# M 172 - Calculus II - Chapter 10
# Sequences and Series

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# Contents

# 10 Sequences and Series

## 10.1 Sequences

**Dichotomy Paradox, Zeno 490-430 BC:** To travel a distance of 1, first one must travel 1/2, then half of what remains, i.e. 1/4, then half of what remains, i.e. 1/8, etc. Since the sequence is infinite, the distance cannot be traveled.

*Remark.* The steps are terms in the **sequence**.

$$\left\{ \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots \right\}$$

Sequences of values of this type is the topic of this first section.

*Remark.* The sum of the steps forms an infinite series, the topic of Section 10.2 and the rest of Chapter 10.

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

We will need to be careful, but it turns out that we can indeed walk across a room!

**Definition 10.1.1.** A **sequence** is a function with domain $\mathbb{N} = \{1, 2, 3, \ldots\}$, the Natural Numbers.

**Examples and Notation:**

**Definition 10.1.2.** We say the sequence $\{a_n\}$ converges to $L$ and write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if for every $\epsilon > 0$, there exists $M$ such that $|a_n - L| < \epsilon$ when $n > M$. If the limit does not exist, we say the sequence diverges. If $L = \infty$, we say the sequence diverges to infinity.

**Examples:**

**Theorem 10.1.1.** *If* $\lim_{x\to\infty} f(x) = L$, *then* $\lim_{n\to\infty} f(n) = L$.

*Remark.* The implication does not work the other direction, i.e.
$\lim_{n\to\infty} f(n) = L \not\Rightarrow \lim_{x\to\infty} f(x) = L$, for example:

**Example 10.1.1.** Show $\left\{ \dfrac{\ln n}{n} \right\}$ converges.

*Remark.* Do not apply L'Hopital's Rule to terms of a sequence. Sequences are not differentiable functions, not even continuous.

Many previous results regarding limits apply in the sequence case as well. For convenience they are summarized here.

**Theorem 10.1.2.** *If* $\lim_{n\to\infty} a_n = A$ *and* $\lim_{n\to\infty} b_n = B$ *then*

- $\lim_{n\to\infty} (a_n + b_n) = A + B$,
- $\lim_{n\to\infty} (a_n b_n) = AB$,
- $\lim_{n\to\infty} \left(\dfrac{a_n}{b_n}\right) = \dfrac{A}{B}$ *provided* $B \neq 0$, *and*
- $\lim_{n\to\infty} (ca_n) = cA$ *for any constant* $c$.

**Theorem 10.1.3** (Squeeze Theorem)**.** *If* $a_n \leq b_n \leq c_n$ *for* $n \geq M$, $\lim_{n\to\infty} a_n = L$, *and* $\lim_{n\to\infty} c_n = L$, *then* $\lim_{n\to\infty} b_n = L$.

**Example 10.1.2.** Since $-|a_n| \leq a_n \leq |a_n|$, by the Squeeze Theorem, if $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

**Example 10.1.3.** Use the Squeeze Theorem to show $\left\{\dfrac{3^n}{n!}\right\}$ converges.

One more result from earlier is useful for us.

**Theorem 10.1.4.** *If $f$ is continuous and $a_n \to L$ as $n \to \infty$, then $f(a_n) \to f(L)$ as $n \to \infty$.*

**Example 10.1.4.** Find $\lim_{n\to\infty} \ln\left(\frac{n}{2n+1}\right)$.

**Definition 10.1.3. Bounded.**

- $\{a_n\}$ is **bounded above** if there exist $M$ such that $a_n \leq M$ for all $n$.
- $\{a_n\}$ is **bounded below** if there exist $N$ such that $a_n \geq N$ for all $n$.
- $\{a_n\}$ is **bounded** if it is bounded above and below.

**Examples:**

**Definition 10.1.4. Monotone.**

- $\{a_n\}$ is **increasing** if $a_n \leq a_{n+1}$ for all $n$.
- $\{a_n\}$ is **decreasing** if $a_n \geq a_{n+1}$ for all $n$.
- $\{a_n\}$ is **monotone** if either of the above hold.

**Examples:**

The two previous definitions are useful by themselves, but combined they give us the one Big Gun of the theory of sequences.

**Theorem 10.1.5** (Monotone Convergence Theorem)**.** *If $\{a_n\}$ is bounded and monotone, then it converges.*

We finish our survey of sequences with five limits, many of which will prove useful for the remainder of this chapter.

