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Liquidity Adjusted Value-at-Risk and Its Applications

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1. Introduction

The recent financial crises have been liquidity crises. Large hedge fund companies held positions that were too large to be liquidated without causing significant price impact. The breakdown of LTCM in 1998 and Amaranth Advisors in 2006 are a few examples. In the sub-prime crisis of 2008, a lot of banks and hedge funds also suffered from liquidity shortage and had to liquidate their assets, which deteriorated the crisis and caused huge losses. These are the signs that people underestimate the liquidity risk.

Nowadays, hedge funds still use Value-at-Risk(VaR) to measure the market risk. However, this measure can cause problems because when the volume of the position is large enough to cause price effect on the spread, the trading price is not at the mid-price. This has as a result that the real price will depend on both the value of the spread and the price effect of the trading volume. And thus, the market liquidity plays an important role. To sum up, we could see that the normal VaR concept lacks a rigorous treatment of liquidity risk.

Regarding the liquidity risk, the fund company examines the fund position in each holding in terms of numbers of shares. Then the holding is compared to the average trading volume of the latest 20 days, to see how much of the position that can be liquidated taking into account that no more than 10 percent of the average trading volume is used in order not to affect the prices too much. But in this way we neither quantify the liquidity risk in a monetary value nor express it as a percentage, which makes it less convenient to use when comparing between different funds and controlling the risk.

So this paper is devoted to incorporation of liquidity risk into VaR model, which could better reflect poor liquidity in the VaR framework. In order to incorporate the liquidity risk with the limited data, this paper used a concept of *LIX,* which is introduced by Oleh Danyliv, Bruce Bland, Daniel Nicholass(2014). It is a new measure of liquidity. With this measure, we could measure different stocks liquidity and predict the liquidity in the future. When we needed to quantify the liquidity risk, we used the concept of Cost of Liquidity (COL), which is half of the spread.

As for VaR, we used at first the variance-covariance method in calculation. Then in order to improve the accuracy, we used the extreme value theory (EVT) with a new quantile estimator, which included all the information in the tail.

Finally, we added the VaR and Cost of Liquidity in order to get the Liquidity adjusted VaR and compared the results from a large cap fund and a small cap fund.

The thesis is organized in the following way: the second chapter introduces all the basic concepts we need to know in order to understand the model. From the generalized inverse to quantile function, and the definitions of VaR and fat tails. It introduces liquidity risk, LIX and cost of liquidity. The third chapter shows the details of Liquidity adjusted VaR model and analyzes each part of the model. The fourth chapter contains the empirical result and conclusion.

2. Background

2.1 Generalized inverses

First of all, quantile function in fact is an application of generalized inverses in financial mathematics. Therefore we will begin with a short introduction of generalized inverses based on an article by Paul Embrecht and Marius Hofert(2013).

The idea of a generalized inverse comes from the fact that, although a real-valued, continuous, and strictly monotone function of a single variable have a unique inverse function on its range, sometimes the requirement is too strong. In order to apply it more easily in real life, we have to drop the assumptions of continuity and strict monotonicity, but still we need to get the inverse, which leads to the notion of a generalized inverse.

Let $T: \mathbb{R} \to \mathbb{R}$ be a non-decreasing function with $T(-\infty) = \lim_{x \to -\infty} T(x)$ and $T(\infty) = \lim_{x \to \infty} T(x)$, *the generalized inverse* $T^{-1}: \mathbb{R} \to \overline{\mathbb{R}} = [-\infty, \infty]$ is defined by

$$T^{-1}(y) = \inf\{x \in \mathbb{R} |\ y \le T(x)\}, y \in \mathbb{R}.$$

If T is a distribution function and the target domain become $[0,1]$, T^{-1} is called quantile function of T. We use the convention that $\inf\emptyset = \infty$.

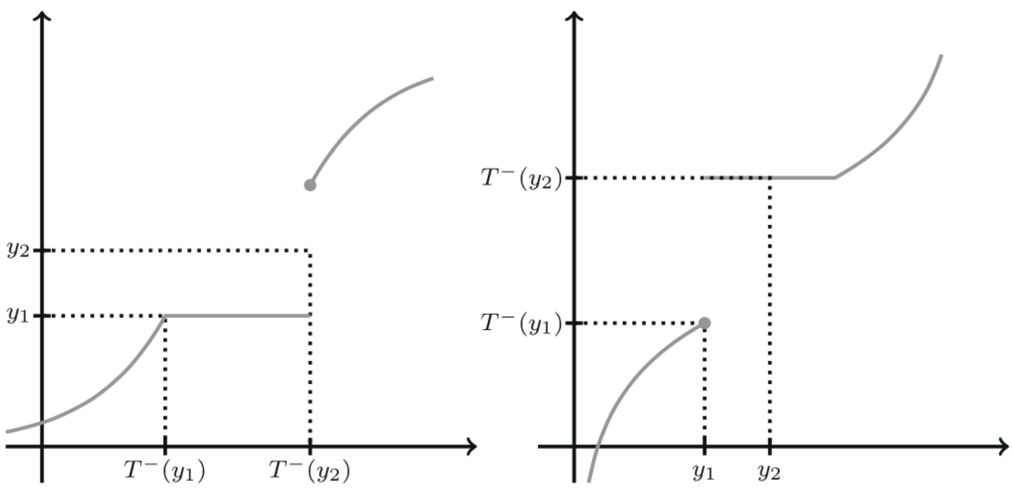


Figure: A non-decreasing function T(left) and its generalized inverse T^{-1}(right)

Source: Paul Embrecht and Marius Hofert(2013, p.425)

We could easily observe the difference between the generalized inverse and normal inverse from the figure. Firstly, we could drop the strictly increasing assumption, so T could be flat. The flat part of T corresponds to the jump in the generalized inverse T^{-1}. Secondly, we could drop the continuity assumption, so T could have jumps. The jump part of T corresponds to the flat in the generalized inverse T^{-1}.

The generalized inverse has the following properties, which are frequently used and proved by Paul Embrecht and Marius Hofert(2013).

Proposition 1: Let $T: \mathbb{R} \to \mathbb{R}$ be a non-decreasing function with $T(-\infty) = \lim_{x \to -\infty} T(x)$ and $T(\infty) = \lim_{x \to \infty} T(x)$, let $x, y \in \mathbb{R}$. Then,

(i) T^{-1} is non-decreasing. If $T^{-1}(y) \in (-\infty, \infty)$, T^{-1} is continuous from the left and has right limits

(ii) If T is continuous from the right, then $T^{-1}(y) < \infty$ implies $T\big(T^{-1}(y)\big) \geq y$. Furthermore, $y \in ran\, T \cup \{\inf ran\, T, \sup ran\, T\}$ implies $T\big(T^{-1}(y)\big) = y$. Moreover, if $y < \inf ran\, T$ then $T\big(T^{-1}(y)\big) > y$ and if $y > \sup ran\, T$ then $T\big(T^{-1}(y)\big) < y$, where *ran* denotes the abbreviation of range.

(iii) $T(x) \geq y \Rightarrow x \geq T^{-1}(y)$. Furthermore, if T is continuous from the right, then $T(x) \geq y \Leftrightarrow x \geq T^{-1}(y)$. Moreover, $T(x) < y \Rightarrow x \leq T^{-1}(y)$.

(iv) If T_1 and T_2 are continuous from the right and have same properties as T, then $(T_1 \circ T_2)^{-1} = T_2^{-1} \circ T_1^{-1}$.

Proof

(i) Let $y_1, y_2 \in \mathbb{R}$ and $y_1 < y_2$. We have
$$\{x \in \mathbb{R} |\, y_1 \leq T(x)\} \supseteq \{x \in \mathbb{R} |\, y_2 \leq T(x)\},$$
so $T^{-1}(y_1) \leq T^{-1}(y_2)$, T^{-1} is non-decreasing.
Let $T^{-1}(y) \in (-\infty, \infty)$ and $y_0 = y$. Suppose $y_n \to y_0$ is a strictly increasing sequence. In order to prove T^{-1} is continuous from the left, we need to prove $\lim_{n \to \infty} T^{-1}(y_n) = T^{-1}(y_0)$. Since T^{-1} is non-decreasing, $x_n := T^{-1}(y_n) \leq x_0 := T^{-1}(y_0)$. Thus
$$\lim_{n \to \infty} x_n = x \leq x_0 \quad \text{for some } x \in \mathbb{R}.$$
Therefore, we need to prove $x = x_0$. We assume $x < x_0$:
From the definition of T^{-1}, we have $\forall \varepsilon > 0$ *and* $n \in \mathbb{N}_0 = \{0,1,2,3 \ldots\}$,
$$T(x_n - \varepsilon) < y_n \leq T(x_n + \varepsilon).$$
let $\varepsilon = \frac{x_0 - x}{2}$, then for $\forall n \in \mathbb{N}$, $y_n \leq T(x_n + \varepsilon) \leq T(x_0 - \varepsilon) < y_0$. So $y_0 = \lim_{n \to \infty} y_n \leq T(x_0 - \varepsilon) < y_0$, which is a contradiction. T^{-1} is continuous from the left.
In order to prove T^{-1} has right limit, suppose $y_n \to y_0$ is a strictly decreasing sequence. $T^{-1}(y_n)$ is a non-increasing sequence and $T^{-1}(y_0) > -\infty$. From the monotone convergence theorem, T^{-1} has right limit.

