.M.	3
2	7 A.M.–11 A.M.	12
3	11 A.M.–3 P.M.	16
4	3 P.M.–7 P.M.	9
5	7 P.M.–11 P.M.	11
6	11 P.M.–3 A.M.	4

Owner Michelle Limoges' scheduling problem is to determine how many servers should report for work at the start of each time period in order to minimize the total staff required for one day's operation. (*Hint:* Let X_i equal the number of servers beginning work in time period i, where $i = 1, 2, 3, 4, 5, 6$.)

• P   **B.19**   A craftsman named Chuck Synovec builds two kinds of birdhouses, one for wrens and a second for bluebirds. Each wren birdhouse takes 4 hours of labor and 4 units of lumber. Each bluebird house requires 2 hours of labor and 12 units of lumber. The craftsman has available 60 hours of labor and 120 units of lumber. Wren houses yield a profit of $6 each and bluebird houses yield a profit of $15 each.
a)   Write out the objective and constraints.
b)   Solve graphically.

: P   **B.20**   Each coffee table produced by Robert West Designers nets the firm a profit of $9. Each bookcase yields a $12 profit. West's firm is small and its resources limited. During any given production period (of 1 week), 10 gallons of varnish and 12 lengths of high-quality redwood are available. Each coffee table requires approximately 1 gallon of varnish and 1 length of redwood. Each bookcase takes 1 gallon of varnish and 2 lengths of wood.

Formulate West's production-mix decision as a linear programming problem, and solve. How many tables and bookcases should be produced each week? What will the maximum profit be?

: P   **B.21**   Graph the following LP problem:

$$\text{Maximize profit} = \$3X_1 + \$2X_2$$
$$\text{Subject to:} \quad 2X_1 + 1X_2 \le 150$$
$$2X_1 + 3X_2 \le 300$$

a) What is the optimal solution?

b) Does the optimal solution change if the profit per unit of X_1 changes to $4.50?

c) What happens if the profit function should have been $3X_1 + 3X_2$?

• **P** **B.22** Solve the following linear programming problem graphically.

$$\text{Minimize cost} = 4X_1 + 5X_2$$
$$\text{Subject to:} \quad X_1 + 2X_2 \geq 80$$
$$3X_1 + X_2 \geq 75$$
$$X_1, X_2 \geq 0$$

: **P** **B.23** Thompson Distributors packages and distributes industrial supplies. A standard shipment can be packaged in a class A container, a class K container, or a class T container. A single class A container yields a profit of $9; a class K container, a profit of $7; and a class T container, a profit of $15. Each shipment prepared requires a certain amount of packing material and a certain amount of time.

Resources Needed per Standard Shipment

CLASS OF CONTAINER	PACKING MATERIAL (POUNDS)	PACKING TIME (HOURS)
A	2	2
K	1	6
T	3	4
Total amount of resource available each week	130 pounds	240 hours

Jason Thompson, head of the firm, must decide the optimal number of each class of container to pack each week. He is bound by the previously mentioned resource restrictions but also decides that he must keep his 6 full-time packers employed all 240 hours (6 workers × 40 hours) each week.
    Formulate and solve this problem using LP software.

• **P** **B.24** Using the data from Problem B.7 on page 714 and LP software:

a) Determine the range within which the unit profit contribution of an air conditioner must fall for the current solution to remain optimal.

b) What is the shadow price for the wiring constraint?

: **P** **B.25** The Denver advertising agency promoting the new Breem dishwashing detergent wants to get the best exposure possible for the product within the $100,000 advertising budget ceiling placed on it. To do so, the agency needs to decide how much of the budget to spend on each of its two most effective media: (1) television spots during the afternoon hours and (2) large ads in the city's Sunday newspaper. Each television spot costs $3,000; each Sunday newspaper ad costs $1,250. The expected exposure, based on industry ratings, is 35,000 viewers for each TV commercial and 20,000 readers for each newspaper advertisement. The agency director, Deborah Kellogg, knows from experience that it is important to use both media in order to reach the broadest spectrum of potential Breem customers. She decides that at least 5 but no more than 25 television spots should be ordered, and that at least 10 newspaper ads should be contracted. How many times should each of the two media be used to obtain maximum exposure while staying within the budget? Use the graphical method to solve.

: **P** **B.26** Libby Temple Manufacturing has three factories (1, 2, and 3) and three warehouses (A, B, and C). The following table shows the shipping costs between each factory and warehouse, the factory manufacturing capabilities (in thousands), and the warehouse capacities (in thousands).

a) Write the objective function and the constraint in equations. Let X_{1A} = 1,000s of units shipped from factory 1 to warehouse A, and so on.

b) Solve by computer.