**Theorem 10.1.6.** *Let $p > 0$.*

1. $\displaystyle\lim_{n\to\infty} \frac{1}{n^p} = 0$
2. $\displaystyle\lim_{n\to\infty} \sqrt[n]{p} = 1$
3. $\displaystyle\lim_{n\to\infty} \sqrt[n]{n} = 1$
4. *If* $|r| < 1$, $\displaystyle\lim_{n\to\infty} r^n = 0$.
5. $\displaystyle\lim_{n\to\infty} \left(1 + \frac{c}{n}\right)^n = e^c$

*Remark.* Parts 2 and 3 imply that if $P(n)$ is any polynomial in $n$, then $\sqrt[n]{|P(n)|} \to 1$ as $n \to \infty$. This will be particularly useful in an upcoming section.

*Selected proofs.*

## 10.2 Introduction to Series

We return to the Dichotomy Paradox. It is clear that I can walk across the room, so we should be able to do something like this.

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=\sum_{n=1}^{\infty}\frac{1}{2^n}=1$$

Of course, we will need to formalize this. We start with the standard sigma notation for finite sums

$$\sum_{n=1}^{N} a_n = a_1 + a_2 + \cdots + a_N.$$

For example,

We will define the **Nth partial sum** to be $S_N = \sum_{n=?}^{N} a_n$.

*Remark.* The starting index is not particularly important. Although many examples will start at $n = 1$ or $n = 0$, there is no reason it cannot start at $n = 47$. Additionally, there is nothing special about the index $n$. We often use $i$, $j$, $k$, or $m$ as well.

For example, the 5th partial sum for the following series are

$$S_5 = \sum_{k=2}^{5} k =$$

$$S_5 = \sum_{k=1}^{5} \frac{1}{k^2} =$$

$$S_5 = \sum_{k=4}^{5} \frac{1}{k^3-1} =$$

We would like to define an **infinite series**, i.e.

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$

The standard 172 idea holds, approximate with partial sums and take the limit.

**Definition 10.2.1.** Let $S_N = \sum_{n=1}^{N} a_n$ be the sequence of Nth partial sums.

- If $\lim_{n\to\infty} S_n = S$, we say $\sum_{n=1}^{\infty} a_n$ **converges** to $S$ and write $\sum_{n=1}^{\infty} a_n = S$.
- If $\lim_{n\to\infty} S_n = \infty$, we say $\sum_{n=1}^{\infty} a_n$ **diverges** to infinity and write $\sum_{n=1}^{\infty} a_n = \infty$.
- If $\lim_{n\to\infty} S_n$ does not exist, we say $\sum_{n=1}^{\infty} a_n$ **diverges**.

One type of series that we can approach using the definition directly are telescoping series.

**Example 10.2.1.** Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{n(n+2)}$.

There is one additional type of series that we can use the definition directly for, they are the topic of the following section. For now, we turn our attention to one issue of theoretical importance and finally one fundamental example.

**Theorem 10.2.1.** *If $\sum a_k$ converges, $a_k \to 0$ as $k \to \infty$.*

*Remark.* Convergence or divergence of a series depends on the behavior of the tail of the series, i.e. we can throw away the first ten, or hundred, or even million terms and not change the convergence of the series. If the index is immaterial to the topic at hand, as in the theorem above, we will often supress the notation for convenience.

*Proof.*

The above theorem is typically used in its contrapositive form.

**Theorem 10.2.2** (Test for Divergence). *If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges.*

*Remark.* We are saying that the terms going to zero is a **necessary** condition for the convergence of a series. However, as the next example shows, it is not a **sufficient** condition.

**Example 10.2.2.** The **Harmonic Series diverges**, i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \infty.$$

*Proof.*

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## 10.3 Geometric Series

An important, perhaps the most important, type of series is the geometric series. We have already seen one example, our walk across the room.

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1$$

Before we dive into the general theory, we should look closely at this example. As before, we consider the Nth partial sum $S_N$.

**Theorem 10.3.1** (Geometric Series)**.** *For $c \neq 0$, $\sum_{n=0}^{\infty} cr^n = \frac{c}{1-r}$ for $|r| < 1$ and diverges otherwise.*

*Proof.*

*Remark.* The more general result is

$$\sum cr^n = \frac{\text{first term}}{1-r}$$

for $|r| < 1$.

**Examples:**

$$\sum_{n=0}^{\infty} \frac{3}{2^n} = 3 + \frac{3}{2} + \frac{3}{4} + \cdots =$$

$$\sum_{n=3}^{\infty} \frac{3}{2^n} = \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \cdots =$$

$$\frac{1}{9} + \frac{8}{9^2} + \frac{64}{9^3} + \cdots =$$

$$\frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots =$$

$$\sum_{n=0}^{\infty} \frac{1 + (-2)^n}{3^{2n}}$$

We will need additional machinery to deal with this last example.

Since finite sums and limits are both linear, so are series.