(ii) We have $T^{-1}(y) < \infty$, then $A = \{x \in \mathbb{R} |\, y \leq T(x)\} \neq \emptyset$. Suppose $x_n \to$

$T^{-1}(y)$ is a strictly decreasing sequence and $(x_n)_{n\in\mathbb{N}} \subseteq A$, thus $T(x_n) \geq y$. Since T is continuous from the right, $T(T^{-1}(y)) = \lim_{n\to\infty} T(x_n) \geq y$. We proved the first part.

At first we consider $y \in ran\, T$, then $B = \{x \in \mathbb{R} |\, y = T(x)\} \neq \emptyset$. We could see $T^{-1}(y) = \inf A = \inf B$. Suppose $x_n \to T^{-1}(y)$ is a non-increasing sequence and $(x_n)_{n\in\mathbb{N}} \subseteq B$, thus $T(x_n) = y$. Since T is continuous from the right, $T(T^{-1}(y)) = \lim_{n\to\infty} T(x_n) = y$. Now let $y = \inf ran\, T$ and $\inf ran\, T \notin ran\, T$ (otherwise we could just use the proof above). Thus $A = \{x \in \mathbb{R} |\, y \leq T(x)\} = \mathbb{R}$, we have $T^{-1}(y) = \inf A = -\infty$. So $T(T^{-1}(y)) = T(-\infty) = \inf ran\, T = y$. At last let $y = \sup ran\, T$ and $\sup ran\, T \notin ran\, T$ (otherwise we could just use the proof of $ran\, T$). Thus $A = \{x \in \mathbb{R} |\, y \leq T(x)\} = \emptyset$, we have $T^{-1}(y) = \inf A = \infty$. So $T(T^{-1}(y)) = T(\infty) = \sup ran\, T = y$. We proved the second part.

If $y < \inf ran\, T$, we have $A = \{x \in \mathbb{R} |\, y \leq T(x)\} = \mathbb{R}$, $T^{-1}(y) = -\infty$, so $T(T^{-1}(y)) = T(-\infty) = \inf ran\, T > y$. If $y > \sup ran\, T$, we have $A = \{x \in \mathbb{R} |\, y \leq T(x)\} = \emptyset$, $T^{-1}(y) = \infty$, so $T(T^{-1}(y)) = T(\infty) = \sup ran\, T < y$. We proved the third part.

(iii) From the definition of T^{-1}, we have $T(x) \geq y \Rightarrow x \geq T^{-1}(y)$. If T is continuous from the right, and $\infty > x \geq T^{-1}(y)$, from property (ii), we have $T(T^{-1}(y)) \geq y$. So $T(x) \geq T(T^{-1}(y)) \geq y$. Thus $T(x) \geq y \Leftrightarrow x \geq T^{-1}(y)$.

If $T(x) < y$ and let $A = \{z \in \mathbb{R} |\, y \leq T(z)\}$, since T is a non-decreasing function, $T(z) \geq y > T(x) \Rightarrow z > x$. So $T^{-1}(y) = \inf A \geq$x.

(iv) Let T_1 and T_2 are continuous from the right and have same properties as T, then $(T_1 \circ T_2)^{-1} = \inf\{x \in \mathbb{R} |\, y \leq T_1(T_2(x))\}$. by property (iii), we have $y \leq T_1(T_2(x)) \Leftrightarrow T_2(x) \geq T_1^{-1}(y) \Leftrightarrow x \geq T_2^{-1}(T_1^{-1}(y))$. So $(T_1 \circ T_2)^{-1} = \inf\{x \in \mathbb{R} |\, x \geq T_2^{-1}(T_1^{-1}(y))\} = T_2^{-1}(T_1^{-1}(y)) = T_2^{-1} \circ T_1^{-1}$.

■

2.2 Quantiles

Let X be a random variable. If F_X is the cumulative distribution function of X, that is

$$F_X(x) = \mathbb{P}[X \le x], \qquad x \in \mathbb{R}.$$

We define the corresponding *quantile function* as the generalized inverse of F:

$$F_X^{-1}(q) = \inf\{x | q \le F_X(x)\}, \qquad q \in (0,1).$$

The number $F_X^{-1}(q)$ is called the *q-quantile* of X. Note that F_X is always non-decreasing, is continuous from the right and has left limits. **The *q-quantile* of X is then the smallest value x such that the probability of X not exceeding x is not smaller than q.**

We will prove the quantile function is equivariant under non-decreasing left continuous transformations. In order to prove it, we need to prove 2 lemmas first.

Lemma 1: (Quantile Value Criterion Lemma)

$F_X^{-1}(q)$ is the only a satisfying (i) and (ii), where

(i) $F_X(a) \ge q$;

(ii) $x < a \Rightarrow F_X(x) < q$.

Proof

Suppose $x_n \to F_X^{-1}(q)$ is a strictly decreasing sequence. According to the definition of $F_X^{-1}(q)$ and $x_n > F_X^{-1}(q)$, we have $F_X(x_n) \ge q$. Since F_X is right continuous

$$\lim_{n\to\infty} F_X(x_n) = F_X(F_X^{-1}(q)).$$

For $\forall n \in \mathbb{N}, F_X(x_n) \ge q$ hence $\lim_{n\to\infty} F_X(x_n) \ge q$ (i) holds. So the $F_X^{-1}(q)$ is the smallest value satisfy $F_X(x) \ge q$, if $x < F_X^{-1}(q)$, then $F_X(x) < q$. So $F_X^{-1}(q)$ satisfies both properties.

Assuming both a and b satisfy them and $a < b$, then $F_X(a) \ge q$ by (i). However b also satisfy both properties and $a < b$, then $F_X(a) < q$ by (ii), which is a contradiction.

■

Let $h(x): \mathbb{R} \to \mathbb{R}$ be a non-decreasing function. We define $h^\star(y)$:

$$h^\star(y) = \sup\{x | h(x) \le y\}.$$

Lemma 2:

If $h(x): \mathbb{R} \to \mathbb{R}$ is a non-decreasing function and left continuous, then

$$h\big(h^\star(y)\big) \le y.$$

Proof

Suppose $x_n \to h^\star(y)$ is a strictly increasing sequence. According to the definition of $h^\star(y)$ and $x_n < h^\star(y)$, we have

$$\forall n \in \mathbb{N}, h(x_n) \le y.$$

Hence, $\lim_{n\to\infty} h(x_n) \le y$. $h(x)$ is left continuous $\Rightarrow$ $\lim_{n\to\infty} h(x_n) = h(h^\star(y))$.

Finally, we have $h\big(h^\star(y)\big) \le y$.

■

Theorem: (Quantile Equivariant Transformation Theorem)
Suppose $h: \mathbb{R} \to \mathbb{R}$ is a non-decreasing function and left continuous, then

$$F_{h(X)}^{-1}(q) = h\big(F_X^{-1}(q)\big).$$

Proof

We use Lemma 1 to prove this. We need to show $h(F_X^{-1}(q))$ satisfies both (i) and (ii) in Lemma 1.

Firstly, we could see $F_{h(X)}\left(h\big(F_X^{-1}(q)\big)\right) = \mathbb{P}\left(h(X) \le h\big(F_X^{-1}(q)\big)\right)$. Since $h(x)$ is a non-decreasing function, $X \le F_X^{-1}(q) \Rightarrow h(X) \le h\big(F_X^{-1}(q)\big)$. Hence we have $\mathbb{P}\left(h(X) \le h\big(F_X^{-1}(q)\big)\right) \ge \mathbb{P}\big(X \le F_X^{-1}(q)\big)$ and from the definition of $F_X^{-1}(q)$ we have $\mathbb{P}\big(X \le F_X^{-1}(q)\big) \ge q$. Therefore:

$$F_{h(X)}\left(h\big(F_X^{-1}(q)\big)\right) = \mathbb{P}\left(h(X) \le h\big(F_X^{-1}(q)\big)\right) \ge \mathbb{P}\big(X \le F_X^{-1}(q)\big) \ge q.$$

(i) holds.

For (ii), let $y < h(F_X^{-1}(q))$. Then we need to show $F_{h(X)}(y) < q$. By the lemma 2 we have

$$h\big(h^\star(y)\big) \le y \;\Rightarrow\; h\big(h^\star(y)\big) \le y < h\big(F_X^{-1}(q)\big).$$

Because h is a non-decreasing function $\Rightarrow h^\star(y) < F_X^{-1}(q)$. Then we have $\mathbb{P}\big(X \le h^\star(y)\big) < q$ and $F_{h(X)}(y) = \mathbb{P}(h(X) \le y)$. According to the definition of $h^\star(y)$ and lemma 2, we know $h^\star(y)$ is the biggest value that satisfies $h(X) \le y$. So we have $h(X) \le y \Rightarrow X \le h^\star(y)$. So:

$$F_{h(X)}(y) = \mathbb{P}(h(X) \le y) \le \mathbb{P}\big(X \le h^\star(y)\big) < q.$$

(ii) holds.

