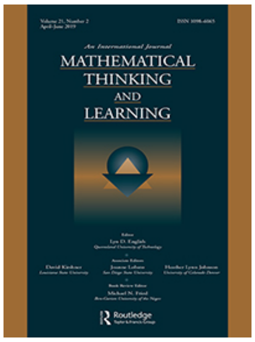

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Linear quantity models in US and Chinese elementary mathematics classrooms

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ABSTRACT

Linear quantity models such as pre-tapes, tape diagrams, and number line diagrams have drawn increasing attention in mathematics education around the world. However, we still know relatively little about how teachers actually use these models in the classroom. This study explores how exemplary US and Chinese elementary teachers use linear quantity models during mathematics instruction. Based on an examination of 64 videotaped lessons on inverse relations, we identified 110 "diagram episodes." An analysis of these episodes reveals that linear quantity models, especially tape diagrams, were used more frequently in US classrooms than in Chinese classrooms. However, Chinese lessons used these models for the sole purpose of modeling the underlying quantitative relationships, whereas US lessons mainly used them to aid in computation. In addition, while US teachers rarely involved students in discussion of linear quantity models, Chinese teachers spent significant time engaging students in co-constructing, comparing, and explaining these models.

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Linear quantity models such as pre-tapes, tape diagrams, and number line diagrams are powerful tools that represent quantitative relationships in a linear manner. As with other diagrams, linear quantity models group relevant information together, which offers perceptual support for problem solving and in turn can enhance the effectiveness of inference-making (Larkin & Simon, 1987). These diagrams can also be used to teach and learn varied mathematical topics across grades, by providing continuity across topics so that students may experience mathematics as a subject of systematic relationships (Murata, 2008). Due to the powerful nature of these diagrams, linear quantity models have been widely used in mathematically high-achieving countries like Singapore (Beckman, 2004; Cai & Moyer, 2008; Ng & Lee, 2009), Japan (Murata, 2008), and China (Ding & Li, 2014). Recently, the Common Core State Standards (Common Core State Standards Initiative [CCSSI], 2010) in the US also embraced these models, expecting students to learn fractions with number lines in third grade and to solve ratio problems using tape diagrams in sixth grade. However, the relative novelty of these models may be problematic: for novice users, this type of model may be seen as relatively abstract and nontransparent in comparison to drawings of realistic pictures. If not used appropriately, these diagrams may interfere with learning objectives. As the classroom praxis of linear quantity models is crucial to their pedagogical impact, there is an urgent need to explore how these models are actually used by teachers in classrooms to support student learning.

Prior studies on linear quantity models have mainly focused on international textbook comparisons (Beckman, 2004; Cai & Moyer, 2008; Ding & Li, 2014; Murata, 2008). Few studies have yet examined how elementary teachers use these diagrams in classroom settings, especially in a cross-cultural context. The current study is a first step in examining how exemplary US and Chinese

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Colour versions of one or more of the figures in the article can be found online at www.tandfonline.com/hmlt.

teachers use these diagrams in elementary mathematics classrooms. In particular, we focus on the fundamental mathematical topic of the inverse relationships between addition and subtraction, and between multiplication and division ("inverse relations" hereafter). We have chosen this topic not only because inverse relations are systematically emphasized by the common core (CCSSI, 2010), but because linear quantity models can use the same diagram to show inverse relations simultaneously. As such, we ask: (a) What types of linear quantity models do teachers use when teaching inverse relations? (b) What is the instructional purpose of teachers' diagram uses as indicated by their teaching of inverse relations? And (c) How are students engaged in the use of diagrams when learning inverse relations? Below, we review the relevant literature that situates the current study.

Types of linear quantity models

Prior studies suggest three types of linear quantity models: tape diagrams, pre-tapes, and number lines (Ding & Li, 2014; Murata, 2008, see Figure 1a–c. Tape diagrams are named differently throughout the literature, referred to variably as strip diagrams (Beckman, 2004), pictorial equations (Cai & Moyer, 2008), or the Singapore model method (Ng & Lee, 2009). In addition, some US textbooks (e.g. *Go Math*) use the term "bar models." Regardless of their names, these diagrams share the common feature of "appropriately sized rectangles" (Cai & Moyer, 2008, p. 284) which are used to represent quantities. Figure 1b illustrates a double tape diagram taken from Murata (2008).

Pre-tapes and number lines are variations of tape diagrams, with the former being slightly more concrete and the latter being more abstract than ordinary tape diagrams. Murata (2008) noted that tape diagrams in Japanese textbooks were developed from pre-tapes, which are linear representations of concrete objects. For example, a set of leaves in a linear arrangement is considered a pre-tape

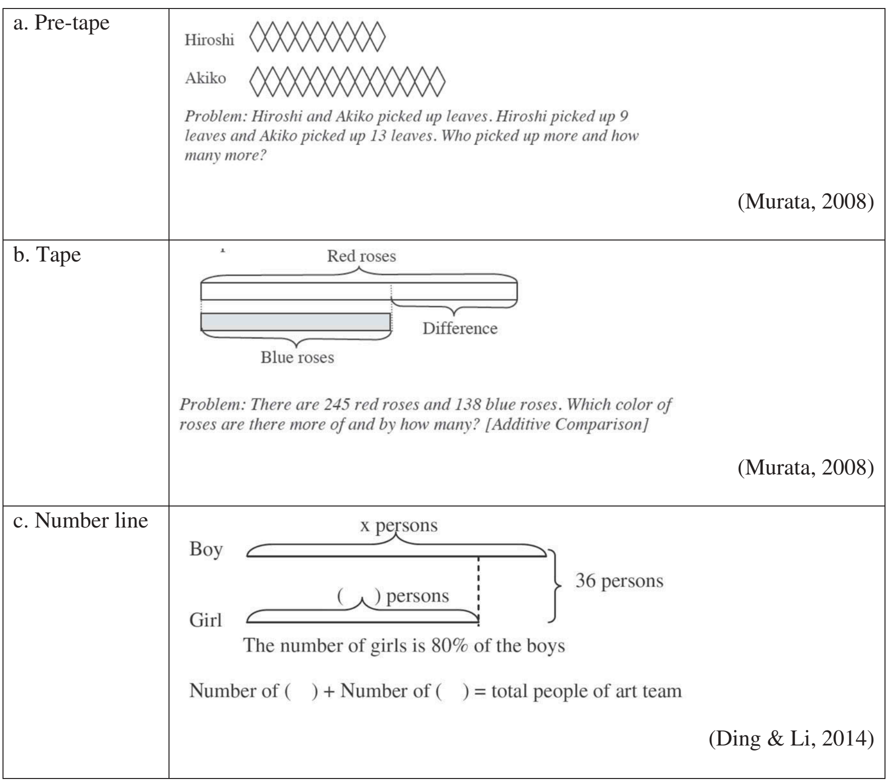


Figure 1. Examples of pre-tape, tape, and number line diagrams in prior studies.

diagram (see Figure 1a). Because of their concrete nature, developing knowledge of pre-tapes in early grades may help students use tape diagrams more effectively. For instance, this set of leaves can be changed from a pre-tape to a tape diagram by simply putting boxes around the leaves and then removing the leaves. In contrast, replacing rectangular boxes of a tape diagram with a line turns the model into a number line diagram which is relatively more abstract because it does not show enclosed area which students could associate with concrete objects. Nevertheless, number line diagrams are widely used in instruction of mathematically high-achieving countries to assist in problem solving (Ding & Li, 2014; Murata, 2008; Ng & Lee, 2009). Note that this study did not distinguish between a number line that implies infinite extent (e.g. adding on a number line) and a line segment that represents a single mathematical quantity. Figure 1c illustrates a number line diagram from a Chinese sixth grade textbook (Ding & Li, 2014). This diagram illustrates quantitative relationships of an algebraic word problem involving ratio and percent, a task similar to what the Common Core expects from US sixth graders. Despite the differences in representational format, all of the above models represent quantitative relationships in a linear manner.

Purpose of using linear quantity models

Literature indicates that the power of a linear quantity model mainly lies in its supportive role for problem solving (Cai & Moyer, 2008; Ding, 2010; Ding & Li, 2014; Murata, 2008). According to (English & Halford 1995), the process of solving word problems contains three components: the problem-text model, the problem-situation model, and the mathematical model. The problem-text model refers to a mental representation of a given text initially read, which is similar to Kintsch's (1986) "textbase." The problem-situation model is a mental representation similar to Kintsch's situation model, containing an inference-making process such as mapping the problem-text onto a familiar analogical situation. Finally, the mathematical model refers to the formal mathematical solution to the problem (e.g. an equation). English and Halford place particular emphasis on the role of the problem-situation model in linking the problem-text to the mathematical model: When the problem-situation model is missing, the resulting direct mapping from the problem-text to a mathematical model often causes difficulties for students (English & Halford 1995). A well-known example of this is the direct translation of "key words" into mathematical operations, which often produces either incorrect solutions, or correct solutions with incorrect reasoning. Accordingly, one may argue the ultimate purpose of using linear quantity models should be for forming a problem-situation model, in which inference-making and mathematical reasoning about quantitative relationships occurs.

In reality, however, linear quantity models are not necessarily used to form problem-situation models. Murata (2008) found that, while Japanese textbooks used tape diagrams to analyze the quantitative relationships embedded in a story problem (e.g. additive comparisons, see examples in Figure 1a,b, US textbooks often used these representations for the sole purpose of finding a computational answer. This may be due to the fact that Japanese textbooks mainly used tape diagrams for contextual tasks (e.g. word problems), while US textbooks often used them for non-contextual tasks (e.g. computation). The above difference may also reflect a broad, cross-cultural difference in teaching style (Stigler & Hiebert, 1999) particularly with the use of representations. For instance, when using the same task 23-17 to teach the concept of "regrouping," Ma (1999) found that some US teachers would ask students to take away 17 objects (e.g. dinosaur eggs) from 23 to obtain the answer of 6, which removed the need for regrouping. In contrast, many Chinese teachers use representations to help students understand the process of regrouping (e.g. how 1 ten is decomposed into 10 ones).