**Theorem 10.3.2** (Linearity of Series)**.** *Assume the following series are convergent, then*

- $\sum ca_n = c \sum a_n$, *and*
- $\sum (a_n + b_n) = \sum a_n + \sum b_n$.

We can now return to the example from the previous page and a similar example.

$$\sum_{n=0}^{\infty} \frac{1 + (-2)^n}{3^{2n}}$$

$$\sum_{n=0}^{\infty} \frac{3^n + (-4)^n}{5^n}$$

*Remark.* The assumption that all the series converged in the theorem is necessary. For example,

$$0 = \sum 0 = \sum (1 - 1) \neq \sum 1 - \sum 1$$

Since neither of the last series converge.

In our discussion of geometric series, the common ratio $r$ was constant. What happens if we let $r$ vary?

**Example 10.3.1.** Find the values of $x$ for which the following series converges and find what it converges to.

$$\sum_{n=0}^{\infty} x^n$$

**Example 10.3.2.** Find the values of $x$ for which the following series converges and find what it converges to.

$$\sum_{n=0}^{\infty} \frac{2(-1)^n x^{2n}}{4^n}$$

*Remark.* The two series on this page are representations of functions. They are examples of series we will refer to to as power series, the topic section 10.5.

**Homework** From section 10.2 in the text, # 23, 25, 27, 29, 33, 39, 43, 47, 49, 57

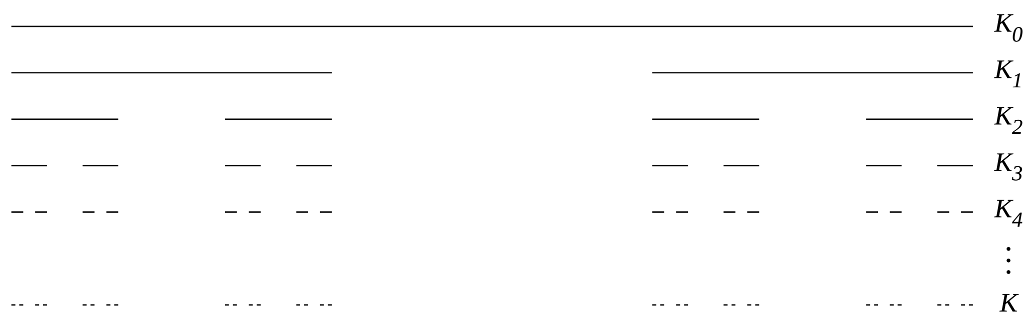


Figure 10.1: Middle third Cantor set

## 10.4 Fractal Dimension

In this brief diversion we consider the question, how big is a fractal?

We begin by discussing four elementary examples that will give us a framework for the upcoming discussion. The examples are classic constructions that date from the late nineteenth and early twentieth century.

One of the simplest fractals to construct is the middle third Cantor set, named for the German mathematician Georg Cantor, see Figure 10.1. Let $K_0$ be the unit interval $[0,1]$. Remove the open middle third $(1/3,2/3)$ to obtain $K_1 = [0,1/3] \cup [2/3,1]$. Removing the open middle third of each remaining interval gives $K_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$. Continuing in this manner, we construct the middle third Cantor set

$$K := \lim_{i \to \infty} K_i$$

Alternatively we could view the construction at each step not as removing intervals, but rather as replacing each previous interval with two intervals of one third the length.

The second example we consider is the von Koch curve, often called a snowflake curve, named after the Swedish mathematician Helge von Koch. While the Cantor set removed an interval at each iteration, the von Koch curve replaces each middle third with two segments of the same length. In essence, replacing a segment with the two other sides of an equilateral triangle. Alternatively, we

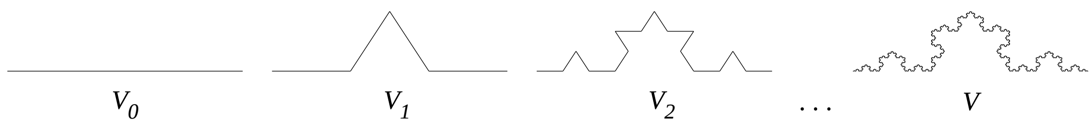


Figure 10.2: von Koch curve

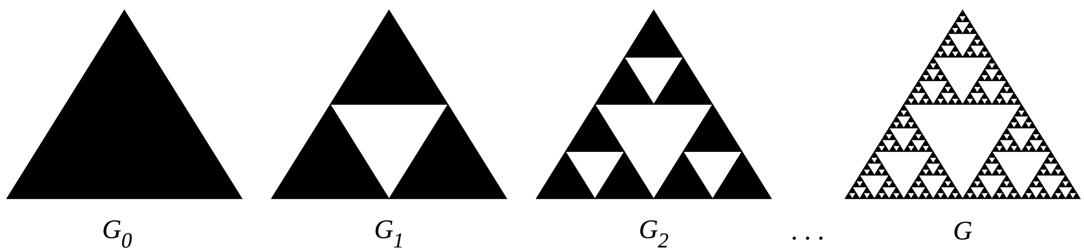


Figure 10.3: Sierpiński gasket

can view the construction as replacing each interval with four intervals of one third the length, see Figure 10.2.