■

2.3 Definition of VaR

From Choudhry and Alexander(2013), we could have a basic understanding of Value-at-Risk. VaR was first introduced to public in October 1994 when JPMorgan launched RiskMetrics free over the internet. With the development of risk management, VaR became an accepted methodology for quantifying market risk and began to be adopted by bank regulators. In 1997, major banks and dealers chose to include VaR information in the notes to their financial statements, because the U.S. Securities and Exchange Commission ruled that public corporations must disclose quantitative information about their derivatives activity. Nowadays, most of the banks and hedge funds use VaR as a measure of market risk, which makes VaR become one of the most popular risk measures in the world.

Let $T > 0$ denote the time horizon. Let P_0 and P_T denote the current price and the future price at time T of a stock. Let $c \in (0,1)$ denote the *confidence level*(which is typically 0.99 or 0.95). The quantity $\alpha = 1 - c$ is called the *tolerance level.* The profit/loss generated by the portfolio at time T is represented by the random variable $P_T - P_0$. The *value-at-risk* with the tolerance level α is defined as

$$VaR_\alpha = -F^{-1}_{P_T - P_0}(\alpha).$$

Therefore, VaR is simply the α quantile of the loss/profit random variable with a minus in front. The minus makes it positive as for small α the quantile itself is usually negative.

Remark 1: In view of Quantile Equivariant Transformation Theorem

$$VaR_\alpha = P_0 - F^{-1}_{P_T}(\alpha) = -P_0 F^{-1}_{R_T}(\alpha),$$

where R_T denotes the rate of return from the stock:

$$R_T = \frac{P_T - P_0}{P_0}.$$

Proof

Let $R_T = h(P_T) = \frac{P_T - P_0}{P_0}$, h be a non-decreasing and continuous function, from the Quantile Equivariant Transformation Theorem, we have:

$$F^{-1}_{h(P_T)}(\alpha) = h(F^{-1}_{P_T}(\alpha))$$

$$\Rightarrow F^{-1}_{R_T}(\alpha) = \frac{F^{-1}_{P_T}(\alpha) - P_0}{P_0}$$

$$\Rightarrow -P_0 F^{-1}_{R_T}(\alpha) = -P_0\left(\frac{F^{-1}_{P_T}(\alpha) - P_0}{P_0}\right) = P_0 - F^{-1}_{P_T}(\alpha).$$

■

This is why **the alternative definition of VaR defines it as the α quantile of the return random variable with a minus in front.**

Remark 2: Alternatively, we can consider the log-return

$$r_T = ln\frac{P_T}{P_0}.$$

Then in view of Quantile Equivariant Transformation Theorem

$$VaR_\alpha = P_0 - F_{P_T}^{-1}(\alpha) = P_0\left(1 - \exp\left(F_{r_T}^{-1}(\alpha)\right)\right).$$

Proof

Let $r_T = h(P_T) = ln\frac{P_T}{P_0}$, h is a non-decreasing and continuous function, from the Quantile Equivariant Transformation Theorem, we have:

$$F_{h(P_T)}^{-1}(\alpha) = h(F_{P_T}^{-1}(\alpha))$$

$$\Rightarrow F_{r_T}^{-1}(\alpha) = ln\frac{F_{P_T}^{-1}(\alpha)}{P_0}$$

$$\Rightarrow P_0\left(1 - \exp\left(F_{r_T}^{-1}(\alpha)\right)\right) = P_0\left(1 - \exp\left(ln\frac{F_{P_T}^{-1}(\alpha)}{P_0}\right)\right)$$

$$= P_0\left(1 - \frac{F_{P_T}^{-1}(\alpha)}{P_0}\right)$$

$$= P_0 - F_{P_T}^{-1}(\alpha).$$

■

Remark 3: If P_T/P_0 is close to 1, then

$$r_T \approx R_T.$$

Proof

$$R_T = \frac{P_T - P_0}{P_0} = \frac{P_T}{P_0} - 1 \Rightarrow \frac{P_T}{P_0} = 1 + R_T$$

According to the Taylor expansion:

$$r_T = ln\frac{P_T}{P_0} = \ln(1 + R_T) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{R_T^n}{n} = R_T - O(R_T^2)$$

And

$$\frac{P_T}{P_0} \to 1 \Rightarrow R_T = \frac{P_T}{P_0} - 1 \to 0$$

Then $O(R_T^2)$ would be negligible; we have $r_T \approx R_T$.

■

But the random variables r_T and R_T have different distributions.

2.4 Fat tails

Empirical studies show that the distribution of asset returns is usually not normal, and instead has fatter tails than normal distribution. From Daníelsson, Jorgensen, Samorodnitsky(2013), we could see a formal definition of a fat-tailed distribution. Fat tails means that the tails vary regularly at infinity so that they approximately follow a Pareto distribution, which like power expansion at infinity.

A cumulative distribution function $F(x)$ varies regularly at $-\infty$ with tail index $\alpha > 0$ if

$$\lim_{n\to\infty} \frac{F(-tx)}{F(-t)} = x^{-\alpha}, \qquad \forall x > 0,$$

and varies regularly at ∞ with tail index $\alpha > 0$ if

$$\lim_{n\to\infty} \frac{1 - F(tx)}{1 - F(t)} = x^{-\alpha}, \qquad \forall x > 0.$$

It gives out that a regularly varying distribution has a tail of the form

$$F(-x) = x^{-\alpha} L(x), \qquad x > 0.$$

Where $L(x)$ is a slowly varying function which means $\forall x > 0, t \to \infty$ we have $L(tx)/L(t) \to 1$. And the constant $\alpha > 0$ is the *tail index*.

2.5 Liquidity risk

In order to understand liquidity risk, we need to understand liquidity first. Liquidity describes the trade-off between how quickly an asset or security can be bought or sold in the market and how large the degree of affecting the asset or security's price. In a liquid market, an asset or security can be sold quickly without reducing the price much. However, in a relatively illiquid market, sell an asset or security quickly will need to reduce its price to some degree.

There are two types of liquidity:

1. Market liquidity, which is the ability that a market allows assets to be bought and sold without affecting the asset's price to a significant degree. Cash is the most liquid asset.
2. Funding liquidity, which is the ability an institution ensures its payment with immediacy.

Therefore, liquidity risk is the risk that an asset or security could not be sold quickly enough without causing great change in the price. There are two types of liquidity risk respectively:

1. Market (asset) liquidity risk, which is the risk that a position cannot be closed (or an asset cannot be sold) quickly enough without influencing the market price to a significant degree.
2. Fund (cash flow) liquidity risk, which is the risk that an institution is unable to repay its liabilities or meet its obligations when they come due, which leads to default.

Here in this paper, we focused on market liquidity risk. Traditionally it is measured by the following 3 aspects:

1. 'Width', which is also called bid-ask spread. It is the difference between the ask price and the bid price. It can also be calculated in percentage term,

which is proportional spread. It is the difference divided by the average of the ask price and the bid price. It is measured in a price dimension.

2. 'Depth'. It is the market ability to absorb the exit of a position without changing the price dramatically, which is that the number of the securities can be bought without having price appreciation. It is measured in a quantity dimension.
3. 'Resiliency'. It is the time the market need to go back from incorrect price. It is measured in a time dimension.

There is also another measure becoming popular recently, which is 'volume'. It is the amount of a certain security traded during a certain time period.

2.6 The definition of *LIX*

This part is a summary of Oleh Danyliv, Bruce Bland, Daniel Nicholass(2014). They give out a new measure of liquidity, which I will use later in the model. So here I give a short summary about how they give out and define *LIX(liquidity index)*.

From the view of a fund company, dealing with liquidity risk could be seen the same as finding an answer to the following question: *what amount of money can one trade/invest without moving the market*? In fact, it is a very hard question to answer, especially when we try to quantify a specific amount of it. It also depends on how much time the trader has to execute it, which strategy the trader uses to close the position and how large the position is compared to average daily volume, etc. However, if we look at the problem in another way, it will give us a more clear and precise definition of liquidity. *What amount of money is needed to create a daily single unit price fluctuation of the stock?*

$$Liquidity \sim \frac{Consideration}{Price\ Range} \equiv \frac{Volume \times Price}{High - Low}. \quad (1)$$

The liquidity measure we got from the formula (1) ranges from thousands to billions. In order to handle the numbers more easily, we take the logarithm of the amount, which reduces the range to manageable numbers. Since the value we got from the formula (1) does not have units of measurement, it is reasonable to do so. And it is called Liquidity Index (*LIX*):

$$LIX_t = \log_{10}\left(\frac{V_t P_{Mid,t}}{P_{High,t} - P_{Low,t}}\right), \quad (2)$$

where V_t is the trading volume today, $P_{High,t}$ is the highest ask price today, $P_{Low,t}$ is the lowest bid price today and $P_{Mid,t}$ is the average of ask price and bid price. A logarithm with the base of 10 makes the LIX_t into a range roughly from very illiquid 5 to very liquid 10. It has a simple meaning: for Sweden stocks the amount of capital needed to create 1kr price fluctuations can be estimated as 10^{LIX_t} kr.

The liquidity measure (2) in non-logged version (1) has already been used in some published literature. In the UK government's Foresight report by Linton(2012), the measure (1) was used to investigate the evolution of market liquidity with the following remark: “in the current environment a plausible alternative to close return is to use intraday high minus low return, since there can be a great deal of intraday movement in the price that ends in no change at the end of the day’”.