Such cross-cultural differences in representational uses may be attributed to teachers' different beliefs about the purpose of concrete representations. Through examining exemplary teachers' representations in lesson planning and their evaluations of student representations in problem solving, Cai (2005) found that US teachers viewed concrete representations as simply problem solutions or used them as ways to find computational answers. In contrast, Chinese teachers treated concrete representations as tools for understanding both the quantitative relationships and the problem structure during the process of problem solving. Consequently, while US teachers accepted students' drawings as mathematical solutions, Chinese teachers expressed disapproval of such concrete solutions because in their view, concrete representations should be shifted to abstract and generalized mathematical solutions. Chinese teachers' beliefs (and Japanese textbook treatment) in the literature seems to align well with the modeling theory of Realistic Mathematics Education, a shifting from a "model of" informal mathematical activities to a "model for" formal mathematical structures (Gravemeijer, 1999). This also echoes the importance of shifting from students' problem-text models to the development of their problem-situation models during the problem solving process (English & Halford 1995).

This above difference in the purpose of representation use – finding computational answers as opposed to understanding quantitative relationships – may occur in the case of using linear quantity models. For instance, a teacher may ask students to count the extra leaves in Figure 1a to obtain the answer. In Figure 1b, even though one cannot count the continuous tape to obtain an answer, a teacher may still focus on the computational aspect (245-138) without using the tape diagram to explain why "subtraction" is the appropriate operation. Indeed, subtraction for novice learners often means "taking away" a part from the whole. However, how can one take away the blue roses from the red roses (see Figure 1a), when the blue roses are not initially part of the red roses? This dilemma indicates a challenge in mapping from textual information to problem structure, especially when the comparison relationship is hard to manipulate (Greeno & Riley, 1987; Nunes, Bryant, & Watson, 2009). To overcome this difficulty, tape diagrams may be used to assist students' understanding. According to Greeno and Riley (1987), in order to understand difficult comparison word problems, students need to understand a critical concept: the one-to-one correspondence (or "the same as"). In Figure 1b, the dashed line indicates the concept of "the same as," which separates the large quantity into two different parts, the "same as" part (equal to the small quantity) and the "more than" part (difference). With this structural understanding, students can visualize that taking away "the same as part" (small quantity) from the large quantity, results in the remaining "more than" part (difference). This explains why "subtraction" is an appropriate operation (large – small = difference). The above structural use of a tape diagram helps transform a difficult comparison problem into a relatively more familiar part-whole problem. In the same vein, a structural use of this tape diagram may also allow students to understand why "large – difference = small" and "small + difference = large." Taken together, these quantitative relationships elucidate the fundamental mathematical idea of inverse relations.

Students engagement with linear quantity models

Although linear quantity models should be primarily for forming problem-situation models through inference-making, students' formation of such a mental model is not simple. To help students form meaningful problem-situation models, English and Halford stressed the importance of engaging students in classroom discussion of "the structure of concrete/pictorial models" (p. 193). Unfortunately, existing literature lacks information on how teachers actually engage students in the process of learning linear quantity models in practice. However, a recent teacher blog (Vanduzer, 2016) provides a quick glimpse at the learning and teaching difficulty associated with linear quantity models. In this blog, the teacher explains how a tape diagram was used to teach a story problem comparing the distances that two students can jump. The teacher first acted the story out with students by jumping on a huge tape placed on the floor. The teacher then placed markers at each jump in order to visually show students the difference in the jumps. After this, students were directed to draw a tape diagram on their paper to solve this problem. However, as described by the teacher, this act-it-out activity did not seem to contribute positively to students' drawings. For instance, one student complained that the paper was too small to draw the jumps and later became disruptive to the class. The above case calls for an understanding of "who" is involved in the creation and discussion of the linear quantity models. Are diagrams presented by teachers without student engagement or do teachers use these models in interactive ways? In addition to this participant structure question (Richland, Holyoak, & Stigler, 2004), this case also calls for an understanding of how teachers can effectively engage students in discussion of the structure of diagrams to form problem-situation models that will enable their problem solving (English & Halford 1995).

Existing literature offers limited research assertion on how teachers may engage students with linear quantity models. Yet, international textbook research does provide some insight into the above questions. Ding and Li (2014) reported that Chinese textbooks involved students in the process of co-constructing and analyzing diagrams. This included having students draw part of a diagram, having students label key quantities on a line, and posing specific questions about a diagram. For example, as seen from Figure 1c, the number line diagram taken from a Chinese textbook only suggested the quantity of boy be noted as "x". Students were expected to complete the diagram by labeling the quantity (80% x) for girls based on the given textual information, "the number of girls is 80% of the boys." After this, the textbook expected students to complete the quantitative relationship sentence, "Number of () + Number of () = total people," which helped to enable students with generating the anticipated algebraic equation ($x + 80\%x = 36$). The above process clearly indicated the purpose of engaging students in reasoning upon the key quantities and quantitative relationships embedded in the number line diagram, which likely facilitated the analogical mapping among the problem-text model, the problem situation model, and the mathematical model (English & Halford, 1995). Such a modeling process – linking real-life situations, diagrams, and written symbols – is supported by previous literature on mathematical modeling (Fruedenthal, 1991; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003).

In addition to the above co-construction strategy, Ding (2018) introduced two Chinese teachers' approaches to engaging students with tape diagrams. Both teachers asked students deep questions involving comparison activities, and both prompted students to gesture towards particular parts of a diagram in order to clarify their thinking and understanding. Further, these two teachers made variations on tape diagrams to enable more comparisons. These strategies echo cognitive research assertions in supporting student learning. For instance, Craig, Sullins, Witherspoon, and Gholson (2006) reported that dialogues and deep-level reasoning questions have a clear effect on improving learning. As such, asking deep questions such as "why, why not, what if, what if not" were recommended as a powerful instructional principle to elicit students' deep explanations of targeted concepts and structural relationships (Richland, Zur, & Holyoak, 2007). Likewise, comparisons were also recognized as powerful instructional strategies that facilitate students' analogical reasoning (CCSSI, 2010; Richland et al., 2007; Rittle-Johnson & Star, 2009). Gestures have been argued to be an effective tool for student engagement in the learning process due to their connection to embodied cognition (e.g. Alibali & DiRusso, 1999; Alibali & Nathan, 2012; Goldin-Meadow & Alibali, 2013). Finally, teachers' use of variations has been reported as a Chinese-based classroom practice, which has been viewed as a critical tool for developing students' mathematical reasoning (Huang & Li, 2017; Sun, 2011). The above strategies suggest potentially rich dimensions for exploring how students in US and Chinese classrooms are engaged in the process of creating and understanding linear quantity models.

Methods

Participants

This study is part of a large research project that seeks to glean effective knowledge for teaching early algebra based on classroom performance of exemplary US and Chinese teachers. In the current study, we analyzed classroom teaching of 8 US and 8 Chinese teachers from the first year of the project. Note that our purpose was to identify potentially effective modeling approaches that may support student learning rather than to generalize findings about the two countries. Our criteria for selecting exemplary teachers included (a) at least 10-year of teaching experience, and (b) a good teaching reputation: All eight of the Chinese teachers have received teaching awards at a local and/or a national level, and three of the eight US teachers are national board certified teachers (NBCT). The remaining five US teachers were highly recommended by their principals and/or their school district. Teachers in each country were selected from six different schools in a large urban city. All except one US teacher were female. Since the targeted mathematical topic was the inverse relations between addition and subtraction (additive inverses) and between multiplication and division (multiplicative inverses), US teachers were selected from grades 1–4 and Chinese teachers from grades 1–3. This difference in grade levels is because multiplication is introduced in grade 2 in China, but not until grade 3 in the US. Consequently, four teachers in each country taught additive and multiplicative inverses, respectively. For convenience, we named these teachers US-T1:T8 and China-T1:T8. All T1-T4 teachers taught additive inverses and T5-T8 teachers taught multiplicative inverses.

Lesson selection

In this study, the lesson selection for US and Chinese classrooms was based on the literature assertions (e.g. Carpenter et al., Ding, 2016; Nunes et al., 2009; Sun & Wang, 2013). That is, inverse relations in elementary schools may be learned through different problem types such as fact families, inverse word problems, initial unknown problems, or using inverse relations for checking or computational purposes. These different problem types may be either contextual or non-contextual. With contextual problems, different word problem structures may be involved. For instance, to illustrate additive inverses, one may use a group of related part-whole word problems where either the "part" or the "whole" is treated as the unknown. The same applies to comparison word problems where the large quantity, or the small quantity, or the difference, may be viewed as the unknown (e.g. the unknown in Figure 1b is the "difference"). The same reasoning can be applied to multiplicative inverses in which equal-groups word problems (with three quantities: group size, number of group, total), multiplicative comparison word problems (large quantity, small quantity, multiples), or array/area models (# of columns, # of rows, and total arrays) may be involved. Aligned with above literature assertions, our selected lessons from existing textbooks contained the types of inverse tasks (e.g. fact family, checking) situated in different types of problems (e.g. part-whole, comparisons).

Each teacher taught four videotaped lessons (total of 64) selected by the project researchers. They were selected through consultation with teachers and were based on their regular textbooks. The majority of lessons were directly related to the targeted topic of inverse relations (e.g. fact family, initial unknown problem, using inverse operation to compute or check). However, a few selected lessons were based on the comparison model, each of which only addressed one type of comparison problems (e.g. to find the difference/multiple, to find the large and/or small quantity). These lessons may not have appeared to target inverse relations; yet, taken together with the relevant prior lessons, they incorporated students' prior knowledge and created new opportunities to strengthen students' inverse understanding.

The actual lessons selected for instruction differ due to the use of different textbooks. In China, all project teachers used *Jiang Su Education Press* (JSEP) textbooks (Sun & Wang, 2012, 2013, 2014). Thus, teachers in the same grade taught exactly the same set of lessons. In contrast, the US teachers in the same grade were from different schools and used different textbooks, either *Go Math* (Dixon, Larson, Burger, Sandoval-Martinez, & Leinwand, 2012) or *Investigations* (Technical Education Research Centers [TERC], 2012). As such, lessons within the same grade were not identical. Nevertheless, we matched all selected lessons along similar topics. For instance, the first grade teachers in both countries taught lessons on (a) fact family/related facts, (b) finding missing numbers, (c) using addition to compute subtraction, and (d) initial unknown problems. This matching process occurred within each grade.