We will soon see that the Cantor set has zero length. Curiously, the snowflake curve has infinite length. Nonetheless, the curve has zero area in the plane. It will be useful to find a measure of the size of this curve, and similar objects, that is a more useful measurement than infinite length or zero area. One such measure of size is the dimension, which we will discuss soon. However, two additional examples are worth familiarizing ourselves with before we dive in.

The final two examples are similar to the construction of the Cantor set and both due to the Polish mathematician Wacław Sierpiński. The Sierpiński gasket is constructed from an equilateral triangle. At each iteration we divide the triangle, or triangles, into four congruent subtriangles and remove the central subtriangle, see Figure 10.3. A Sierpiński carpet is constructed from a square. At each iteration we divide the square, or squares, into nine congruent subsquares and remove the central subsquare, see Figure 10.4.

We will soon see that the area of the gasket and the carpet are both zero. Again we see that area is a poor measure of these objects. A more useful analytic tool

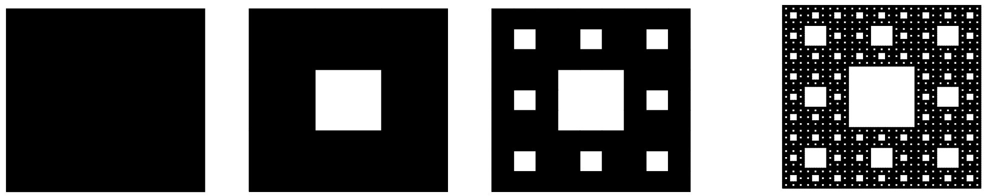


Figure 10.4: Sierpiński carpet

for these types of fractals is their dimension.

Before we consider the concept of dimension, we should verify the claims made earlier.

**Exercise 10.4.1.** Show the length of the Cantor set is 0.

This is homework.

**Exercise 10.4.2.** Show the lenght of the von Koch curve is infinite.

*Solution.* It is clear the line segment $V_0$ has length 1. At the next iteration we replace the segment by four pieces one third the length, so the length of $V_1$ is $(4/3)$. The length grows by this factor at each step, so $V_2$ has length $(4/3)^2$, $V_3$ has length $(4/3)^3$, and in general $V_n$ has length $(4/3)^n$. Since $(4/3)^n \to \infty$ as $n \to \infty$ we see the length of the von Koch curve $V$ is infinite. This is an example of a divergent geometric sequence, see Theorem 10.1.6.

**Exercise 10.4.3.** Show the area of the Sierpiński gasket is 0.

This is homework.

**Exercise 10.4.4.** Show the area of the Sierpiński carpet is 0.

*Solution.* The area can be computed by subtracting the removed squares from the total area. Conviently, the area of removed squares form a geometric series.

$$\text{Area} = 1 - \left(\frac{1}{9} + \frac{8}{9^2} + \frac{8^2}{9^3} + \cdots\right) = 1 - \frac{1/9}{1 - 8/9} = 1 - 1 = 0$$

For these types of mathematical toys, and many real world objects that have self-simliar structures at various scales, a useful measure is the dimension. There are many concepts of dimension, we will discuss a very basic version.

**What is dimension?**

Roughly,

$$\text{number of pieces} = (1/(\text{`size' of pieces}))^{\text{dimension}}$$

Solving for the dimension in the previous gives.

$$\dim X = \frac{\ln(\text{number of pieces})}{-\ln(\text{'size' of pieces})}$$

Or formally, if $X$ is self similar shape made of $N$ copies of itself, each scaled by a similarity with contraction factor $r$ then we define the **similarity dimension** as

$$\dim_S X = \frac{\ln N}{-\ln r}$$

The examples we have seen have the following similarity dimensions.

- Cantor set $K$, $\dim_S K = \frac{\ln 2}{\ln 3} \approx 0.63$
- von Koch curve $V$, $\dim_S V = \frac{\ln 4}{\ln 3} \approx 1.26$
- Sierpiński gasket$G$, $\dim_S G = \frac{\ln 3}{\ln 2} \approx 1.58$
- Sierpiński carpet $S$, $\dim_S S = \frac{\ln 8}{\ln 3} \approx 1.89$

Although similarity dimension is very easy to compute for our examples, it is not very flexible. For more complicated objects, mathematical or real world, other more rigorous methods are needed. But, we are already beyond the scope of the class, so we should return to the topic at hand.