LIX as a liquidity measure has the following 2 advantages:

1. The currency value is eliminated from calculation, so we can compare stocks on different international markets directly,
2. The data that it requires is easy to have access to, comparing with other measures.

2.7 Cost of liquidity

In order to calculate the cost of liquidity (COL), we need use bid-ask *spread*. It is the difference between the highest ask price today and the lowest bid price today:

$$Spread_t = P_{High,t} - P_{Low,t}.$$

It can also be calculated in percentage term, which is *proportional spread*:

$$Proportional\ spread_t = \frac{P_{High,t} - P_{Low,t}}{P_{Mid,t}},$$

where $P_{Mid,t}$ is the average of ask price and bid price. The full spread represents the liquidity cost of a round trip, which means the cost of buying and selling the stock today. Here, we only need to consider the liquidity cost of selling the stock, hence the cost of liquidity (COL) is only half of the spread.

$$COL_t = \frac{1}{2} \times Spread_t = \frac{1}{2}\left(P_{High,t} - P_{Low,t}\right);$$

$$COL_t = \frac{1}{2} \times P_{Mid,t} \times Proportional\ spread_t;$$

$$Percentage\ COL_t = \frac{1}{2} \times Proportional\ spread_t = \frac{1}{2}\left(\frac{P_{High,t} - P_{Low,t}}{P_{Mid,t}}\right).$$

3. Liquidity Adjusted VaR Model

The purpose of a Liquidity adjusted VaR model is to incorporate the liquidity risk into VaR model. Bangia, Diebold, Schuermann and Stroughair(1999) proposed a model, which incorporated the exogenous liquidity risk into VaR model. They modeled the transaction price as mid-price plus a half of the proportional bid-ask spread:

$$P_{mid,t+1} = P_{mid,t} \exp(r_{t+1}) - \frac{1}{2} P_t S_{t+1},$$

where P_{mid} is the middle price of the ask price and the bid price, r_{t+1} is the daily return between t and t+1 and S is a time-varying proportional bid-ask spread. Liquidity adjusted VaR is the sum of the normal VaR and Cost of liquidity(COL):

$$LaVaR = \overbrace{P_{mid,t}(1 - \exp(z_\alpha \sigma_r))}^{Normal\,VaR} + \overbrace{\frac{1}{2} P_{mid,t}(\mu_S + \hat{z}_\alpha \sigma_S)}^{COL},$$

where σ_r is the variance of the daily return, μ_S is the mean of the proportional bid-ask spread, σ_s is the standard deviation of the proportional bid-ask spread. z_α and $\hat{z}_\alpha$ are the α-percentile of the daily return distribution and the proportional spread distribution.

Le Saout (2002) extends the model of Bangia et al. for including the endogenous risk, by substituting the proportional bid-ask spread which is used for Cost of liquidity calculation by Weighted Average Spread (WAS):

$$LaVaR = \overbrace{P_{mid,t}(1 - \exp(z_\alpha \sigma_r))}^{Normal\,VaR} + \overbrace{\frac{1}{2} P_{mid,t}\left(\mu_{\bar{S}(V)} + \hat{z}_\alpha \sigma_{\bar{S}(V)}\right)}^{COL},$$

where $\bar{S}(V)$ is the proportional Weighted Average Spread, which is a function of the volume of the certain stock we have in the portfolio.

However, in order to get Weighted Average Spread (WAS), we need the order book data from the stock market, which is quite extensive and expensive.

Based on the idea above, so instead of using Weighted Average Spread (WAS), we use the liquidity measure LIX to forecast the spread. However, using LIX to forecast spread will produce unreasonable spread prediction under extreme situation. In order to give out more realistic prediction, we need to scale it:

$$LaVaR = Normal\,VaR + \widetilde{COL};$$
$$\widetilde{COL} = A * COL,$$

where $A > 0$ is a scale coefficient, Cost of liquidity (COL) is half of the Spread, the Spread is a function of LIX and volume of certain stock we have in the portfolio.

3.1 Model of Normal VaR

Firstly we calculated the one-day stock return r_t by taking the logarithm of the ratio of two adjacent prices. The price P_t is estimated by taking the average of bid and ask price:

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right).$$

Then we assumed the daily return at t+1 is normally distributed with the mean $E(r_t)$ and the variance σ_t^2

$$r_{t+1} \sim N(E(r_t), \sigma_t^2).$$

For the given confidence level we use both 99% and 95%. We calculate the standard normal distribution percentile of 99% and 95%. The 1% left tail of normal distribution, Norminv(0.01)=-2.326. The 5% left tail of normal distribution, Norminv(0.05)=-1.645. With the given 99% confidence level, the worst daily return at time t+1 will be $r_{t+1}^w = E(r_t) - 2.326\sigma_t$. Similarly with 95% confidence level $r_{t+1}^w = E(r_t) - 1.645\sigma_t$.

Here, we consider the one-day horizon and thus we take the expected daily return to be $E(r_t) = 0$. Hence, the worst price tomorrow will be

$$p_{t+1}^w = p_t e^{r_{t+1}^w}.$$

Therefore, the normal Value at risk for one share of the stock will be

$$99\%\ VaR = p_t - p_{t+1}^w = p_t\left(1 - e^{r_{t+1}^w}\right) = p_t(1 - e^{-2.326\sigma_t});$$

$$95\%\ VaR = p_t - p_{t+1}^w = p_t\left(1 - e^{r_{t+1}^w}\right) = p_t(1 - e^{-1.645\sigma_t}).$$

The VaR in percentage will be

$$99\%\ VaR = 1 - e^{-2.326\sigma_t};$$

$$95\%\ VaR = 1 - e^{-1.645\sigma_t}.$$

We have the price directly on the market. The only parameter we need to estimate here is σ_t. From the empirical analysis, it will give out nice results by using exponentially weighted moving average.

Exponentially weighted moving average gives less weight to the further data in the past and more weight to the more recent data:

$$\sigma_t = \sqrt{\left(\frac{1-\lambda}{1-\lambda^T}\right)\sum_{t=1}^{T}\lambda^{t-1}\big(r_t - E(r)\big)^2}.$$

T is the history time period we considered. λ is the decay factor which is between (0,1). From the industry experience, we choose the decay factor to be 0.94.

For the portfolio VaR, We also need to estimate the covariance σ_{ij}. Here we still use exponentially weighted moving average way to do it:

$$\sigma_{ij} = \sqrt{\left(\frac{1-\lambda}{1-\lambda^T}\right)\sum_{t=1}^{T}\lambda^{t-1}\big(r_{it} - E(r_{it})\big)\left(r_{jt} - E(r_{jt})\right)}.$$

In order to get the portfolio standard deviation σ, we will need the weight of each stock in the portfolio and the covariance matrix:

$$W = (w_1, w_2 \dots w_n)$$

$$V = \begin{pmatrix} \sigma_{11}^2 & \cdots & \sigma_{1n}^2 \\ \vdots & \ddots & \vdots \\ \sigma_{n1}^2 & \cdots & \sigma_{nn}^2 \end{pmatrix}$$

As for the weight, VaR of the cash and the management fee is considered to be zero. So we only consider the weight of the stocks.

Then we could get

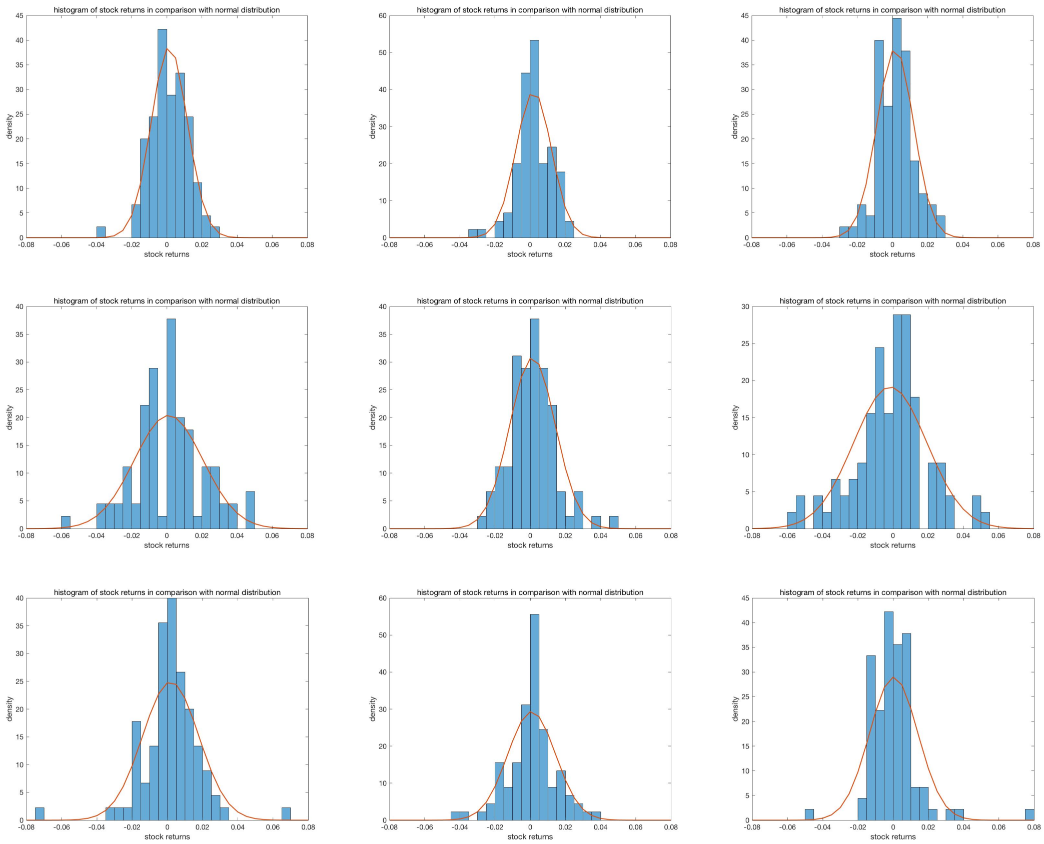
$$\sigma^2 = wvw^T.$$

The portfolio percentage VaR:

$$99\%\ VaR = 1 - e^{-2.326\sigma};$$

$$95\%\ VaR = 1 - e^{-1.645\sigma}.$$

However the assumption that the daily return at t+1 is normally distributed may not be true. Empirical studies show that the asset returns is not normally distributed and in fact the distribution has fat tails. Let’s check the histogram of different stock returns in comparison with normal distribution.