Despite the effort to match lesson topics, we found that textbook presentations differed in contextual support and their use of linear quantity models. Overall, Chinese textbook lessons always situated a new lesson in real-world situations (Ding & Li, 2014). Yet, both US textbook series were less consistent. For instance, *Go Math* sometimes asked students to use cubes or bar-models to model a story situation, but other times asked students to model abstract number sentences without contextual support. Similarly, *Investigations* sometimes provided lengthy story situations, but other times only presented a lesson that was context-free (e.g. math games). With regard to linear quantity models, Chinese textbooks mainly introduced tape diagrams in second grade lessons where the topic appeared non-straightforward; yet, *Go Math* introduced tape diagrams (bar models) across many different lessons in first grade. In fact, *Go Math* contained many more bar models than *Investigations*.

Data sources and procedures

The selected lessons were taught by teachers who may or may not have followed textbook suggestions. Overall, Chinese teachers were generally loyal to textbook presentations, similar to what was reported in the literature (Li & Huang, 2013). However, when enacting the lessons, some Chinese teachers did make changes to their textbook diagram suggestions. This included incorporating tape diagrams not found in their textbook and using different models than were suggested. The situation in US classrooms was more diverse. For instance, teachers who used *Go Math* either paid close attention to all bar models suggested by the textbooks or completely discarded them. For *Investigations*, since this textbook series contained a significantly fewer number of linear quantity models, teachers added their own (mainly pre-tape) diagrams. Overall, it seems that textbook presentation had some influence on teachers' classroom instruction; but teachers also demonstrated flexibility and their own choice in classroom instruction related to the use of linear quantity models.

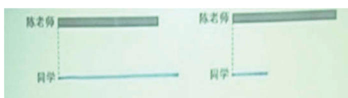
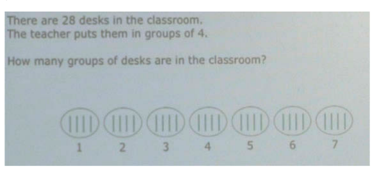
We suggested to teachers in both countries that they teach the selected lessons in the same way that they ordinarily do. All Chinese lessons lasted approximately 40 minutes. The US lessons ranged from 34 to 84 minutes, with the average being 58 minutes. All lessons were videotaped. The videotaping in US and China were conducted by the trained project staff and graduate students (the authors were not physically present for all lessons). For each lesson, we used two cameras for videotaping, one focusing on the teacher and the other following the students. After each lesson, teachers were interviewed with a few semi-structured questions. We started with asking teachers their teaching objectives and whether they believed that they had achieved those objectives. The next three questions focused on teachers' use of example tasks, representations, and questioning, which were the targeted instructional aspects of the larger project. Particular to representations, we asked, "What do you think about the representations you or students used during this lesson? Please explain. Did using the representations communicate mathematical ideas the way you thought they would? Did you use them as you had planned? Explain." Even if teacher's responses did not specifically involve linear quantity models, they shed light on general teacher views on representation uses. In addition, we asked teachers about unexpected events, their satisfaction with student reasoning, and to briefly describe their next planned lesson. Throughout the interview, follow-up questions for elaboration and clarification were asked as necessary. All videotaped lessons and interviews were transcribed by the trained student workers prior to data analysis.

Coding and data analysis

Our overall data analysis followed Chi's (1997) method of quantifying qualitative verbal data. This included developing and refining operational definitions, which was followed by coding and reliability checking. Based on the literature and screening of the videotaped lessons, a "diagram episode" was defined as one complete classroom discourse during which teachers sought to use linear quantity models (e.g. pre-tapes, tapes, and number lines). A new episode was considered if classroom discussion shifted to a new task or activity. A single "complete" episode therefore included all continuous classroom talk that involved sub-activities on the same diagram (e.g. discussion about how to draw a diagram, students' drawing of the diagram, comparisons of student products). Note that we did not consider the sole use of physical concrete objects as a diagram episode. For instance, if manipulatives (e.g. a cube train) were used, we did not code them; however, if a teacher drew cubes in a row or projected a cube train onto the board, this was considered a diagram episode because the image was presented as a pre-tape diagram. We also did not code any use of an array diagram (columns and rows) because this is a two-dimensional model, not a linear arrangement of objects.

Each identified diagram episode was coded for five aspects (type, context, purpose, who, and how) as reviewed in the conceptual framework, which aimed to address the three research questions. For the first research question, we coded the "type" of linear quantity models as pre-tape, tape, or number line (Murata, 2008). When an episode contained more than one type of model, we coded them as "multiple." Table 1 illustrates examples of these codes. For instance, Ch-T4's diagram was considered as a tape diagram while US-T5's diagram was coded as a pre-tape. For the second research question, we coded both the "context" and the "purpose" of using the linear model, because of the possible association between these aspects as suggested by Murata (2008). With regard to context, if an episode was purely computational, we coded it as "non-contextual." In contrast, if an

Table 1. An example coding-sheet.

Teacher	Lesson (Episode)	Time (Duration)	Type/Context/Purpose	Who/How
Ch-T4	L2 (1)	3:08–8:22 (5′14″)	*Type*: Tape diagram  *Context/Purpose*: Contextual & relationship. This pair of real-world problems were about the students' and their teacher's favorite numbers – Ms. Chen's favorite number is 45. My favorite number is 3 more than (left)/35 less than (right) Ms. Chen's. Tape diagrams were used to model the quantitative relationship (comparison). The screen shot was a student work selected by the teacher for class discussion.	*Who*: Class. Teacher provided the first tape; Students were asked to draw the second tape. *How*: Co-construction and deep questioning. Students drew the second tape. During the discussion of a selected student work, the teacher asked seemingly trivial questions that targeted the key aspects: - Why do you all draw the second tape longer? - What do you mean by "a bit longer"? - Why should it only be a little bit longer?
US-T5	L2 (3)	17:00–17:30 (30″)	Type: Pre-tape.  *Context/purpose*: Contextual & answer. This is a real-world context about dividing 28 desks into groups of 4. After students' group work, T brought this tally diagram with a description of drawing procedures, leading toward the conclusion, "they counted out the groups." The follow-up teaching revealed that T viewed this tally diagram as one strategy to obtain the answer, which was parallel to 28 ÷ 4 and 7 × □ = 28. No connections were made between this diagram and the equations.	*Who*: Teacher. No student talk was involved. *How*: Direct telling. T directly showed this tally diagram to class, " I am going to show you what they came up with and this is the tally strategy. And I noticed that some of you were using that strategy. *Where you drew out 28 tallies or 28 sticks or 28 lines and then they circled 4. Because they are 4 what, desks in each group. And then they counted out the groups."* …And it is a little more efficient then drawing out all of the desks with all of the legs, so this might be a strategy that would help you. In fact, T asked a few questions but all were about checking. For instance, she asked, "Tara did that help you? Anybody else use this strategy today? Did it help you?" As such, we did not code "questioning" as an instructional aid for this diagram episode.

episode was situated in a word problem context or accompanied by real-word pictures, we considered it "contextual." To code purpose, we differentiated between finding the answer (simply "answer"), understanding the relationships (simply "relationship"), or a combination of "both." As seen in Table 1, Ch-T4 and US-T5 both discussed contextual tasks, but with different purposes. Ch-T4 asked a set of questions that targeted the relationship between two quantities illustrated by the tapes (e.g. why is the second tape longer?) and as such, the purpose was coded as "relationship." In contrast, US-T5 explained the procedures of drawing tallies to find the answer – first draw 28 tallies, then circle every 4 tallies, and finally, count the groups to obtain the answers of "7." Given that no relationship among these quantities was stressed, we coded the purpose of this model as "just finding an answer." If US-T5 had asked questions (or even explained herself) about the meaning of the tally diagram, or about the relationships between the quantities embedded in the diagram (total, group, group size), we would have coded the purpose of this linear quantity diagram as "both" finding the answer and understanding the relationships.

To address the third research question, we first coded "who" was engaged in the discussion of the diagram. If a diagram was presented and explained by the teacher without involving the students, it was coded as "teacher." On the other hand, if a diagram was generated and explained by a student(s) without teacher input, it was coded as "student." Otherwise, if a diagram was discussed by both the teacher and students, it was coded as "class." In Table 1, the episodes of Ch-T4 and US-T5 were coded as "class" and "teacher," respectively. In addition to the dimension of "who," we also inspected "how" students were involved in the discussion. Based on the literature (e.g. Ding, 2018; Ding & Li, 2014) and actual data screening, we coded the following instructional aids as how the discussion occurred: co-construction, deep questioning, direct telling, comparison, variation, gesturing, and other concrete aids. For instance, in Table 1, Ch-T4 presented the first tape and students were asked to draw the second tape, which was coded as "co-construction." Ch-T4 also asked a set of "why" questions that appeared to orient students' attention to the targeted quantitative relationships. We therefore coded this episode as "deep questioning." Note that questions that focused on only computational answers or simple checking were not coded as "deep questioning." For instance, US-T5 in Table 1 asked a few checking questions; yet the explanations of relationships were completely provided by the teacher. This was therefore coded as "direct telling." Other instructional aids such as comparison, gesturing, and variation were coded in alignment with the literature assertions (Ding, 2018). For instance, when introducing the bar model in a lesson about multiplicative inverses, US-T6 asked students to compare two bar models by asking, "How is it the same and how is it different from the one we just talked about?" Similarly, Chinese teachers often asked students to compare different linear quantity diagrams (coded as "comparison"). They even asked students to come to the board to point out the exact section of a diagram, in order to clarify their thinking (coded as "gesturing"). In addition, some Chinese teachers uniquely made "variations" on linear diagrams (e.g. changing the location of a question mark, removing the context), which provided further opportunities for comparisons. Interestingly, we noted that some US teachers introduced other types of representational tools (e.g. folding papers, using cubes, drawing a fact triangle) along with the linear quantity diagrams. We captured this uniqueness by including a code for "other concrete aids." Infrequent but distinguished strategies (e.g. using a metaphor) that did not fall into one of the above codes were classified as "other."

The entire coding framework was developed based on the three authors' independent analysis of common sample lessons, along with ongoing discussions of disagreements. Initially, reliability (percentage of common codes out of the total codes) exceeded 90%. Discussions of disagreements enabled a refinement of several operational definitions. For instance, although both US and Chinese teachers frequently used pointing gestures, only Chinese teachers tended to ask students to gesture on the diagrams. To capture this cross-cultural difference, we decided to only code student gestures as an instructional aid. After shared understanding was reached, the three authors divided and coded/re-coded all lessons. After all diagram episodes were coded, relevant percentages for type of diagrams, instructional purposes under different contexts, participant structures, and instructional aids on the diagrams were computed.