## 10.5 Power Series

An important application of series is that of a power series, i.e. a way to represent a function by a series. In essence, turning a function like $e^x$ into something computable. Although it is useful to think of $e^x$ as the function that is its own derivative and that has slope 1 when $x = 0$, that is hard to use from a computational point of view.

We saw two examples of power series at the end of section 10.3. The most important being

$$F(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots \quad \text{for } |x| < 1. \tag{10.1}$$

The goal of this chapter is to be able to find series representations for functions. In essence, finding 'infinite polynomial' representations of functions.

**Definition 10.5.1.** A **power series** with center $c$ is a series of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

In the power series (10.1) above, $a_n = 1$ for all $n$ and the center is $c = 0$. One thing to note is that the series only converges for $|x| < 1$. This type of restriction on $x$ is typical.

**Theorem 10.5.1** (Radius of Convergence)**.** *Every power series*

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

*has a* **radius of convergence** $R$ *with* $R = 0, R > 0$, *or* $R = \infty$.

- *If* $R = 0$, *the series only converges at* $x = c$.
- *If* $R > 0$, *the series converges for* $|x - c| < R$.
- *If* $R = \infty$, *the series converges for all* $x \in \mathbb{R}$.

We will be interested in determining the radius of convergence for power series soon, but we start with generating some power series based off of (10.1).

**Games we can play with power series.**

1. **Multiplication.** Find a power series representation for

$$f(x) = \frac{2x}{1-x}.$$

2. **Substitution.** Find a power series representation for

$$g(x) = \frac{1}{1+x^2}.$$

3. **Shifting the center.** Find a power series representation centered at $x = 2$ for
$$h(x) = \frac{1}{1-x}.$$

One reason power series are so useful is that they are very easy to differentiate and integrate.

**Theorem 10.5.2.** *Assume*

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots$$

*has radius of convergence $R$. Then $F$ is differentiable and integrable on $(c-R, c+R)$ with*

$$F'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

*and, for any constant $A$*

$$\int F(x)\,dx = A + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-c)^{n+1}.$$

*Furthermore, all three series have the same radius of convergence.*

**Example 10.5.1.** Find a power series representation for $\ln(1 + x)$.

**Example 10.5.2.** Find a power series representation for $\arctan x$.

*Remark.* Although the radius of convergence does not change when we integrate, it is possible that the **interval of convergence** does. We will discuss the convergence at endpoints soon, but for now we simply note that the representation of $\arctan x$ is valid for $x \in [-1, 1]$ which gives the following curious result.

$$\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

**Example 10.5.3.** Find a power series representation for $e^x$.

We start by a quick diversion into differential equations, in particular it is major result in the subject that there is a unique solution to an Inital Value Problem (IVP)of the following type

$$y' = y, \quad y(0) = 1$$

i.e. there is a unique function that is its own derivative with slope one when $x = 1$. The function $e^x$ clearly satifies the IVP and so is the unique solution.

We will put that knowledge on hold for a minute and search for a power series solution to the IVP. Using power series to solve differential equations is a fundamental tool in the subject.

Which leads us to the recursive relationship

$$a_n = \frac{a_{n-1}}{n}. \tag{10.2}$$

We can choose $a_0$ arbitrarily, and use (10.2) to find the other coefficients in terms of $a_0$.

We conclude that

$$F(x) = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We also want it to satisfy the inital data, i.e. $F(0) = 1$. Since $F(0) = a_0$ we conclude that $a_0 = 1$ and

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{10.3}$$

We are left with one major concern; for what values of $x$ does this series converge? That question will be the topic of the following section.

**Homework.**

**Exercise 10.5.4.** True / False

1. **T / F :** If $\sum a_n(x-3)^n$ converges for $x=5$ it converges for $x=0$.
2. **T / F :** If $\sum a_n(x+3)^n$ converges for $x=5$ it converges for $x=0$.
3. **T / F :** There exists a power series that only converges for $x>0$.

**Exercise 10.5.5.** If a power series converges for $-4<x<2$, what is the center and radius of convergence?

**Exercise 10.5.6.** Use (10.1) to find a power series representation for the following functions. Specify where each series converges. Unless otherwise specified, center the series at $x=0$.