TO FROM	WAREHOUSE A	WAREHOUSE B	WAREHOUSE C	PRODUCTION CAPABILITY
FACTORY 1	$ 6	$ 5	$ 3	6
FACTORY 2	$ 8	$10	$ 8	8
FACTORY 3	$11	$14	$18	10
CAPACITY	7	12	5	

B.27 A fertilizer manufacturer has to fulfill supply contracts to its two main customers (650 tons to Customer A and 800 tons to Customer B). It can meet this demand by shipping existing inventory from any of its three warehouses. Warehouse 1 (W1) has 400 tons of inventory onhand, Warehouse 2 (W2) has 500 tons, and Warehouse 3 (W3) has 600 tons. The company would like to arrange the shipping for the lowest cost possible, where the per-ton transit costs are as follows:

	W1	W3	W3
CUSTOMER A	$7.50	$6.25	$6.50
CUSTOMER B	6.75	7.00	8.00

a) Explain what each of the six decision variables (V) is: *(Hint: Look at the Solver report below.)*

V1: _____

V2: _____

V3: _____

V4: _____

V5: _____

V6: _____

b) Write out the objective function in terms of the variables (V1, V2, etc.) and the objective coefficients.

c) Aside from nonnegativity of the variables, what are the five constraints? Write a short description for each constraint, and write out the formula (and circle the type of equality/inequality).

Description	Variables and Coefficients	What Type?	RHS
C1:_____	Formula: _____	(=> | = | =<)	_____
C2: _____	Formula: _____	(=> | = | =<)	_____
C3: _____	Formula: _____	(=> | = | =<)	_____
C4: _____	Formula: _____	(=> | = | =<)	_____
C5: _____	Formula: _____	(=> | = | =<	_____

After you formulate and enter the linear program in Excel, the Solver gives you the following sensitivity report:

Adjustable Cells

CELL	NAME	FINAL VALUE	REDUCED COST	OBJECTIVE COEFFICIENT	ALLOWABLE INCREASE	ALLOWABLE DECREASE
$B$6	V1	0	1.5	7.5	1E+30	1.5
$C$6	V2	100	0	6.25	0.25	0.75
$D$6	V3	550	0	6.5	0.75	0.25
$E$6	V4	400	0	6.75	0.5	1E+30
$F$6	V5	400	0	7	0.75	0.5
$G$6	V6	0	0.75	8	1E+30	0.75

Constraints

CELL	NAME	FINAL VALUE	SHADOW PRICE	CONSTRAINT R.H. SIDE	ALLOWABLE INCREASE	ALLOWABLE DECREASE
$H$7	C1	650	6.5	650	50	550
$H$8	C2	800	7.25	800	50	400
$H$9	C3	400	−0.5	400	400	50
$H$10	C4	500	−0.25	500	550	50
$H$11	C5	550	0	600	1E+30	50

d) How many of the constraints are binding?

e) How much slack/surplus is there with the nonbinding constraint(s)?

f) What is the range of optimality on variable V3?   _____ to _____.

g) If we could ship 10 tons less to Customer A, how much money *might* we be able to save?   _____
If we could chose to short *either* Customer A or Customer B by 10 tons, which would we prefer to short? Why?

B.28 New Orleans's Mt. Sinai Hospital is a large, private, 600-bed facility complete with laboratories, operating rooms, and X-ray equipment. In seeking to increase revenues, Mt. Sinai's administration has decided to make a 90-bed addition on a portion of adjacent land currently used for staff parking. The administrators feel that the labs, operating rooms, and X-ray department are not being fully utilized at present and do not need to be expanded to handle additional patients. The addition of 90 beds, however, involves deciding how many beds should be allocated to the medical staff (for medical patients) and how many to the surgical staff (for surgical patients).

The hospital's accounting and medical records departments have provided the following pertinent information. The average hospital stay for a medical patient is 8 days, and the average medical patient generates

$2,280 in revenues. The average surgical patient is in the hospital 5 days and generates $1,515 in revenues. The laboratory is capable of handling 15,000 tests per year more than it *was* handling. The average medical patient requires 3.1 lab tests, the average surgical patient 2.6 lab tests. Furthermore, the average medical patient uses 1 X ray, the average surgical patient 2 X rays. If the hospital were expanded by 90 beds, the X-ray department could handle up to 7,000 X rays without significant additional cost. Finally, the administration estimates that up to 2,800 additional operations could be performed in existing operating-room facilities. Medical patients, of course, require no surgery, whereas each surgical patient generally has one surgery performed.

Formulate this problem so as to determine how many medical beds and how many surgical beds should be added to maximize revenues. Assume that the hospital is open 365 days per year.