Following this initial coding, all linear quantity diagram episodes from all lessons were systematically inspected for enriched understanding of emerging instructional patterns. To further enhance understanding of instructional decisions, all teacher interviews were also qualitatively analyzed for teachers' views on representational uses (e.g. purpose). This contained a within-lesson examination for critical aspects of teacher views and across-lesson comparisons for common themes. A further comparison between US and Chinese teachers' representational views was also made. Note that whenever possible, we paid particular attention to teachers' discussions on linear quantity models. Nevertheless, we acknowledge that our teacher interviews did not focus solely on linear quantity models and thus the identified teacher voice may be limited.

Results

Types of tape diagrams in US and Chinese classrooms

A total of 110 diagram episodes were identified from the US and Chinese lessons, including 59 US episodes and 51 Chinese episodes. Each episode involved either pre-tape, tape, number line, or a combination of these diagrams (US: $N_{\text{pre-tape}} = 14$, $N_{\text{tape}} = 26$, $N_{\text{number line}} = 10$, $N_{\text{multiple}} = 9$; China: $N_{\text{pre-tape}} = 31$, $N_{\text{tape}} = 12$, $N_{\text{number line}} = 7$, $N_{\text{multiple}} = 1$). Figure 2 indicates that US lessons contained a higher proportion of tape diagram episodes than pre-tapes, while Chinese lessons showed an opposite pattern. One possible explanation for why pre-tapes may occur more frequently in the Chinese lessons is due to the fact that lower Chinese elementary grades (G1-G3) were involved in this study (as opposed to US G1-G4) because of what grade level inverse relations are taught.

Table 2 lists the total number of episodes used by each teacher. As indicated, one US teacher (US-T3) and two Chinese teachers (Ch-T3, Ch-T4) used the linear quantity models with high frequency. It is also interesting to note the difference in multiple diagram use between the US and Chinese teachers. A total of nine US episodes (15.3%) contained multiple diagrams (pre-tape/number line: $n = 1$; pre-tape/tape: $n = 2$; pre-tape/tape/number line: $n = 1$; tape/number line: $n = 5$); yet, only one Chinese episode (2%) contained multiple representations (see Table 2, Figure 2). Figure 4 illustrates typical examples. In the example of US-T3-Example 2, this teacher first used a tape diagram to illustrate the problem context and then drew a number line diagram to aid computation. To illustrate the inverse relationship, a different representation called a "fact triangle" was also drawn. As such, multiple representations were involved in the discussion of this word problem. This was typical among the sampled US lessons, but rare in the Chinese

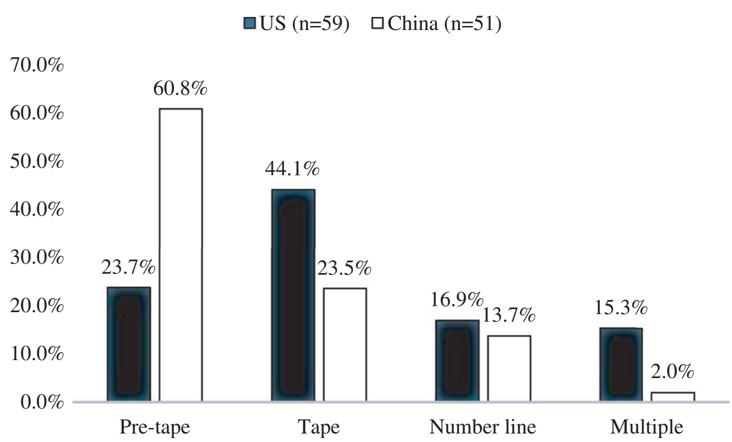


Figure 2. Episodes that involve types of linear quantity models in the US and Chinese lessons.

Table 2. Number of diagram episodes in each teacher's four lessons.

			Pre-tape	Tape	Number Line	Multiple	Total
US	Additive Inverse	T1	4	2	0	0	6
		T2	0	0	0	0	0
		T3	0	12	5	7	24
		T4	0	1	2	1	4
	Multiplicative Inverse	T5	7	0	2	0	9
		T6	1	7	0	1	9
		T7	1	2	1	0	4
		T8	1	2	0	0	3
		Total	**14**	**26**	**10**	**9**	**59**
China	Additive Inverse	T1	3	0	0	0	3
		T2	3	0	0	0	3
		T3	8	3	0	0	11
		T4	4	6	3	0	13
	Multiplicative Inverse	T5	1	1	0	0	2
		T6	4	0	0	0	4
		T7	3	2	3	0	8
		T8	5	0	1	1	7
		Total	**31**	**12**	**7**	**1**	**51**

For additive topics, grade level matches in both sides. That is, T1/T2 taught G1; T3/T4 taught G2. For multiplicative topics, grade level did not match. In US, T5/T6 taught G3 while T7/T8 taught G4; In China, T5/T6 taught G2 while T7/T8 taught G3.

lessons. Figure 4 illustrates the only Chinese example, involving two pre-tapes and a number line diagram. In this example, Ch-T8 used one screen to simultaneously present two example problems (top) and one practice problem (bottom) so students could compare and articulate the problem structure underlying these three tasks. Even though we coded this diagram episode as multiple representations, a closer inspection reveals that each task was indeed solved by just one single representation, which was essentially different from the US examples.

Purpose of using linear quantity models in US and Chinese classrooms

Overall, the Chinese teachers used linear quantity models only contextually (e.g. story problems) with a clear purpose of modeling the quantitative relationships. In contrast, the linear quantity models found in the US classrooms were used for both contextual and non-contextual situations (see Figure 3), with a main purpose of finding computational answers. Figure 4 presents typical examples for each type of linear quantity model. Elaboration follows.

Purpose for using pre-tapes across contexts

As indicated by Figure 4, US-T1-Example 1, the teacher asked students to use images of ten-sticks (pre-tapes) to solve non-contextual tasks involving missing numbers (e.g. __ + 6 = 10). To initiate discussion, she used a ten-stick formed by 10 connected cubes.

T: Alright, and I know that 10 is going to be the sum because that's how many we have altogether…. So I know that I already have 6, so I'm going to go "1, 2, 3, 4, 5, 6" (counts out 6 cubes and removes them) and now I can figure out what's left. How many do I have left?
Ss Four!
T: 1, 2, 3, 4.
T: Huh! I can put a 4 there, okay? (Teacher fills "4" in the equation)
T: What did I do? I found out what was left over. Hmm. And yet this says add, but I didn't put anything together. I want you to try one now.

In the above excerpt, US-T1 performed procedures of "taking away" (removing 6 cubes from 10) and "counting the leftovers" to obtain the answer of "4". Assuming that students could map

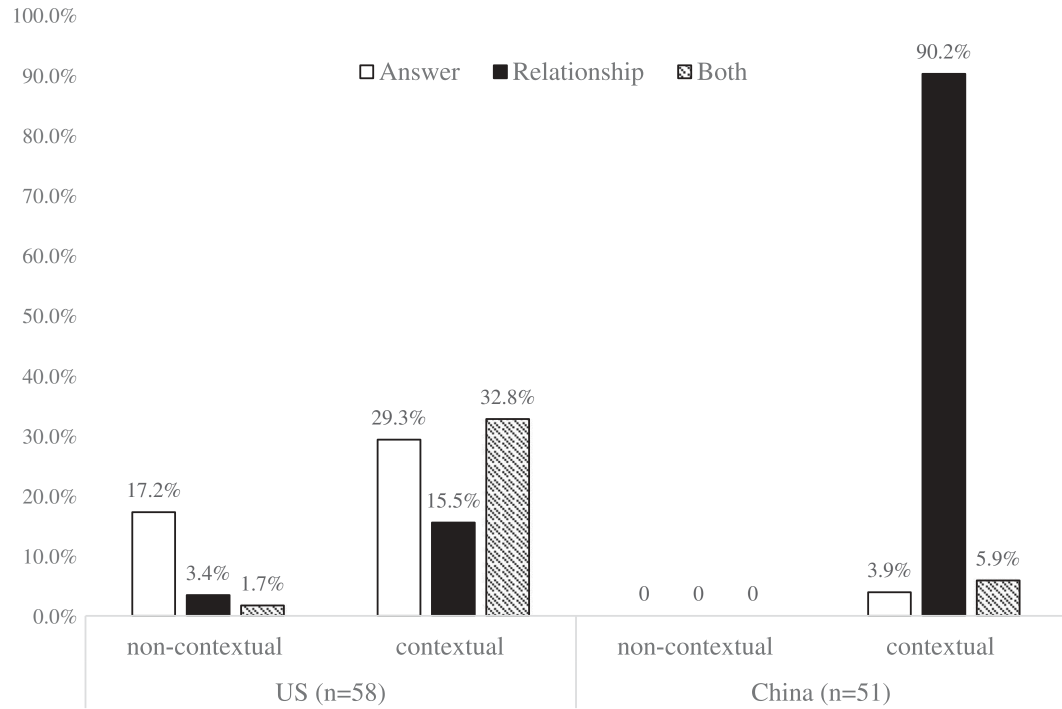


Figure 3. Different purposes of using linear quantity models across contexts.

manipulatives to diagrams to perform the same procedure, it is still unclear whether students could understand why this subtraction process (taking away) could be used to solve this problem. As the teacher reminded students, this task indicated "add" but she did not "put things together." Rather, she "found out what was left over." Although this was a great teaching moment that could have potentially activated students' curiosity and thus draw their attention to inverse relations, no further classroom discussion occurred. In her post-instructional interviews, US-T1 expressed that her use of representations was effective because her students used a variety of tools to arrive at their answers. While students' use of variety of tools is indeed encouraging, the nature and purpose of using these tools is arguably more important. Unlike US-T1's use of pre-tapes for answer seeking, Ch-T1 demonstrated a different purpose for using pre-tapes (see Figure 4). To teach a similar missing number problem, □ + □ = 8, 8 − □ = □, Ch-T1 presented a set of butterflies provided by the textbook and asked students how they could separate, in order, these butterflies based on what they had learned about number combinations. According to Ding (2016), the concept of number combinations (e.g. 3 and 5 are composed into 8, 8 is decomposed into 3 and 5) was uniquely presented by Chinese textbooks before addition and subtraction, which is expected to lay a foundation for students' learning of additive inverses. Ch-T1's request for number combinations elicited varied responses (e.g. separating 8 butterflies into1 and 7, 2 and 6, 3 and 5, and 4 and 4, and vice versa for each pair). Her requests for all combinations "in order" might have contributed to students' flexible and logical thinking. Later in her post-instructional interview, Ch-T1 indicated her focus was on the "relationships among number sentences." This explicitly illustrated that Ch-T1's purpose for using pre-tapes was different from US-T1, who focused mainly on obtaining computational answers.