1. $f(x) = \dfrac{1}{1+4x}$
2. $f(x) = \dfrac{1}{8+x^3}$
3. $f(x) = \dfrac{x}{1-x}$
4. $f(x) = \dfrac{3x^2}{2+x^4}$
5. $f(x) = \dfrac{1}{1-x}$ centered at $x=4$
6. $f(x) = \dfrac{4}{2+x}$ centered at $x=2$
7. $f(x) = \dfrac{1}{(1+x)^2}$
8. $f(x) = \dfrac{1}{(1+x)^3}$

**Exercise 10.5.7.** Use (10.3) to integrate $\displaystyle\int e^{x^2}\,dx$.

## 10.6 Ratio and Root Tests

In this section we develop two tests useful for determining the convergence or divergence of series with a particular emphasis on power series. Both are generalizations of the geometric series from section 10.3.

**Theorem 10.6.1** (The Ratio Test). *Let* $\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

1. *if* $\rho < 1$, $\sum a_n$ *converges.*
2. *if* $\rho > 1$, $\sum a_n$ *diverges.*
3. *if* $\rho = 1$ *or the limit does not exist, the test is inconclusive.*

*Sketch of Proof.*

**Theorem 10.6.2** (The Root Test). *Let* $L = \lim_{n\to\infty} \sqrt[n]{|a_n|}$

1. *if* $L < 1$, $\sum a_n$ *converges.*
2. *if* $L > 1$, $\sum a_n$ *diverges.*
3. *if* $L = 1$ *or the limit does not exist, the test is inconclusive.*

*Sketch of Proof.*

**Initial Examples.** Show the following series converge.

$$\sum_{n=0}^{\infty} \left( \frac{n+1}{2n+3} \right)^n$$

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}$$

**Example 10.6.1.** In the previous section we showed that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Show that the series converges for all $x$, i.e. the radius of convergence is infinite.

Both tests are useful for determining the radius of convergence of power series. For know power series, there is no need to reinvent the wheel, but to determine the radius of convergence of an unknown series either of the tests is our first step.

There are two important points to note. First, both tests are inconclusive at the endpoints. Neither will give us any insight into the **interval of convergence**. To address that issue we will have to develop more subtle tests; that will be the topic of the next few sections. Second, although the Root Test seems trickier, we have some nice results that streamline the process; see Theorem 10.1.6. In particular...

**Power Series Examples.** Find the radius of convergence for the following.

$$\sum_{n=1}^{\infty} \frac{n^2(x-2)^{2n}}{9^n}$$

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+2}\right)^{n^2} (x+1)^n$$

**Two Additional Examples.** Find the radius of convergence for the following.

$$\sum_{n=0}^{\infty} \frac{(2x)^n}{n^2+1}$$

$$\sum_{n=0}^{\infty} \frac{n!x^n}{n^n}$$

**Homework**

In section 10.5 do #5, 11, 17, 19, 23, 29, 37, 41

**Exercise 10.6.2.** Find the radius of convergence for the following. Express your solution in the form $|x-c|<R$, where possible.

1. $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n2^n}$
2. $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n(x+4)^n}{3^n}$
3. $\displaystyle\sum_{n=0}^{\infty}\frac{2^n(x-1)^n}{n+2}$
4. $\displaystyle\sum_{n=0}^{\infty}\frac{n^2x^n}{(2n)!}$
5. $\displaystyle\sum_{n=0}^{\infty}\frac{n!x^n}{n^3+1}$
6. $\displaystyle\sum_{n=0}^{\infty}\frac{(-2)^n(x+2)^n}{3^n}$
7. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^nx^{2n}}{(2n)!}$

## 10.7 Convergence of Positive Series

In this section we do a brief survey of methods for testing convergence at endpoints of intervals of convergence.

**Theorem 10.7.1** (Integral Test)**.** *Let* $a_n = f(n)$*, where* $f$ *is positive and decreasing for* $x \geq 1$*.* $\displaystyle\int_1^\infty f(x)\,dx$ *converges if and only if* $\displaystyle\sum_{n=1}^\infty a_n$ *converges.*

*Sketch of Proof.*

**Example 10.7.1.** Convergence of p-Series.

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \text{ converges if and only if } p > 1.$$

**Theorem 10.7.2** (Direct Comparison Test)**.** *Assume* $0 \leq a_n \leq b_n$ *for* $n \geq M$*.*

1. *If* $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges.*
2. *If* $\sum_{n=1}^{\infty} a_n$ *diverges, then* $\sum_{n=1}^{\infty} b_n$ *diverges.*

**Examples.**

Show $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2n^2 - 1}$ diverges.

Show $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 3n + 2}$ converges.

**Theorem 10.7.3** (Limit Comparison Test)**.** *Let $a_n, b_n \geq 0$. Assume the following limit exists, or is $\infty$.*

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

1. *If $0 < L < \infty$, the $\sum a_n$ converges if and only if $\sum b_n$ converges.*
2. *If $L = 0$, then $\sum a_n$ converges if $\sum b_n$ converges.*
3. *If $L = \infty$, then $\sum b_n$ converges if $\sum a_n$ converges.*

The limits of 0 and $\infty$ are rarely used. Typically we try to compare with a series that is of a similar size and hence have a limit that is positive and finite, i.e. $0 < L < \infty$. Generally it is easier to write an argument for the Direct Comparison Test, but in cases where the inequalities go the wrong direction we make use of the Limit Comparison Test.