We could see from the figure above, the distribution of the stock returns has fat tails. It is also called the tail coarseness problem. In order to deal with the problem, by Daníelsson and de Vries (2000), we could use extreme value theory (EVT), which gives out another quantile estimator.

For a distribution with fat tails, the tail asymptotically follows a power law, i.e. the Pareto distribution:

$$F(x) = 1 - Ax^{-\alpha}.$$

We take the sample size to be n and let $m < n$ sufficiently small. Here we take the sample size to be 262, which is one year history instead of 90 days, because this method requires a relatively large sample size to accurately estimate the quantile. There is no good way to decide which m is the best, so we will try different m. Then we could use the Hill estimator to estimate the *tail index* α:

$$\frac{1}{\hat{\alpha}(m)} = \frac{1}{m} \sum_{i=1}^{m} ln \frac{X_{(i)}}{X_{(m+1)}},$$

where $X_{(i)}$ indicates order statistics of stock daily return. So we need to rearrange all the history stock daily return from smallest to largest. Then we will get the quasi maximum likelihood VaR estimator

$$99\%\ VaR = X_{(m+1)} \left(\frac{m/n}{0.01}\right)^{\frac{1}{\hat{\alpha}(m)}};$$

$$95\%\ VaR = X_{(m+1)} \left(\frac{m/n}{0.05}\right)^{\frac{1}{\hat{\alpha}(m)}}.$$

As for the portfolio VaR, instead of rearranging all the history single stock daily return from smallest to largest and give the value to $X_{(i)}$, we calculate all the history portfolio daily return. And let $X_{(i)}$ be the order statistics of portfolio daily return. Then the formula will give out portfolio VaR.

The quasi maximum likelihood VaR estimator is estimated by using all observations in the tail, which makes it less sensitive to the tail coarseness problem. Hence, it produces more accurate results.

3.2 Model of Cost of liquidity(COL)

Firstly, we know that Cost of liquidity depends on the spread

$$COL_{t+1} = \frac{1}{2} \times Spread_{t+1} = \frac{1}{2}\left(P_{High,t+1} - P_{Low,t+1}\right).$$

In order to estimate the cost of liquidity at $t+1$, we only need to predict the spread at $t+1$. According to the definition of LIX, we see that

$$LIX_t = \log_{10}\left(\frac{V_t P_{Mid,t}}{P_{High,t} - P_{Low,t}}\right);$$

$$Spread_t = P_{High,t} - P_{Low,t} = \frac{V_t P_{Mid,t}}{10^{LIX_t}}.$$

Then we see the cost of liquidity as

$$COL_{t+1} = \frac{1}{2}\left(\frac{VP_{Mid,t+1}}{10^{LIX_{t+1}}}\right);$$

$$Percentage\ COL_{t+1} = \frac{COL}{P_{Mid,t+1}} = \frac{1}{2}\left(\frac{V}{10^{LIX_{t+1}}}\right),$$

where V is the volume of stock we have in the portfolio, and LIX_{t+1} is the liquidity index at $t+1$. We could see it is a good measure to liquidity risk, because the higher the volume we have in the portfolio, the higher the liquidity cost will be. And the higher the LIX we have (which means the more liquid the stock is), the lower the liquidity cost will be. However, the volume we have in the portfolio sometimes is much larger than the trading volume in the market for small cap stocks under extreme condition, which will produce unreasonable $Percentage\ COL$ over 100%. In order to deal with this situation, we introduce a scale coefficient to scale it:

$$\widetilde{COL_{t+1}} = A * COL_{t+1};$$
$$Percentage\ \widetilde{COL_{t+1}} = A * Percentage\ COL_{t+1},$$

where the constant $A > 0$ is the *scale coefficient*. Then we have two problems here: one is deciding A, the other is estimating the LIX at $t + 1$. For the first problem, it is quite subjective to choose a scale since we lack order book data to check how the spread will look like under both crisis situation and colossal transaction volume. So, we used the experience from industry and discussed with the experienced risk controller in the hedge fund. We decided to set $A = 1/10$. It is just an experience based choice. When we could have access to more data in the future, this coefficient can be calibrated to provide a better result.
For the second problem, we experimented two ways:

1) Estimate LIX_{t+1} by taking average

Nowadays in the industry, we use the ratio between the volume we have in the portfolio and past 20 days average daily market trading volume to measure liquidity risk. With the similar idea, we could use the past 20 days average LIX to estimate the LIX at $t + 1$:

$$LIX_{t+1} = \frac{LIX_t + LIX_{t-1} + \cdots + LIX_{t-19}}{20};$$

$$LIX_t = \log_{10}\left(\frac{V_t P_{Mid,t}}{P_{High,t} - P_{Low,t}}\right),$$

where V is the trading volume at t, $P_{High,t}$ is the highest ask price at t, $P_{Low,t}$ is the lowest bid price at t and $P_{Mid,t}$ is the average of ask price and bid price.

2) Estimate LIX_{t+1} by assuming normal distribution

Instead of simply taking average of the past 20 days data, we could assume that the LIX_{t+1} is normally distributed with the mean $E(LIX_t)$ and the variance $\sigma^2_{LIX,t}$

$$LIX_{t+1} \sim N\left(E(LIX_t), \sigma^2_{LIX,t}\right).$$

We also assume that under adverse market situation, the extreme change in the stock price and liquidity index happen at the same time.

Therefore, similar to the normal VaR model, we use confidence level both 99% and 95%. The 1% left tail of normal distribution, Norminv(0.01)=-2.326. The 5% left tail of normal distribution, Norminv(0.05)=-1.645. With the given 99% confidence level, the worst LIX at time t+1 will be

$$LIX_{t+1}^{w} = E(LIX_t) - 2.326\sigma_{LIX,t}.$$

Similarly with 95% confidence level

$$LIX_{t+1}^{w} = E(LIX_t) - 1.645\sigma_{LIX,t},$$

where the equally weighted σ estimator

$$\sigma_{LIX,t} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}\left(LIX_t - E(LIX)\right)^2}.$$

As for the portfolio cost of liquidity, it is just the sum of each stock's cost of liquidity.

4. Results and conclusion

4.1 Numerical results

In order to compare the liquidity effect on different funds, we perform the calculation on two funds, one is a large cap fund and another is a small cap fund. It is known that the small cap fund has higher liquidity risk than the large cap fund. We want to quantify the liquidity risk and compare the results of Liquidity adjusted VaR between different funds. Running the MATLAB(R2016b) code in Appendix, we could get the results.

Firstly, let us analyze the result from the large cap fund of calculating the VaR without considering tail coarseness problem and the result of both methods of predicting LIX.

We could see from the Table 1: the first result of a large cap fund, 'NETIB' has a relatively high Cost of liquidity compared with other stocks in the fund. If we check the ratio between the volume we have in the portfolio and past 20 days average daily market trading volume, 'NETIB' has an ratio of 11.47, which means the share we have in the portfolio is 11.47 times of the average daily trading volume in the market. Indeed, it yields a relatively high liquidity risk. This corroborate our way of quantifying liquidity risk. We could see from the result that for the large cap fund the liquidity risk has a much smaller impact than VaR during adverse market situation.

The assumption that under adverse market situation, the extreme change in the stock price and liquidity index happens at the same time may not be true in real world. In fact, when the stock price drops down extremely, the liquidity goes up firstly and then it goes down. This is why the second method of calculating the liquidity risk produces higher outcome comparing to the experience from the industry. Therefore, we prefer the first method of estimating the cost of liquidity. When we deal with the small cap fund, we only keep the method of taking the average.