The cultural difference in the purpose for using pre-tapes – answers versus relationships – was also apparent in the teaching of comparison problems. In Figure 4, US-T4 used a pre-tape (cube sticks on the smart board) for a contextual task comparing the loss of two children's teeth. Even

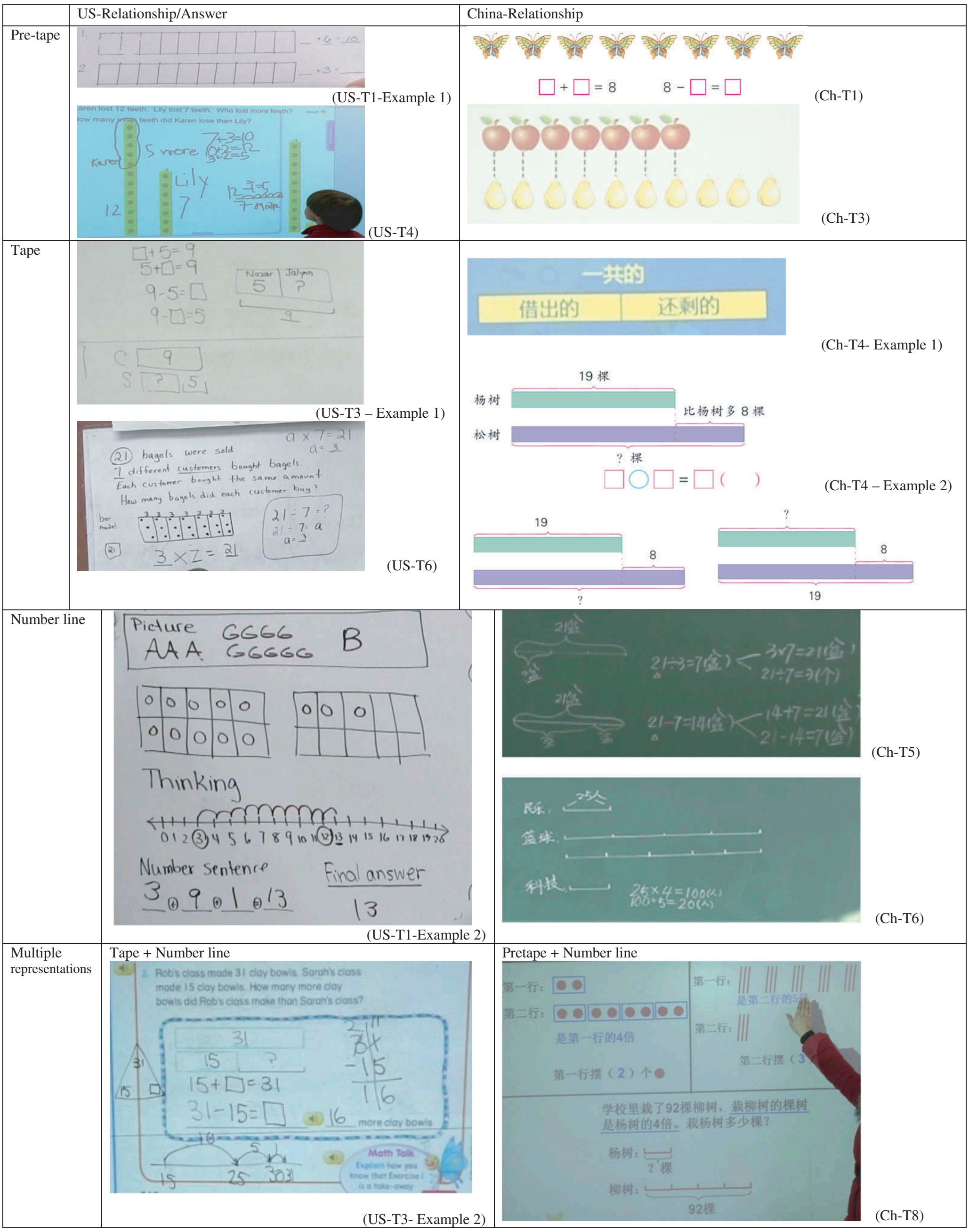


Figure 4. Tape diagrams used in contextual and non-textual contexts in US lessons.

though this teacher discussed the meaning of each cube stick, she only asked students to circle the part that showed "how many more" for the purpose of obtaining the answer of "5." In contrast, Ch-T3 stressed the concept of "the same as" through a one-to-one correspondence and asked students to verbalize the quantitative relationships: the large quantity (pear) contains the same as part (same as the small quantity apple) and the more than part. As reviewed in the literature, this relationship purpose is consistent with the research assertion on how to teach comparison problems (Greeno &

Riley, 1987). The above difference in representational purpose was confirmed by US-T4 and Ch-T3 interviews. For instance, US-T4 explained that her use of concrete materials was mainly to address the issue of diverse learners. Consequently, the illustration of cubes on the smart board was mainly for helping struggling students who could not solve a problem with numbers alone. Ch-T3 however, used concrete situations and images as a starting point for all students before transitioning to abstract representations during instruction. She explained how she used pre-tapes (pear and apple) to stress a "one-to-one correspondence" and to show students how the two parts (same as, more than) could be mapped to the later abstract number sentences. This indicates a structural use of the pre-tape, which may facilitate students' problem-situation model (English & Halford, 1995). Ch-T3 also pointed out that the process of "from concrete to abstract (从具体到抽象)" was to prevent students from using key words to solve a problem, a strategy that has been cautioned against in the literature.

Purpose for using tapes across contexts

Similar cultural differences were found with the use of tape diagrams. For US teachers, tape diagrams were used to show both relationships and to find answers. In Figure 4-tape, US-T3 started her lesson with a fact family that involved 4, 5, and 9. She then replaced "4" with a "□" in each number sentence and created both a comparison story problem (Carly and Sophia's pokemon cards) and a part-whole problem (Nazar and Jalyn's pretzels) for this fact family. For both story problems, US-T3 drew a tape diagram (called bar model in this lesson) to illustrate the problem situation, which was suggested by the *Go Math* textbook. She then asked the class to pick a number sentence from the fact family to solve the story problems. US-T3's purpose for using bar models became evident when she announced in class:

> And by filling in what we know and what we don't know, it's going to help you know whether we are adding or subtracting and maybe which part of a fact family, which equation from a fact family will be the easiest to use. (US-T3)

That is, the tape diagram was used for the purpose of helping students organize individual quantities. By "filling in what we know and what we don't know," US-T3 thought that students would then be able to come up with the correct equation needed to solve the problem. Although the teacher's phrasing of "whether we are adding or subtracting" is implicitly suggestive of the diagram's ability to represent the problem's underlying quantitative relationships, this teacher's focus on individual quantities rather than interactions between these quantities may have not contributed to students' development of problem-situation models (English & Halford, 1995). In fact, students in this episode actually did suggest both solutions ($15 + \square = 31$, $31–15 = \square$) to the inverse relationship that could have been illustrated by the tape diagram. Instead, the teacher drew an extra fact triangle (a triangle with 15, 31, and □ on its three vertices) to show this inverse relationship. Since the fact triangle was not linked to the tape diagrams to meaningfully model the quantitative structures of the problem situation, it may have only served to facilitate the obtaining of an answer. In other words, the quantitative relationships embedded in the tape diagram (e.g. small/large/difference, part/part/whole) were never made explicit for students, and thus the purpose of using tape diagrams to facilitate problem situation models was not evident. During her interview, US-T3 explained that her second graders did not use the *Go Math* textbook in Grade 1. Instead, they used *Everyday Mathematics* and thus were familiar with the "fact triangle." As such, she used the fact triangle instead of the bar model, even though she expressed a preference for the bar model because of its clarity in illustrating relationships (e.g. part-whole). Nevertheless, by only using the tape diagram to organize information, US T3's relational focus appeared to be at most implicit. This was typical among US teachers in this study. For instance, after drawing a tape diagram to illustrate and generate a number sentence ($21 \div 7 =$) for equal sharing of 21 bagels with 7 customers (see Figure 4), US-T6 guided the class with distributing 21 dots evenly into 7 sections. This resulted in the correct answer of 3 but, given that she used the tape diagram to show the sharing process, we coded the purpose of using this tape diagram as "both answer and relationship." It is worth noting that the depth of this "relationship" in US T6's modeling process was limited due to the lack of explicit discussion about the interactions among these quantities.

In contrast with the US teachers' use of tape diagrams to mainly find answers and occasionally to illustrate relationships, Chinese teachers used tape diagrams solely to illustrate quantitative relationships. In fact, the depth of discussion surrounding tape diagrams in the Chinese classrooms always went beyond organizing known and unknown information to include interactions between individual quantities. As seen in Figure 4, in Ch-T4's first example, a tape diagram was used to illustrate the quantitative relationship so as to justify why one can use addition to check for subtraction. Uniquely, the teacher labeled the two parts of the tape diagram with the words "lent-out" and "leftover" and the whole tape with "total." This clearly indicates the part-whole relationship "Total = lent-out + leftover" (see Figure 4). In her second example, she discussed a problem comparing poplar trees (19) and pine trees (8 more than poplar trees). After the discussion, she made two variations, which indicated her focus on generalizable quantitative relationships. First, she removed the tree context (See Figure 4) and asked, "If I remove poplar and pine trees from this diagram, what other problems may this diagram help solve? Can you create a story problem?" This teaching move likely helped students make meaning of the tape diagram, as indicated by the fact that two students posed different story problems based on the same tape diagram. Ch-T4 then explicitly generalized the quantitative relationship as "Small quantity (the same as part) + difference = large quantity." "Now, I want to change this diagram again," Ch-T4 continued, as she changed the known (small quantity) into the unknown (marked with "?") to make another variation (see Figure 4). Discussions on this new tape diagram led to further revelations of the inverse quantitative relationship "large quantity – difference = small quantity." In fact, Ch-T4 prompted further comparisons between these two diagrams generated from variation (How are they the same and different? When do we use addition and when do we use subtraction?), which made the inverse quantitative relations even more explicit for students. During her post-instructional interview, Ch-T4 stressed that her purpose in using tape diagrams was to illustrate the essence of concepts by using representations to help students "build mathematical models" (数学建模). In brief, all Chinese tape diagram episodes did not involve using the model for computational purposes, but rather concept-related discussions of more generalized quantitative relationships.