**Two examples.**

Show $\sum \sin\left(\dfrac{1}{n}\right)$ converges.

Show $\sum \frac{n+1}{2n^3-2}$ converges.

**Example 10.7.2.** Using the Ratio Test we can show that $\sum_{n=1}^{\infty} \frac{(x-2)^n}{2^n n}$ converges for $|x-2| < 2$ and diverges for $|x-2| > 2$. Does it converge when $|x-2| = 2$?

When $x = 4$ we have

When $x = 0$ we have

Our current tools do not address this situation, but the following section will.

**Exercise 10.7.3.** From section 10.3 in the text do # 79 using the Integral Test, 19, 21, 25, 31, 39, 41, 43, 47.

## 10.8 Conditional and Absolute Convergence

Initially we note that everything we discussed regarding positive series $\sum a_n$, i.e. $a_n > 0$, in the previous section applies to negative series $\sum b_n$, i.e. $b_n < 0$, after we factor out the negative (series are linear). For series with both signs, we will need additional tools.

**Definition 10.8.1.** $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges.

*Remark.* A convergent positive series is absolutely convergent by definition.

**Theorem 10.8.1.** *If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.*

*Proof.*

**Examples.**

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \cdots$$

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2+1}$$

*Remark.* To use either of the comparison tests we must have nonnegative terms, so often testing for absolute convergence is necessary.

**Example 10.8.1.**
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \cdots$$
is the **Alternating Harmonic Series**. Since the Harmonic Series diverges, the Alternating Harmonic Series does not converge absolutely. But, does it converge? This was the question we left off with in Example 10.7.2.

**Definition 10.8.2.** $\sum a_n$ **converges conditionally** if it converges, but not absolutely.

**Definition 10.8.3.** For $a_n > 0$, a series of the form $\sum(-1)^n a_n$ or $\sum(-1)^{n+1} a_n$ is an **alternating series**.

**Theorem 10.8.2.** *[Alternating Series Test] If $a_n > 0$, $a_n$ decreasing, and $a_n \to 0$ as $n \to \infty$, then $S = \sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges. Additionally, we have the following estimates for $N \geq 1$:*

1. $0 < S < a_1$,
2. $S_{2N} < S < S_{2N+1}$, *and*
3. $|S - S_N| < a_{N+1}$.

*Sketch of Proof.*

To finish Example 10.8.1 above,

**Example 10.8.2.** Find the interval of convergence for the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (x+2)^n}{(n+1)3^n}.$$

As before, we use either the Ratio or Root test to find the radius of convergence.

Now we check the endpoints individually.

For $x = -5$ we have

For $x = 1$ we have

**Example 10.8.3.** In Example 10.5.2 we found a power series representation for $\arctan x$ by integrating the power series for $1/(1+x^2)$, in particular

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \tag{10.4}$$

which was valid for $|x| < 1$. Theorem 10.5.2 assured us that the radius of convergence remained 1, but it does not tell us anything about the convergence or divergence at the endpoints of the interval. We test the two endpoints now.

At $x = 1$ we have

At $x = -1$ we have

We conclude that (10.4) is valid for $-1 \leq x \leq 1$.

**Example 10.8.4.** In Example 10.5.1 we found a power series representation for $\ln(1+x)$ by integrating the power series for $1/(1+x)$, in particular

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{10.5}$$

which was valid for $|x| < 1$. As in the previous example, we would like to check the endpoints for convergence.

At $x = 1$ we have

At $x = -1$ we have

We conclude that (10.5) is valid for $-1 < x \leq 1$. It would be a problem if the series converged at $x = -1$ since we would be computing the value of $\ln 0$.

**Example 10.8.5.** Show

$$\sum_{n=1}^{\infty} \frac{n+1}{n^2+1} \tag{10.6}$$

converges conditionally. This argument requires two parts.

First, we must show (10.6) does not converge absolutely.

Second, we must show (10.6) does converge.

**Conditional Convergence is Curious,** or, a brief bit a mathematical weirdness.

In the previous example we saw that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots .$$

We consider rearranging the terms as follows

This turns out to be the case with all conditionally convergent series, if we rearrange the terms we can change the sum, or even make the series diverge.