B.29 Charles Watts Electronics manufactures the following six peripheral devices used in computers especially designed for jet fighter planes: internal modems, external modems, graphics circuit boards, floppy disk drives, hard disk drives, and memory expansion boards. Each of these technical products requires time, in minutes, on three types of electronic testing equipment as shown in the following table:

	INTERNAL MODEM	EXTERNAL MODEM	CIRCUIT BOARD	FLOPPY DRIVES	HARD DRIVES	MEMORY BOARDS
Test device 1	7	3	12	6	18	17
Test device 2	2	5	3	2	15	17
Test device 3	5	1	3	2	9	2

The first two test devices are available 120 hours per week. The third (device 3) requires more preventive maintenance and may be used only 100 hours each week. The market for all six computer components is vast, and Watts Electronics believes that it can sell as many units of each product as it can manufacture. The table that follows summarizes the revenues and material costs for each product:

DEVICE	REVENUE PER UNIT SOLD ($)	MATERIAL COST PER UNIT ($)
Internal modem	200	35
External modem	120	25
Graphics circuit board	180	40
Floppy disk drive	130	45
Hard disk drive	430	170
Memory expansion board	260	60

In addition, variable labor costs are $15 per hour for test device 1, $12 per hour for test device 2, and $18 per hour for test device 3. Watts Electronics wants to maximize its profits.
a) Formulate this problem as an LP model.
b) Solve the problem by computer. What is the best product mix?
c) What is the value of an additional minute of time per week on test device 1? Test device 2? Test device 3? Should Watts Electronics add more test device time? If so, on which equipment?

B.30 Susan Chollette, campus dietitian for a small California college, is responsible for formulating a nutritious meal plan for students. For an evening meal, she feels that the following five meal-content requirements should be met: (1) between 900 and 1,500 calories; (2) at least 4 milligrams of iron; (3) no more than 50 grams of fat; (4) at least 26 grams of protein; and (5) no more than 50 grams of carbohydrates. On a particular day, Susan's food stock includes seven items that can be prepared and served for supper to meet these requirements. The cost per pound for each food item and its contribution to each of the five nutritional requirements are given in the accompanying table:

	TABLE OF FOOD VALUES[a] AND COSTS					
FOOD ITEM	CALORIES/ POUND	IRON (MG/LB)	FAT (G/LB)	PROTEIN (G/LB)	CARBOHYDRATES (G/LB)	COST/ POUND ($)
Milk	295	0.2	16	16	22	0.60
Ground meat	1,216	0.2	96	81	0	2.35
Chicken	394	4.3	9	74	0	1.15
Fish	358	3.2	0.5	83	0	2.25
Beans	128	3.2	0.8	7	28	0.58
Spinach	118	14.1	1.4	14	19	1.17
Potatoes	279	2.2	0.5	8	63	0.33

[a]From C. F. Church and H. N. Church, Bowes and Church's *Food Values of Portions Commonly Used*, 12th ed. Philadelphia: J. B. Lippincott, 1975.

What combination and amounts of food items will provide the nutrition Susan requires at the least total food cost?

a) Formulate as an LP problem.
b) What is the cost per meal?
c) Is this a well-balanced diet?
d) How sensitive is the solution to price changes in milk, ground meat, fish, and chicken?

INTERNET HOMEWORK PROBLEMS

See our Companion Web site at **www.prenhall.com/heizer** for these additional homework problems: B.31 through B.40.

CASE STUDY

Golding Landscaping and Plants, Inc.

Kenneth and Patricia Golding spent a career as a husband-and-wife real estate investment partnership in Washington, DC. When they finally retired to a 25-acre farm in northern Virginia's Fairfax County, they became ardent amateur gardeners. Kenneth planted shrubs and fruit trees, and Patricia spent her hours potting all sizes of plants. When the volume of shrubs and plants reached the point that the Goldings began to think of their hobby in a serious vein, they built a greenhouse adjacent to their home and installed heating and watering systems.

By 2005, the Goldings realized their retirement from real estate had really only led to a second career—in the plant and shrub business—and they filed for a Virginia business license. Within a matter of months, they asked their attorney to file incorporation documents and formed the firm Golding Landscaping and Plants, Inc.

Early in the new business's existence, Kenneth Golding recognized the need for a high-quality commercial fertilizer that he could blend himself, both for sale and for his own nursery. His goal was to keep his costs to a minimum while producing a top-notch product that was especially suited to the northern Virginia climate.

Working with chemists at George Mason University, Golding blended "Golding-Grow." It consists of four chemical compounds, C-30, C-92, D-21, and E-11. The cost per pound for each compound is indicated in the following table:

CHEMICAL COMPOUND	COST PER POUND
C-30	$.12
C-92	.09
D-21	.11
E-11	.04

The specifications for Golding-Grow are established as:

a. Chemical E-11 must constitute at least 15% of the blend.
b. C-92 and C-30 must together constitute at least 45% of the blend.
c. D-21 and C-92 can together constitute no more than 30% of the blend.
d. Golding-Grow is packaged and sold in 50-lb bags.

Discussion Questions

1. Formulate an LP problem to determine what blend of the four chemicals will allow Golding to minimize the cost of a 50-lb bag of the fertilizer.
2. Solve to find the best solution.

ADDITIONAL CASE STUDIES

See our Companion Web site at www.prenhall.com/heizer for these three additional case studies:

- **Mexicana Wire Works**: This capacity case at a Mexican electrical manufacturer addresses back-ordering versus using temporary workers.

- **Coastal States Chemical**: This production planning case deals with handling cuts in natural gas availability.

- **Chase Manhattan Bank**: This scheduling case involves finding the optimal number of full-time versus part-time employees at a bank.

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