A large cap fund

Stock	weight	Volume		99%VaR	95%VaR	LIX_avg	COL_avg	99%LIX_nor	99%COL_nor	95%LIX_nor	95%COL_nor
SWEDA	5.69%	1,130,000		2.25%	1.60%	8.20	0.036%	7.82	0.085%	7.94	0.065%
NDA	7.56%	3,150,000		1.99%	1.41%	8.72	0.030%	8.32	0.075%	8.43	0.059%
SEBA	4.08%	1,800,000		2.46%	1.74%	8.58	0.024%	8.13	0.067%	8.25	0.051%
NETIB	2.82%	14,930,000		5.11%	3.64%	7.47	2.534%	6.95	8.407%	7.09	6.064%
ATCOB	4.24%	730,000		2.72%	1.93%	7.66	0.080%	7.18	0.242%	7.30	0.184%
SAND	0.53%	200,000		2.78%	1.97%	8.39	0.004%	8.00	0.010%	8.11	0.008%
NOKIASEK	4.15%	4,300,000		4.56%	3.25%	7.80	0.339%	7.38	0.906%	7.51	0.662%
INVEB	4.53%	573,000		1.66%	1.18%	7.90	0.036%	7.50	0.090%	7.63	0.067%
ERICB	2.37%	2,100,000		3.39%	2.41%	8.76	0.018%	8.38	0.044%	8.50	0.033%
VOLVB	8.24%	3,160,000		2.96%	2.10%	8.56	0.044%	8.16	0.109%	8.28	0.084%
AZN	4.36%	387,000		3.39%	2.41%	7.58	0.051%	7.09	0.157%	7.20	0.122%
HOLMB	2.14%	279,000		3.13%	2.22%	6.74	0.254%	6.34	0.632%	6.48	0.466%
ELUX	5.02%	978,000		2.95%	2.09%	7.84	0.071%	7.35	0.220%	7.48	0.163%
Tel2B	3.93%	2,250,000		3.87%	2.75%	8.20	0.072%	7.78	0.185%	7.89	0.144%
STER	3.40%	1,850,000		4.89%	3.48%	7.44	0.333%	7.06	0.812%	7.20	0.590%
NOBI	3.54%	1,180,000		4.24%	3.02%	7.15	0.413%	6.64	1.365%	6.79	0.950%
SKFB	4.41%	1,665,000		3.39%	2.41%	8.17	0.056%	7.76	0.145%	7.86	0.116%
SOBI	4.05%	1,550,000		3.95%	2.81%	7.62	0.184%	7.27	0.420%	7.37	0.327%
SAABB	4.07%	498,800		1.95%	1.38%	7.14	0.182%	6.73	0.469%	6.84	0.362%
HMB	9.08%	1,667,000		3.90%	2.77%	8.41	0.032%	7.91	0.103%	8.03	0.078%
ALIV	3.54%	172,000		4.69%	3.34%	7.26	0.048%	6.81	0.133%	6.93	0.101%
						method of average		method of assuming normal distribution			
				portfolio 99%VaR: 1.34%		portfolio cost of liquidity: 0.16%		99%portfolio cost of liquidity: 0.48%		95%portfolio cost of liquidity: 0.35%	
				portfolio 95%VaR: 0.95%							
						99%LA-VaR: 1.50%		99%LA-VaR: 1.82%		95%LA-VaR: 1.3%	
						95%LA-VaR: 1.11%					

Table 1: the first result of a large cap fund
Where LIX_avg means the Liquidity Index got by using method 1)
99%LIX_nor means the Liquidity Index got by using method 2) with 99%
95%LIX_nor means the Liquidity Index got by using method 2) with 95%
COL_avg means the Cost of liquidity got by using method 1)
99%COL_nor means the Cost of liquidity got by using method 2) with 99%
95%COL_nor means the Cost of liquidity got by using method 2) with 95%

A small cap fund							
Stock	weight	Volume		99%VaR	95%VaR	LIX_avg	COL_avg
Alimak	3.50%	825,048		4.22%	3.00%	6.34	1.90%
Beijer Alma	1.18%	148,964		2.72%	1.93%	5.69	1.53%
Beijer Ref B	4.43%	631,118		1.95%	1.38%	4.96	34.81%
Bonava B	4.90%	1,086,258		3.51%	2.50%	6.96	0.59%
Bravida	2.67%	1,393,514		2.13%	1.51%	6.85	0.98%
Dometic	2.96%	1,316,241		1.97%	1.40%	7.52	0.20%
Duni	2.63%	633,381		2.20%	1.56%	5.91	3.89%
Fabege	5.98%	1,169,424		2.25%	1.60%	7.42	0.22%
Fagerhult	5.31%	561,274		4.35%	3.10%	5.54	8.14%
Holmen B	2.05%	184,321		3.13%	2.22%	6.74	0.17%
Intrum Justi	2.72%	275,349		4.26%	3.03%	7.22	0.08%
KappAhl	1.18%	707,747		3.48%	2.48%	6.79	0.58%
Kungsleden	2.71%	1,481,116		3.02%	2.14%	7.40	0.29%
Lindab	3.18%	1,280,952		3.26%	2.31%	6.87	0.87%
Mekonomen	3.41%	590,640		2.87%	2.04%	6.32	1.43%
Munksjö	3.25%	738,815		3.32%	2.36%	5.74	6.80%
NCC B	1.72%	242,730		3.02%	2.15%	7.40	0.05%
NetEnt B	2.44%	1,020,000		3.41%	2.43%	7.49	0.17%
Nobia	2.51%	899,407		4.25%	3.02%	7.15	0.32%
OEM Internat	6.68%	1,302,055		2.01%	1.42%	4.88	85.25%
RaySearch B	2.96%	402,157		8.14%	5.83%	5.95	2.24%
Recipharm B	2.26%	573,334		2.63%	1.86%	6.24	1.66%
Securitas B	2.06%	470,407		3.48%	2.47%	7.89	0.03%
Sweco B	3.71%	569,900		2.63%	1.87%	6.47	0.97%
Swedish Orph	2.62%	689,083		3.96%	2.82%	7.62	0.08%
Tele2 B	2.60%	1,024,190		3.88%	2.76%	8.20	0.03%
Thule	1.99%	430,589		2.20%	1.56%	6.72	0.41%
Tomra System	1.86%	589,316		2.90%	2.06%	6.58	0.77%
VBG Group B	1.63%	400,000		12.46%	8.98%	5.88	2.65%
VBG Group BT	1.63%	400,000		12.46%	8.98%	5.88	2.65%
Vitrolife	2.74%	208,672		4.19%	2.98%	6.04	0.96%
Össur hf	2.01%	1,982,669		1.97%	1.39%	6.43	3.68%
						method of average	
				portfolio 99%VaR:	1.25%	portfolio cost of	
				portfolio 95%VaR:	0.89%	liquidity: 8.61%	
						99%LA-VaR: 9.86%	
						95%LA-VaR: 9.50%	

Table 2: the first result of a small cap fund
Where LIX_avg means the Liquidity Index got by taking the average
COL_avg means the Cost of liquidity got by taking the average

We could see from the tables 1 and 2 that both funds have similar VaR, but the small cap fund has much higher Cost of liquidity than the large cap fund, from 8.61% to 0.16%. ‘OEM International B’ presents a very high Cost of liquidity. If we check the ratio between the volume we have in the portfolio and past 20 days average daily market trading volume, ‘OEM International B’ has an ratio of 701.16, which means the share we have in the portfolio is 701.16 times of the average daily trading volume in the market. According to the empirical industry experience, selling 10% of the average daily trading volume will not affect the stock price significantly. It means that it will take 7010 trading days (26 years) to exit the position, which definitely gives out a huge liquidity risk. Therefore, dealing with large cap fund, the liquidity risk is not so significant. Dealing with small cap fund, liquidity risk is something we need to pay attention.

Secondly, since we can see from the histogram of stock returns that the distribution of the stock returns has fat tails. We used the quasi maximum likelihood VaR estimator to get more accurate VaR, and we did some experiments

of different m on both funds.
We can see from the Table 3 and 4 that when m/n is near the quantile, it gives out a more accurate estimation of VaR. For the small cap fund, since we have a newly listed company, we only have a trading history of 173 days. Instead of choosing $m = 12,13,14$, we chose $m = 8,9,10$ for small cap fund when predicting the $95\%\ VaR$. We can also see that the VaR estimation usually is a little bit higher when we consider the fat tail of the stock return distribution.