Purpose of using number lines across contexts

The difference in representational purpose – as seen with the other diagram types – was most clearly apparent with number lines. All US teachers (except for one) used number lines to find computational answers while Chinese teachers used number line diagrams only to illustrate quantitative relationships for problem solving (see Figure 4, number line), but never for computation. The most common use of number lines in the US classroom was to find answers even within contextual tasks. Solving this type of question was often illustrated by jumping the number of steps indicated by the number to be added. For instance, in a story problem involving 3 adults, 9 girls and 1 boy, the teacher used multiple representations (picture, ten frame, number line) as means to find the answer for 9 + 3 + 1. Students counted the number of jumps on the number line in the same way as counting the letters in the picture and the cells in the ten-frame (see Figure 4, US-T1-Example 2). Likewise, in the examples of "multiple representations" (see Figure 4), US-T3 first drew tape diagrams to represent a story problem that compared 31 and 15 clay bowls. To find the answer of 31–15, she then drew a number line and taught students to jump from 15 to 25 (10 steps), then to 30 (5 steps), and finally to 31 (1 step), resulting in the answer of 16. US-T3's approach to number lines was similar to US-T4 who explained that her purpose of using a number line diagram to teach word problems was to help students make sense of the "jumping strategy" and to lay a foundation for their follow-up learning of using a hundreds chart to do computations, where the same strategy would be used. Such a use of number line diagrams may explain why 46.5% of the US diagram episodes (17.2% for non-contextual, 29.3% for contextual, see Figure 3) were coded as finding answers. There was only one case (US-T7) where a number line was used to illustrate a multiplicative comparison relationship, 3 times as tall.

Unlike the majority of US cases, number lines in Chinese examples were always used to represent quantitative relationships, including instruction of complex situations. For instance, Ch-T5 used number line diagrams to compare two seemingly related but structurally different story problems. One problem stated that “21 basins of flowers were shared evenly with 3 classes. How many basins of flowers did each class receive?” The second problem stated, “There were 21 basins of flowers. Seven of them were given away to a class. How many basins of flowers were left?” Ch-T5 guided the class to draw two tape diagrams for each problem (see Figure 4), which led to corresponding solutions, 21 ÷ 3 and 21–7. The teacher also asked students to compare both diagrams to explain why both problems were about dividing flowers even though one problem used division and the other used subtraction. During her interview, Ch-T5 explained that her teaching move was to address a common mistake and to deepen students’ understanding of the concepts:

> … I have noted that students often list 21 ÷ 7 (not 21 – 7) once they see the second problem because they can use the rhythm “Three Seven Twenty-one”[1] (三七二十一) to solve this problem. Today, my purpose is to help them differentiate between these two: Even though we both know the total, we need to know when we should use division and when to use subtraction. In the first problem, the total is divided evenly into many equal groups, thus, we use division. In the second problem, the total is divided into two parts, it is about the part and whole relationships, we should use subtraction.

Note that once students reached this understanding, Ch-T5 further requested students to change each problem to its corresponding inverse word problem. As seen from Figure 4, a set of numerical solutions to students’ generated word problems were recorded on the board, which indicated both multiplicative and additive inverses (21 ÷ 3 = 7, 3 × 7 = 21, 21 ÷ 7 = 3; 21–7 = 14, 14 + 7 = 21, 21–14 = 7). Likewise, Ch-T6 (Figure 4) used number lines to illustrate a two-step word problem where the first step used multiplication and the second step used division. Ch-T6 clarified in her interview that her purpose behind using these number lines was to stress the inverse relationships. She also stated that the use of number lines is a process of abstraction which aligned with the method of “number-shape combination” (数形结合), a term mentioned by most of the Chinese teachers in this project. According to Ding (2010), number-shape combination is a mathematical thinking method that is useful for effective problem solving. This method stresses the integration of abstract numbers and concrete shapes/figures and thus links concrete to abstract thinking. In the same vein, Ch-T8 (Figure 4, multiple representations) compared a number line diagram (bottom) to two pre-tape diagrams (top), the purpose of which was stated to help students understand the concept of multiplicative comparisons.

The pattern across the different types of linear quantity models was clear: diagrams in the sampled US lessons were for either organizing given information or for finding computational answers; yet similar diagrams in sampled Chinese lessons were mainly used for understanding the concept of quantitative relationships. In general, we found that the Chinese lessons focused mainly on the modeling process with only a brief discussion of computations. In contrast, the US lessons often quickly shifted from a discussion of diagrams to focus on computational strategies.

Student engagement with linear quantity models in US and chinese classrooms

Overall findings

Table 3 indicates who was involved with the presentation and discussion of the linear quantity models. It further indicates how teachers involved students in the process of understanding the linear quantity modes.

As indicated by Table 3, while all Chinese episodes (100%) involved teacher-student conversation (coded as “class”), only 34.5% of the US episodes contain classroom discussion that focused on the diagrams. In fact, more than half (55.2%) of the US diagram episodes included only teacher talks. In these episodes, the US teacher may have posed questions to students, but those questions were often not related to the diagrams (see Table 1, US-T5 for an example). This explains why only 22.4% of the

Table 3. Student involvement in diagram uses in the US and Chinese classrooms.

		US (n = 58)	China (n = 51)
Who	Teacher	55.2%	0
	Student	10.3%	0
	Class	34.5%	100%
How[1]	Co-construction	19.0%	21.6%
	Direct telling	69.0%	0
	Deep questioning	22.4%	98.0%
	Comparison	13.8%	58.8%
	Variation	0	17.6%
	Student gesturing	1.7%	27.5%
	Other concrete aids	25.9%	0
	Other	10.3%	7.8%

[1]For the "how" dimension, each episode may be coded for multiple categories. Therefore, the percentages for "how" do not add up to 100%.

US episodes involved deep questioning. In addition, 10.3% of the US episodes were coded as student only. In such cases, students were asked to work on the linear quantity models as independent practice without any teacher feedback provided to the class.

When focusing on how teachers' approaches engaged students in their diagram use, we noted that a similar proportion of US and Chinese teachers (19.0% vs. 21.6%) asked students to co-construct diagrams (see Table 3). However, an examination of these actual diagram episodes indicated that the general co-construction styles were very different. While the US teachers tended to ask questions such as "how should I draw the tallies" or recorded students' suggestions on the board (e.g. teachers drew out the diagrams), the Chinese teachers often asked students to draw part of the diagrams themselves. In fact, even though co-construction occurred in both US and Chinese classrooms, the overall student involvement in US classrooms appeared to be much more passive. For instance, 69% of the US diagram episodes employed teachers' direct telling, which is drastically different from the Chinese case (0%). Instead, Chinese teachers almost always (98%) asked deep questions, which was rarely seen in the US episodes (22.4%). An inspection of the deep questions asked during the US and Chinese diagram episodes revealed four general types. Questions focused on (a) the meaning of a particular quantity on a diagram, (b) relationships among individual quantities embedded in a diagram, (c) mapping from a diagram to numerical equations, and (d) comparison between the diagram and either another linear quantity diagram or another representation. Example questions included "In this number line, which quantity was viewed as one unit?" (Ch-T6), "What do these dotted lines represent?" (US-T6), "Looking at these drawings, which method matches with which method? (Ch-T4), "What equation would match which picture?" (US-T5). Requests for comparisons were found to be particularly common in Chinese episodes, in which a large proportion of the instruction involved comparison (58.8%). In contrast, only 13.8% of the US diagram episodes involved comparison. During the teacher interviews, Chinese teachers shared that they asked questions that were purposefully designed to promote students' deep thinking. Several Chinese teachers (Ch-T1, Ch-T5, Ch-T6, Ch-T7) also discussed their comparison questions. For example, Ch-T1 explained why she asked students to compare problems with problems that had either similar or different structures. The former was to help students build mathematical models (数学建模) and the latter was to prevent students from being unable to think out of the box (防止思维定势). Chinese teachers' unique use of variations on problems (17.6%) may have increased their frequency of using comparisons. As previously reported, Ch-T4 (Figure 4) made two variations on a tape diagram – removing the context and changing the location of the question mark – which enabled a follow-up comparison of these two newly generated diagrams. In addition to the above instructional aids, some Chinese teachers asked students to come to the board to articulate and clarify their thinking by gesturing on the diagrams (27.5%). This strategy was used only once (1.7%) by only one US teacher (US-T4) who asked a student to circle "how many more" in an image of a cube train. US teachers also incorporated unique approaches, in that 25.9% of the US episodes involved other concrete aids such as cubes, fact triangles or even tape diagrams created by paper-folding. During the teacher interviews, the majority of the US teachers expressed that because of the diverse learners in their classrooms, they viewed multiple representations as an indicator of learning and teaching success. Despite using rich concrete representations, the mapping between concrete aids and linear quantity models was often missing. The effect of using multiple representations on developing student understanding thus remains unknown.

Typical episodes

To further illustrate the differences of teachers' approaches to engaging students with linear quantity models, we present a typical Chinese episode from Ch-T4, and contrast it with a typical US episode by US-T3. Both episodes involved additive comparison problems. During Ch-T4's instruction, a pair of example tasks on comparisons was presented side by side on the board. The first problem stated, "Teacher Chen's favorite number is 45. This student's favorite number is 3 bigger than 45. What is this student's favorite number?" The second problem replaced "3 bigger" with "35 smaller." To help students understand these word problems, Ch-T4 presented the first tape that represented the teacher's favorite number 45. She then asked students to watch the movement of her computer's cursor as she pretended to draw the second tape that represented "3 bigger." Students were asked to inform her of when and why she should end the second tape. Ch-T4's modeling of the second tape along with deep questions such as "why not stop as it already arrived at 45," provided instructional aids for her second graders who had just started learning to draw tape diagrams themselves. After this discussion, she asked the class to draw out the second tape on their individual worksheets.

Using students' generated drawings, Ch-T4 selected typical ones for class discussion. She first presented one student drawing and discussed why the second tape was longer or shorter than the first tape. This was a seemingly trivial question; however, it may have oriented students' attention to the relationship between the two compared quantities. Next, Ch-T4 presented another student's work that appeared to be different (see Figure 5):

T: I found another student draw this way. There is an extra dot on the first picture (picture on the left). Do you know what he wants to convey? And, here is an extra point too (picture on the right). Can you discuss with your desk mate?

T: Let me ask someone to explain.

S1: (go to the board; T passed S1 the teaching stick). From here (the start) to here (the dot), it is 45, which is the same amount as teacher Chen's. From here (the dot) to here (the end), it shows 3 more than teacher Chen's (pointing to the second picture). This part (second tape) represents the student's favorite number. This is the part (the empty) that shows teacher Chen has more.

T: What does the dot mean?

S: This dot means to line up with teacher Chen's tape or to show the part that is the same as Teacher Chen's. Here (the empty), it refers to the part that Teacher Chen has more.

T: Isn't her explanation great? Applause! Let's have another student to explain.

S2: (go to the board and get the teaching sticks from T) The function of this dot is to compare with the tape of teacher Chen, that is, from here (the start) to here (the dot). And the extra part shows 3 more (pointing to the second picture). Here is a dot. It means that until here, it is 45. However, this is an empty part. And, the marked is 10.

T: (to the class) Do you all understand?

Ss: Yes.

T: However, I have a place that I don't understand. What does this empty part represent?

S: 35.

T: Okay, it looks like you all understand. So, here (point to the picture on the left), we should first draw the part that is the same as teacher Chen, and then add the 3 more. Here (point to the picture on the right), we should first imagine it the same as teacher Chen's; yet, it is 35 less. Now, can you write a math expression to solve it?

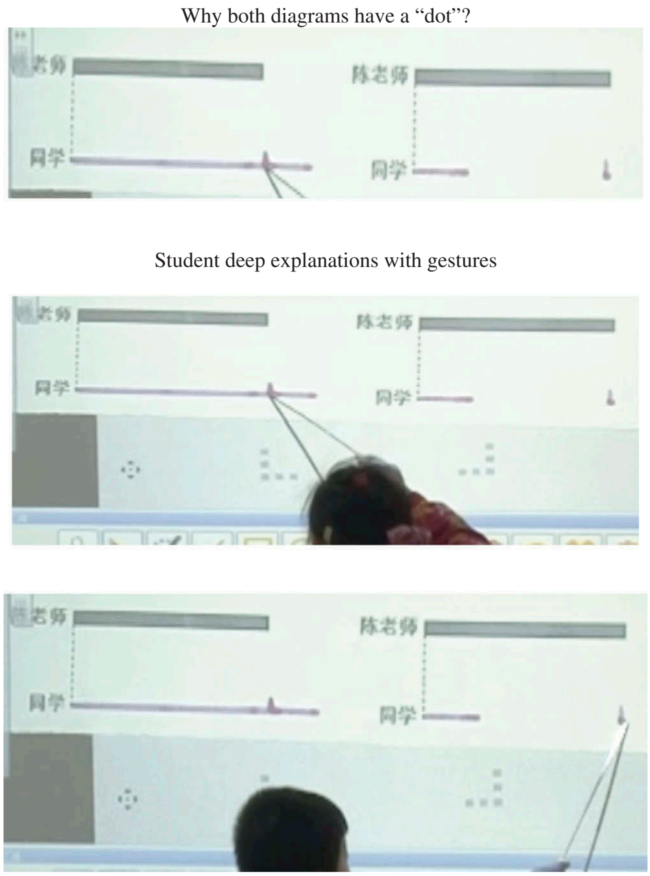


Figure 5. Engaging students with tape diagrams in Ch-T4’s lesson.

In the above excerpt, Ch-T4’s question – “What does the dot mean?” – seemed to be trivial, yet was actually a deep question. This question elicited students’ deep explanations of the precise meaning of each part of the tape diagram separated by the dot and the relationships between the different parts. This likely enabled students to better understand that the larger quantity contained “the same as part” and “the difference” (in the left picture). In other words, adding the difference to the smaller quantity would result in the larger quantity (in the right picture). In addition to deep questions, the above episode revealed that Ch-T4 asked students to gesture on the tape diagrams written on the board, which may have not only enabled students to demonstrate their precise understanding but also may have enabled the whole class to reach a shared understanding. Furthermore, arranging this pair of tape diagrams side-by-side may have enabled students to easily compare this pair of problems so that they could realize the inverse relationship between addition and subtraction.

The above episode from Ch-T4 illustrates how the sampled Chinese teachers spent a significant amount of time engaging students in discussion of the embedded concepts and relationships in linear quantity models. This engagement included the use of varied aids such as co-construction, deep questioning, comparisons, variations, and student gestures. This is in contrast to many US episodes where discussion time was not devoted to the linear quantity diagrams. For instance, US-T3 taught the following comparative word problem: “Rob’s class made 31 clay bowls. Sarah’s class made 15 clay bowls. How many more clay bowls did Rob’s class make than Sarah’s class?” (see Figure 4, US-T3 example 2). US-T3 first projected onto the board the textbook page that contained the tape diagram (see Figure 4). After going over the known and unknown information in the story situation, she asked students to fill in the corresponding numbers on the tape:

T: So where do you think the 31 goes in this diagram?
S21: The big box.
T: In the big box, good. And where do you think 15 goes?
S21: The small box.
T: Good. Because this line over here is showing us where this is different than this. So that's what we need to find. So what equation do you think you need to write?
S21: 31-15 = box

As previously analyzed, this use of tape diagram was mainly for organizing information. During this process, even though US-T3 asked a few questions about the tape, the questions focused on individual quantities and only facilitated a mapping from the given text to the tape diagram. Right after this, she moved to requesting a numerical equation, which elicited 31–15 = □. However, the process of mapping from the tape diagram to the numerical solution was not discussed. In other words, why "subtraction" was an appropriate operation was not connected to the tape diagram. This was in sharp contrast with the Ch-T4's discussion of the "dot" as detailed above and which focused on the embedded quantitative relationships.

Encouragingly, after students worked on solving 31–15 = □ for a few minutes, US-T3 refocused the class's attention by asking the deep question: "Who can explain why 31 and 15 had to go in the boxes?" Even though this question still focused on individual quantities, this "why" question had a potential to facilitate meaning making and thus was considered as deep. One student explained that this diagram was a part-whole model, "because 31 has to go in the top box because 31 is the total and 15 has to go in the bottom left box because it is a part." Faced with this mistake, US-T3 quickly stopped this student and provided her own lengthy explanation:

T: Okay, let's stop right there because this problem is a little different than the one we did on the other page, because these are not parts and total. These are both what we call quantity. Everybody say that word, quantity. (Ss: quantity). Quantities mean amounts. We are comparing two different classes here. When you are comparing two things, each one of those things has a quantity. We are comparing Rob's class's bowls to Sarah's class's bowls. That's why those numbers need to be in those boxes (point to the diagram). We are comparing those two amounts. This little bracket over here, the line with the little edges on it, that is showing us, that it is the difference between those two. This is showing us that Sarah's class has this much less. If you line up all the bowls from Rob's class it would be that many (pointing to the top tape). Sarah's class, they only have this many (pointing to the bottom tape), they would need this many more to make it the same (pointing to the bracket). So this would be our difference (writes a "?" on the bracket). Okay, who can tell me what equation they used to solve?

The above excerpt illustrates how a potential opportunity for engaging students in deep thinking about tape diagrams was missed. When the student made a mistake, US-T3 quickly reclaimed the responsibility of explanation. This was different from Ch-T4 who engaged students in comparison and explanation of the tape diagrams along with gestures. Indeed, even though US-T3 demonstrated sound understanding of different types of word problems (e.g. part-whole versus comparison), her explanation of the tape diagram still focused on individual quantities (the big, the small, difference) rather than the interactions among these quantitates.

Discussion

This study is among the very first to examine how linear quantity models are used in US and Chinese elementary mathematics classrooms. Given that diagrams are powerful learning tools (Lakin & Simon, 1987) and that linear quantity models are expected to be used by the Common Core for solving challenging problems such as ratio and proportions (CCSSI, 2010), an examination of how these models are currently being used in classrooms bears great importance. Our findings indicate differences in the type and purpose of using linear quantity models as well as how teachers engage students with these models between the sampled US and Chinese classrooms. Below, we discuss our findings relative to our three research questions.

Types of linear quantity models: learning opportunities and progression

We found that overall US lessons contained more linear quantity models than Chinese lessons. Given that linear quantity models are relatively new for US classrooms, this finding indicates the presence of encouraging learning opportunities. We also found that US teachers concurrently use multiple linear quantity models and other concrete aids to supplement linear diagrams. This was different from Chinese teachers' focus on one model during the teaching of one mathematics problem. The existing literature offers different positions on the use of single and multiple representations (e.g. National Council of Teachers of Mathematics [NCTM], 2000; Murata, 2008). For instance, NCTM (2000) calls for using multiple representations to facilitate student learning and to address diverse student needs; yet Murata (2008) reported that a systematic use of a single representation (the tape diagram) potentially provided Japanese students with a familiar tool to aid their problem solving. Our teacher interviews indicate that US teachers' rationales for using multiple representations aligned more with the NCTM position, while the Chinese teachers' views aligned more with Murata. We think both positions have their advantages; yet, whether these advantages are achieved may depend on how these diagrams (regardless of single or multiple) are used. In particular, when non-transparent tape or number line diagrams are used, some students may need additional instructional support in order to gain full comprehension. This might include the use of other concrete aids such as cubes or paper folding, both of which were tools used by US teachers in our study for assisting students with understanding. In this sense, US teachers' use of multiple representations, offers insight for how Chinese teachers may promote understanding for all students. In fact, instruction involving various representations has repetitively been shown to increase comprehension (Ainsworth, Bibby, & Wood, 2002; Goldstone & Sakamoto, 2003; Richland et al., 2007). Unfortunately, the US teachers often introduced multiple representations sequentially without making connections between various representations, which may actually decrease the potential for supporting student learning. Future studies may explore under what condition each approach would yield ideal learning opportunities.