A large cap fund											
Stock	weight	Volume	m		m=2	m=3	m=4		m=12	m=13	m=14
			confidence	99%VaR	99%VaR	99%VaR	99%VaR	95%VaR	95%VaR	95%VaR	95%VaR
SWEDA	5.69%	1,130,000		2.25%	4.05%	5.26%	7.77%	1.60%	1.81%	2.00%	2.09%
NDA	7.56%	3,150,000		1.99%	3.75%	6.34%	7.38%	1.41%	2.01%	2.16%	2.36%
SEBA	4.08%	1,800,000		2.46%	4.84%	5.61%	9.18%	1.74%	2.21%	2.53%	2.77%
NETIB	2.82%	14,930,000		5.11%	5.49%	8.11%	10.97%	3.64%	3.53%	3.87%	4.13%
ATCOB	4.24%	730,000		2.72%	2.95%	3.87%	6.33%	1.93%	1.73%	1.93%	2.16%
SAND	0.53%	200,000		2.78%	2.11%	4.56%	7.09%	1.97%	1.80%	2.03%	2.18%
NOKIASEK	4.15%	4,300,000		4.56%	4.02%	6.43%	9.13%	3.25%	2.81%	3.07%	3.16%
INVEB	4.53%	573,000		1.66%	2.35%	3.89%	6.20%	1.18%	1.52%	1.70%	1.94%
ERICB	2.37%	2,100,000		3.39%	4.37%	8.38%	15.21%	2.41%	2.14%	2.58%	2.81%
VOLVB	8.24%	3,160,000		2.96%	1.35%	4.35%	7.48%	2.10%	1.78%	2.03%	2.29%
AZN	4.36%	387,000		3.39%	1.73%	3.02%	4.34%	2.41%	1.76%	1.95%	2.08%
HOLMB	2.14%	279,000		3.13%	1.96%	3.75%	5.40%	2.22%	1.45%	1.56%	1.73%
ELUX	5.02%	978,000		2.95%	3.53%	5.57%	8.23%	2.09%	1.98%	2.16%	2.43%
Tel2B	3.93%	2,250,000		3.87%	3.11%	5.20%	6.96%	2.75%	1.46%	1.72%	1.96%
STER	3.40%	1,850,000		4.89%	4.52%	4.80%	7.92%	3.48%	2.45%	2.63%	2.88%
NOBI	3.54%	1,180,000		4.24%	1.67%	4.96%	8.44%	3.02%	2.26%	2.45%	2.73%
SKFB	4.41%	1,665,000		3.39%	2.25%	5.11%	8.08%	2.41%	1.97%	2.26%	2.57%
SOBI	4.05%	1,550,000		3.95%	4.74%	5.77%	8.15%	2.81%	2.84%	3.03%	2.78%
SAABB	4.07%	498,800		1.95%	2.44%	4.20%	5.84%	1.38%	1.52%	1.75%	1.93%
HMB	9.08%	1,667,000		3.90%	3.04%	5.28%	6.82%	2.77%	2.24%	2.39%	2.61%
ALIV	3.54%	172,000		4.69%	5.81%	5.19%	8.71%	3.34%	1.93%	2.25%	2.30%
			Portfolio:	1.34%	1.48%	2.90%	4.80%	0.95%	1.22%	1.40%	1.56%

Table 3: the second result of a large cap fund

A small cap fund											
Stock	weight	Volume	m		m=2	m=3	m=4		m=8	m=9	m=10
			confidence	99%VaR	99%VaR	99%VaR	99%VaR	95%VaR	95%VaR	95%VaR	95%VaR
Alimak	3.50%	825,048		4.22%	4.88%	8.75%	12.79%	3.00%	2.16%	2.53%	2.86%
Beijer Alma	1.18%	148,964		2.72%	3.41%	5.35%	7.61%	1.93%	1.78%	1.95%	2.22%
Beijer Ref	4.43%	631,118		1.95%	4.94%	7.72%	11.11%	1.38%	1.77%	1.90%	2.28%
Bonava B	4.90%	1,086,258		3.51%	4.38%	6.29%	8.87%	2.50%	1.81%	2.06%	2.33%
Bravida	2.67%	1,393,514		2.13%	5.26%	7.80%	11.55%	1.51%	2.13%	2.50%	2.60%
Dometic	2.96%	1,316,241		1.97%	4.68%	7.47%	10.22%	1.40%	2.14%	2.54%	2.93%
Duni	2.63%	633,381		2.20%	4.02%	6.64%	9.33%	1.56%	1.92%	2.24%	2.45%
Fabege	5.98%	1,169,424		2.25%	3.99%	6.22%	9.01%	1.60%	1.87%	1.97%	2.22%
Fagerhult	5.31%	561,274		4.35%	6.68%	9.62%	12.95%	3.10%	3.67%	4.12%	4.56%
Holmen B	2.05%	184,321		3.13%	3.95%	7.01%	11.48%	2.22%	1.37%	1.63%	1.91%
Intrum Jus	2.72%	275,349		4.26%	4.66%	9.44%	14.52%	3.03%	1.69%	2.09%	2.14%
KappAhl	1.18%	707,747		3.48%	4.21%	6.58%	8.93%	2.48%	2.10%	2.36%	2.62%
Kungsleden	2.71%	1,481,116		3.02%	4.38%	6.68%	8.95%	2.14%	2.11%	2.37%	2.67%
Lindab	3.18%	1,280,952		3.26%	6.75%	13.69%	22.49%	2.31%	2.31%	2.86%	3.24%
Mekonomen	3.41%	590,640		2.87%	4.79%	9.44%	14.97%	2.04%	2.15%	2.41%	2.66%
Munksjö	3.25%	738,815		3.32%	3.34%	4.94%	6.68%	2.36%	2.19%	2.47%	2.76%
NCC B	1.72%	242,730		3.02%	4.04%	7.35%	10.59%	2.15%	1.62%	1.97%	2.33%
NetEnt B	2.44%	1,020,000		3.41%	6.48%	11.71%	18.97%	2.43%	2.41%	2.95%	3.46%
Nobia	2.51%	899,407		4.25%	5.66%	13.18%	19.83%	3.02%	2.24%	2.69%	3.22%
OEM Interna	6.68%	1,302,055		2.01%	4.10%	5.91%	8.02%	1.42%	2.26%	2.49%	2.74%
RaySearch I	2.96%	402,157		8.14%	5.21%	7.78%	10.65%	5.83%	2.80%	3.21%	3.60%
Recipharm I	2.26%	573,334		2.63%	4.35%	7.08%	9.67%	1.86%	2.04%	2.44%	2.69%
Securitas I	2.06%	470,407		3.48%	5.85%	7.86%	12.12%	2.47%	1.42%	1.77%	2.08%
Sweco B	3.71%	569,900		2.63%	3.92%	6.74%	10.19%	1.87%	1.53%	1.86%	2.12%
Swedish Or	2.62%	689,083		3.96%	5.05%	7.96%	11.05%	2.82%	2.08%	2.40%	2.81%
Tele2 B	2.60%	1,024,190		3.88%	5.18%	7.91%	15.75%	2.76%	1.48%	1.70%	2.01%
Thule	1.99%	430,589		2.20%	4.80%	8.62%	12.49%	1.56%	1.97%	2.06%	2.42%
Tomra Syst	1.86%	589,316		2.90%	3.74%	5.61%	7.55%	2.06%	2.05%	2.09%	2.37%
VBG Group I	1.63%	400,000		12.46%	8.25%	38.55%	73.25%	8.98%	1.82%	2.30%	3.17%
VBG Group I	1.63%	400,000		12.46%	8.25%	38.55%	73.25%	8.98%	1.82%	2.30%	3.17%
Vitrolife	2.74%	208,672		4.19%	8.84%	14.31%	23.03%	2.98%	3.15%	3.82%	4.14%
Össur hf	2.01%	1,982,669		1.97%	3.45%	5.46%	7.38%	1.39%	2.00%	2.18%	2.35%
			Portfolio:	1.25%	2.43%	4.76%	6.92%	0.89%	0.87%	1.07%	1.27%

Table 4: the second result of a small cap fund

In the end, we decided to combine the quasi maximum likelihood VaR estimator and LIX estimated by taking the average to get the final liquidity adjusted VaR for the portfolio.

A small cap fund							
Stock	weight	Volume		99%VaR	95%VaR	LIX	COL
Alimak	3.50%	825,048		4.88%	2.16%	6.34	1.90%
Beijer Alm	1.18%	148,964		3.41%	1.78%	5.69	1.53%
Beijer Ref	4.43%	631,118		4.94%	1.77%	4.96	34.81%
Bonava B	4.90%	1,086,258		4.38%	1.81%	6.96	0.59%
Bravida	2.67%	1,393,514		5.26%	2.13%	6.85	0.98%
Dometic	2.96%	1,316,241		4.68%	2.14%	7.52	0.20%
Duni	2.63%	633,381		4.02%	1.92%	5.91	3.89%
Fabege	5.98%	1,169,424		3.99%	1.87%	7.42	0.22%
Fagerhult	5.31%	561,274		6.68%	3.67%	5.54	8.14%
Holmen B	2.05%	184,321		3.95%	1.37%	6.74	0.17%
Intrum Jus	2.72%	275,349		4.66%	1.69%	7.22	0.08%
KappAhl	1.18%	707,747		4.21%	2.10%	6.79	0.58%
Kungsleden	2.71%	1,481,116		4.38%	2.11%	7.40	0.29%
Lindab	3.18%	1,280,952		6.75%	2.31%	6.87	0.87%
Mekonomen	3.41%	590,640		4.79%	2.15%	6.32	1.43%
Munksjö	3.25%	738,815		3.34%	2.19%	5.74	6.80%
NCC B	1.72%	242,730		4.04%	1.62%	7.40	0.05%
NetEnt B	2.44%	1,020,000		6.48%	2.41%	7.49	0.17%
Nobia	2.51%	899,407		5.66%	2.24%	7.15	0.32%
OEM Intern	6.68%	1,302,055		4.10%	2.26%	4.88	85.25%
RaySearch l	2.96%	402,157		5.21%	2.80%	5.95	2.24%
Recipharm l	2.26%	573,334		4.35%	2.04%	6.24	1.66%
Securitas l	2.06%	470,407		5.85%	1.42%	7.89	0.03%
Sweco B	3.71%	569,900		3.92%	1.53%	6.47	0.97%
Swedish Or	2.62%	689,083		5.05%	2.08%	7.62	0.08%
Tele2 B	2.60%	1,024,190		5.18%	1.48%	8.20	0.03%
Thule	1.99%	430,589		4.80%	1.97%	6.72	0.41%
Tomra Syst	1.86%	589,316		3.74%	2.05%	6.58	0.77%
VBG Group l	1.63%	400,000		8.25%	1.82%	5.88	2.65%
VBG Group l	1.63%	400,000		8.25%	1.82%	5.88	2.65%
Vitrolife	2.74%	208,672		8.84%	3.15%	6.04	0.96%
Össur hf	2.01%	1,982,669		3.45%	2.00%	6.43	3.68%
	portfolio 99%VaR: 2.43%		portfolio cost of liquidity: 8.61%			99%LA-VaR: 11.04%	
	portfolio 95%VaR: 0.87%					95%LA-VaR: 9.48%	

Table 5: the final result of a small cap fund

We can see from the table 5 and 6 that the large cap 99% Liquidity Adjusted VaR is 1.64%, which 1.48% comes from VaR and 0.16% comes from liquidity risk, the small cap 99% Liquidity Adjusted VaR is 11.04%, which 2.43% comes from VaR and 8.61% comes from liquidity risk. By comparing the stock LIX in two funds, we can observe that the stocks in the small cap fund are more illiquid than the stocks in the large cap fund. It also explains the reason why the small cap fund suffers higher liquidity risk than the large cap fund.

A large cap fund							
Stock	weight	Volume		99%VaR	95%VaR	LIX	COL
SWEDA	5. 69%	1, 130, 000		4. 05%	1. 81%	8. 20	0. 036%
NDA	7. 56%	3, 150, 000		3. 75%	2. 01%	8. 72	0. 030%
SEBA	4. 08%	1, 800, 000		4. 84%	2. 21%	8. 58	0. 024%
NETIB	2. 82%	14, 930, 000		5. 49%	3. 53%	7. 47	2. 534%
ATCOB	4. 24%	730, 000		2. 95%	1. 73%	7. 66	0. 080%
SAND	0. 53%	200, 000		2. 11%	1. 80%	8. 39	0. 004%
NOKIASEK	4. 15%	4, 300, 000		4. 02%	2. 81%	7. 80	0. 339%
INVEB	4. 53%	573, 000		2. 35%	1. 52%	7. 90	0. 036%
ERICB	2. 37%	2, 100, 000		4. 37%	2. 14%	8. 76	0. 018%
VOLVB	8. 24%	3, 160, 000		1. 35%	1. 78%	8. 56	0. 044%
AZN	4. 36%	387, 000		1. 73%	1. 76%	7. 58	0. 051%
HOLMB	2. 14%	279, 000		1. 96%	1. 45%	6. 74	0. 254%
ELUX	5. 02%	978, 000		3. 53%	1. 98%	7. 84	0. 071%
Tel2B	3. 93%	2, 250, 000		3. 11%	1. 46%	8. 20	0. 072%
STER	3. 40%	1, 850, 000		4. 52%	2. 45%	7. 44	0. 333%
NOBI	3. 54%	1, 180, 000		1. 67%	2. 26%	7. 15	0. 413%
SKFB	4. 41%	1, 665, 000		2. 25%	1. 97%	8. 17	0. 056%
SOBI	4. 05%	1, 550, 000		4. 74%	2. 84%	7. 62	0. 184%
SAABB	4. 07%	498, 800		2. 44%	1. 52%	7. 14	0. 182%
HMB	9. 08%	1, 667, 000		3. 04%	2. 24%	8. 41	0. 032%
ALIV	3. 54%	172, 000		5. 81%	1. 93%	7. 26	0. 048%
	portfolio 99%VaR: 1. 48%		portfolio cost of			99%LA-VaR: 1. 64%	
	portfolio 95%VaR: 1. 22%		liquidity: 0. 16%			95%LA-VaR: 1. 38%	

Table 6: the final result of a large cap fund

4.2 Conclusion

In this thesis, the Liquidity adjusted VaR model managed to avoid the weakness of the normal VaR model by incorporating the liquidity risk. It makes the result more realistic since it is considered that the real price of transaction deviates from the mid price of the spread. Due to the limited data we accessed, we used the LIX to predict the liquidity and spread of the stock. It yields results that match the industry experience. By comparing the cost of liquidity, we managed to distinguish liquid portfolio and illiquid portfolio. So the Liquidity adjusted VaR reflected the market risk more precisely.

For a future work, if we have access to the limit order book data, we can use it to predict the liquidity and spread from another aspect. Then we can compare the results we get from both aspects.

Appendix:

The MATLAB code

```
% Peiyu Wang master thesis
clear all
close all
clc

%load the data from excel
load StockReturnHistory
load PriceHistory
load weight
load LIX_history
load market_trading_volume
load Volume_in_portfolio

%%Model of normal VaR
T=90; %The history period
lamda=0.94; %The decay factor
a=norminv(0.01);%The 1% or 5% left tail of normal distribution
N=21;%how many stocks in the portfolio

%Calculate the exponential weight
w=(1-lamda)/(1-lamda^T)*lamda.^(0:T-1);
%We get the last 90 days stock daily return
SRhistory=StockReturnHistory(end:-1:end-(T-1),:);
%Calculate the mean of the daily return of all the stocks
E_sr=mean(SRhistory);
M_sr=SRhistory-E_sr;
%Calculate the Exponentially weighted moving average sigma
sigma_sr=sqrt(w*M_sr.^2);

%Then we can estimate the percentage VaR for each stock
VaR=1-exp(a*sigma_sr);

%Calculate the exponentially weighted Covarience matrix V
V=zeros(N,N);
for i=1:N
    for j=1:N
        V(i,j)=w*(M_sr(:,i).*M_sr(:,j));
    end
end
```

```
sigma_portfolio=sqrt(weight'*V*weight);
Var_portfolio=1-exp(a*sigma_portfolio)

%for write the result into the excel
%xlswrite('weight','VaR','C1:C21');

%%Model of Quasi maximum likelihood VaR
m=14;
%rearrange the Stock return history with one year
Rear_Ln_dailyreturn=sort(StockReturnHistory);
%rearrange the Stock return history with 90 days
RA_SRhistory=sort(SRhistory);
al=zeros(m,21);
for i=1:m
    al(i,:)=Rear_Ln_dailyreturn(i,:)./Rear_Ln_dailyreturn(m+1,:);
    %al(i,:)=RA_SRhistory(i,:)./RA_SRhistory(m+1,:)
end
%calculate the one over alpha, which alpha is the tail index
one_over_alpha=sum(al)/m;
%calculate the quasi maximum likelihood VaR estimator
QML_VaR=-Rear_Ln_dailyreturn(m+1,:).*(m/262/0.05).^one_over_alpha;
%QML_VaR=RA_SRhistory(m+1,:).*(m/T/0.01).^one_over_alpha;

%calculate the history portfolio daily return
Por_Rhistory=StockReturnHistory*weight;
%rearrange the Portfolio return history with one year
Rear_Por_Rhistory=sort(Por_Rhistory);
bl=zeros(m,1);
for i=1:m
    bl(i)=Rear_Por_Rhistory(i)./Rear_Por_Rhistory(m+1);
end
%calculate the one over alpha for the portfolio, which alpha is the tail
index
Por_one_over_alpha=sum(bl)/m;
%calculate the quasi maximum likelihood VaR estimator for portfolio
Por_QML_VaR=-Rear_Por_Rhistory(m+1).*(m/262/0.05).^Por_one_over_alpha
;

%%Model of Cost of liquidity

% 1.by taking the past 20 days average

%set the scale coefficient
A=1/10;
```

```
%We get the last 20 days LIX
Lhistory=LIX_history(end:-1:end-19,:);
%get the past 20 days average LIX
LIX_average=mean(Lhistory);
%Calculate the cost of liquidity
COL_average=A.*Volume_in_portfolio'./(2*10.^LIX_average);
COL_average_port=COL_average*weight

% 2.by assuming normal distribution

%We get the last 90 days LIX
LIhistory=LIX_history(end:-1:end-(T-1),:);
%Calculate the mean of LIX of all the stocks
E_LIX=mean(LIhistory);
M_LIX=LIhistory-E_LIX;
%Calculate the sigma of LIX of all the stocks
w_LIX=ones(1,T)*1/T;
sigma_LIX=sqrt(w_LIX*M_LIX.^2);
%Then we can estimate the LIX for each stock
LIX_nor=E_LIX+a*sigma_LIX;
%Calculate the cost of liquidity
COL_nor=A.*Volume_in_portfolio'./(2*10.^LIX_nor);
COL_nor_port=COL_nor*weight

%draw histogram of stock returns in comparison with normal distribution
% for i=1:21
%     figure('Name','stock returns in comparison with normal
distribution')
%
histogram(SRhistory(:,i),[-0.08:0.005:0.08],'Normalization','pdf')
%     hold on
%     y = -0.08:0.005:0.08;
%     mu = mean(SRhistory(:,i));
%     sigma=sqrt((ones(1,T)*1/T)*(SRhistory(:,i)-mu).^2);
%     f = exp(-(y-mu).^2./(2*sigma^2))./(sigma*sqrt(2*pi));
%     title('histogram of stock returns in comparison with normal
distribution')
%     xlabel('stock returns')
%     ylabel('density')
%     plot(y,f,'LineWidth',1.5)
% end
```

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