Chinese teachers in this study used more pre-tapes than tape diagrams. As previously mentioned, this may be due to the fact that the participating Chinese teachers are from lower elementary grades (G1-3). According to Murata (2008), pre-tapes in Japanese textbooks were introduced at the beginning of schooling to develop students' readiness with tape diagrams. From this view, Chinese teachers' frequent use of pre-tapes in lower grades provides students with a reasonable introduction to linear quantity models. This progression from pre-tape to tape diagrams in Chinese early grades may contribute to students' readiness of tape diagrams so that Chinese sixth graders can be expected to solve complex ratio and percent problems based on the number line models (see Figure 1, Ding & Li, 2014). In contrast, US teachers used tape diagrams more frequently than pre-tape diagrams. In some classrooms, students were directly exposed to tape diagrams without seeing them in prior grades. Given that tape diagrams are relatively new for most common-core based US curricula, it is therefore unknown how well the learning opportunities with tape diagrams was actually grasped. This observed difference in using pre-tapes may also partially explain why some US classroom teachers struggle with making meaning of tape diagrams (Vanduzer, 2016). When students lack the readiness to work with tape diagrams, a linear quantity model becomes a roadblock rather than a cognitive aid for learning. Of course, as seen from Ch-T4's case, when students do have the readiness for reasoning, teachers should exercise their instructional flexibility by introducing tape diagrams or number line diagrams (rather than pre-tapes) to challenge student learning.

Purpose of linear quantity models: computation or quantitative reasoning

Regardless of the type of linear quantity model, a major cross-cultural difference occurred with the representational purposes behind all types of diagrams. According to English and Halford's (1995) problem solving model, it is important to shift students' problem-text models to problem-situation models, which will lead to a mathematical model that is a numerical solution of a problem. In this study, Chinese teachers consistently used these diagrams to model quantitative relationships in contextual tasks. This most likely facilitated students' creation of problem-situation models. In contrast, US teachers often used these diagrams to find or confirm answers in addition to representing the problem contexts, which is unlikely to help students form problem-situation models. With pre-tapes that contain discrete objects, it is understandable that one may count the objects to obtain the answers. However, for those non-countable tape diagrams, it is surprising that some US teachers still used them to find answers. For instance, some teachers distributed individual dots to each section of a tape diagram and then counted the resulting dots in each section. In such a case, the tape diagram only served as a placeholder for manipulatives, with the goal to obtain answers rather than to understand quantitative relationships. This is consistent with Ma's (1999) finding about US teachers' procedural use of manipulatives and with the general US teaching style described by Stigler and Hiebert (1999).

It should be acknowledged that some US teachers first used tape diagrams to represent story situations (see Figure 4 for T3, T4, T5, T6, T7) and then asked students to generate number sentences to solve those story problems. This finding is encouraging because this representational sequence (story situation, tape diagram, numerical solution) is consistent with the recent research assertion of concreteness fading, which was found to be effective in student learning (Goldstone & Son, 2005). However, in these cases, tape diagrams were mainly used to organize the given information of a story problem. In other words, the textual information was most often directly translated to corresponding sections of a tape diagram, without further discussion. While this is a necessary step of problem solving, a direct translation of textual information to a linear quantity model with limited discussion may not promote a shifting from a "model of" informal mathematical activities to a "model for" formal mathematical structures (Gravemeijer, 1999) to form a problem-situation model (English & Halford, 1995). This is because such a direct translation process deals with only individual quantities; yet, deep initial learning demands going beyond individual quantitates to focus on quantitative relationships (Chi & van Lehn, 2012) and inference-making resulting in problem situation models. In fact, research (Murata & Kattubadi, 2012) found that when students lacked well-grounded understanding of situation models, it took them longer to generate and understand numerical solutions.

With regard to number lines, US teachers' approaches clearly show answer-seeking purposes with both contextual and non-contextual tasks. Except for one teacher (US-T7) who drew number lines to represent quantitative relationships, the other teachers treated number lines as a strategy to find numerical answers by counting each step on the number line. This may be related to cross-cultural curriculum differences – for instance, the US textbooks teach the above counting strategies with number lines (CCSSI, 2010), yet Chinese textbooks do not. While developing computational skills through number lines is clearly important, number line diagrams have the same power as tape diagrams for illustrating quantitative relationships during problem solving. In our study, number lines were not used for this purpose in the sampled US lessons. Rather, teacher interviews revealed that many US teachers valued concrete representations including diagrams as a tool to address diverse learners' computational struggles – for instance, some teachers expressed satisfaction in seeing students draw tallies to obtain the answer for a word problem. Two issues arise from US teachers' use of linear quantity models for solely finding answers. First, it decreases the pedagogical power of these diagrams to support students' quantitative reasoning and problem solving skills. Secondly, it decreases students' chances of mastering basic facts, as students may grow to depend on these diagrams (e.g. counting the pre-tapes) to obtain answers. These findings call for paying closer attention to the purpose for which linear quantity models are used.

Student involvement in linear quantity models: unique insights from China

The Chinese teachers in this study clearly engaged students in the process of constructing and discussing linear quantity models – an aspect that was notably lacking in the US instruction. Chinese teachers employed a variety of strategies to mentally and physically engage students in the process of learning – such as co-constructing, deep questioning, comparison, variation, and student gesturing on the diagrams – all of which are well supported by the cognitive literature and shed light for the mathematics education field. In fact, co-construction in Chinese classrooms often differed from US classrooms because Chinese teachers tended to ask students to draw part of the diagrams as opposed to simply requesting oral responses from students. The process of actually drawing part of the diagram may foster students' thinking about the relationships embedded in the tape diagram. Such a co-construction process may also shift students learning from passive receiving to active constructing, because students become a co-owner of the diagram as they take more responsibility for their learning (Fukawa-Connelly, 2012). In addition to co-construction, Chinese teachers uniformly asked deep questions about tape diagrams. Even seemingly-trivial questions often proved critical to orienting students' attention to the underlying quantitative relationships (Ma, 1999). Other questions prompted student comparisons, which have been found to be a powerful tool to extract underlying relationships and general problem solving methods (CCSSI, 2010; Rittle-Johnson & Star, 2009). Chinese teachers' comparative questioning specifically drew attention to the analogical mapping between the problem text, the linear diagram, and the corresponding equations. They also frequently asked students to compare selected student drawings so as to highlight the key concepts and relationships. These analogical reasoning processes are critical for problem solving (English & Halford, 1995). Moreover, during the process of student explanation, Chinese teachers often uniquely requested students to gesture on the diagrams to explicitly articulate their thinking. Since gestures embody cognition (Alibali & Nathan, 2012), Chinese students' articulation of thinking along with these gestures may in turn promote deeper understanding of key concepts and quantitative relationships. Additionally, students who gestured via pointing on the diagrams, likely provided teachers timely assessment of students' understanding of the diagrams and embedded relationships. Finally, some Chinese teachers employed variation by altering part of a tape diagram to change it to other relevant diagrams. Variation is a cultural-based practice unique to Chinese classrooms (Sun, 2011), which generates rich opportunities for comparisons. When the diagrams were altered, the original and derived tape diagrams may share either structural similarities or structural differences. As reported, Chinese teachers in their post-instructional interviews clearly stressed importance of comparing such diagrams to promote students' structural thinking and to build students' mathematical models (数学建模). In summary, the varied instructional aids used by Chinese teachers to discuss linear quantity models seemed to work together to achieve the same purpose of facilitating students' deep understanding of the key concepts and relationships rather than to obtain computation answers for specific problems.

Limitation, implications and future direction

Findings about differences in teachers' classroom use of linear quantity models seem to be consistent with prior report on teachers' beliefs about representation use (e.g. Cai, 2005). For instance, many Chinese teachers in their post-instructional interviews stressed the method of "from concrete to abstract" while US teachers favored using concrete models to address learners' diverse needs. Findings from teacher interviews also reflected textbook influences on teachers' diagram uses (e.g. Ding, 2016; Ding & Li, 2014). As acknowledged earlier, textbook presentations (e.g. whether or not they suggested the used of linear quantity models) appeared to have an influence on teachers' classroom influence; yet, teachers did have their own instructional choices when enacting the lessons. For instance, teachers may have shift instruction away from tape diagrams to other models that may be more familiar to students. These findings have practical implications because they suggest a need for US teachers to re-evaluate their purpose for using linear quantity models and ways to engage students in making meaning of these models. Meanwhile, these findings also suggest a need to better support teachers who use these types of models. This support may be delivered by teacher educators and curriculum designers through professional development, teacher education programs, and textbook revisions.

Despite these findings and potential implications, we acknowledge that our teacher interviews did not focus on linear quantity models, and thus we have limited understanding of teachers' instructional decisions, which suggests some directions for future research. Even though we noticed possible influence of teacher beliefs on teachers' diagram use, it is unclear what role teacher beliefs actually play in their diagram use. Many US teachers in this study embrace the goal of conceptual learning; yet, they did not engage students in the process of meaning-making and their use of linear models did not support the achievement of this goal. Future studies may explore the gap between teachers' perceptions and their enacted beliefs in using linear quantity models. Prior studies (e.g. Ma, 1999) also indicated that teachers' knowledge plays an important role in their representation use – this suggests the differences seen may also be in part due to the teachers' own understanding of these models. Likewise, we noticed a possible textbook influence on teachers' diagram use. However, due to the scope of this study, we did not closely investigate how textbooks may play a role in teachers' instruction and understanding with linear quantity models. Future studies may continue this line of research.

We also acknowledge that our study did not directly link instructional approaches to student learning. As such, another research direction is to explore the feasibility and transferability of our findings in a cross-cultural settings in terms of student learning: How do US teachers view Chinese instructional approaches to tape diagrams (co-constructing, deep questioning, comparison, variation, student gesturing) and are any of these methods effective in supporting US students' learning? Likewise, how do Chinese teachers view the US strategies of using other concrete aids (e.g. cubes, paper-folding) along with tape diagrams? Is this strategy effective in supporting student learning with diverse needs and what might be cultural barriers when employing this approach? Continuing research on linear quantity models, and more broadly the use of representations, may shed light on better ways to help students further develop mathematical thinking and learning skills.

Note

1. To solve division problems, Chinese students are taught to use multiplication rhythm (乘法口诀)。This is a common mathematical practice generation by generation.

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