**Theorem 10.8.3** (Riemann Rearrangement Theorem)**.** *If $\sum a_n$ is a conditionally convergent series, then for any $M \in \mathbb{R}$ there exists a rearrangement of $\{a_n\}$ into $\{b_n\}$, i.e. a one-to-one onto mapping, such that $\sum b_n = M$. Furthermore, there is a different rearrangement of $\{a_n\}$ into $\{c_n\}$ such that $\sum c_n = \infty$.*

**Theorem 10.8.4.** *If $\sum a_n$ is absolutely convergent, then any rearrangement also converges absolutely and to the same value.*

*Remark.* Now that we have the appropriate language it should be noted that the conclusion in both the Root and Ratio test is that the series in question converges absolutely when the appropriate limit is less than one. Similarly, power series converge absolutely on their radius of convergence. Since absolute convergence implies convergence, the results as written are true, but not as strong as they are properly.

**Homework**

**Exercise 10.8.6.** **T**rue or **F**alse.

1. **T / F :** If $\sum a_n$ converges conditionally, then $\sum |a_n|$ converges.
2. **T / F :** If $\sum a_n$ converges absolutely, then $\sum |a_n|$ converges.
3. **T / F :** If $\sum |a_n|$ converges, then $\sum a_n$ converges.
4. **T / F :** If $a_n \to 0$ as $n \to \infty$, then $\sum a_n$ converges.
5. **T / F :** If $|a_n| \to 0$ as $n \to \infty$, then $\sum a_n$ converges.

**Exercise 10.8.7.** Determine if the following CONVERGE ABSOLUTELY, CONVERGE CONDITIONALLY, OR DIVERGE. Proper justification will be expected on quizzes and exams, now is a good time to practice.

1. $\sum \dfrac{\sin n \cos 5n}{n^2+1}$
2. $\sum \dfrac{(-1)^n}{\sqrt{n^2+1}}$

**Exercise 10.8.8.** Find an error estimate for $\left|\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5}\right)\right| <$________________. Hint: See Theorem 10.8.2 and Example 10.8.3.

**Exercise 10.8.9.** From section 10.6 in the text do # 11, 13, 19, 23 (Note: for $n \geq 1$, $\ln n < n$), 29

## 10.9 Intuition

In this section we concentrate on trying to build and intuition and understanding of the behavior of series. Although we should be able to craft proper arguments to verify all of the claims we will make, it is important to be able to identify a convergent series. Of equal importance, being able to identify the correct method of showing convergence or divergence.

**Methods we have discussed**.

1. Test for Divergence.

   Examples.

2. Geometric Series.

   Examples.

3. Ratio Test.

   Examples.

4. Root Test.

   Examples.

Question: For what types of series are the Root and Ratio Test inconclusive?

5. Integral Test.

   Examples.

6. Comparison Tests.

   Examples.

**Exercise 10.9.1.** For each of the following series determine if they converge or diverge and then choose a test that can be used to show that.

1. $\sum \left(1+\frac{1}{n}\right)^{-n^2}$ Converges / Diverges by the Root Test / Divergence Test.
2. $\sum \left(1+\frac{1}{n}\right)^{n}$ Converges / Diverges by the Root Test / Divergence Test.
3. $\sum \frac{1}{n \ln n}$ Converges / Diverges by the Ratio Test / Integral Test.
4. $\sum \frac{n+1}{n^2+2}$ Converges / Diverges by the LCT / Ratio Test.
5. $\sum \frac{(-1)^n}{n}$ Converges / Diverges by the Integral Test / AST.
6. $\sum (-n)^n$ Converges / Diverges by the Divergence Test / AST.

**Exercise 10.9.2.** For the following series, specify what series you would compare each to (either direct or limit comparison) and based on your comparison, decide if it converges or diverges. No formal justification is needed.

1. $\sum_{n=2}^{\infty} \frac{\sqrt{n^2+1}}{n^3+4n}$ compare to so it CONVERGES / DIVERGES
2. $\sum_{n=2}^{\infty} \frac{3}{2^n\sqrt{n}}$ compare to so it CONVERGES / DIVERGES
3. $\sum_{n=2}^{\infty} \frac{1}{n^2+\sqrt{n}}$ compare to so it CONVERGES / DIVERGES
4. $\sum_{n=2}^{\infty} \frac{\sqrt{n^3+3}}{n^2+n}$ compare to so it CONVERGES / DIVERGES
5. $\sum_{n=2}^{\infty} \frac{2^n}{n3^n}$ compare to so it CONVERGES / DIVERGES

**Exercise 10.9.3.** Choose all series below that...

1. can be shown to **converge** using the Divergence Test.

   (a) $\sum \frac{4 \ln n}{n^2 + 1}$ (b) $\sum \frac{2}{n}$ (c) $\sum \frac{1}{e^n + n^2}$

2. can be shown to **diverge** using the Divergence Test.

   (a) $\sum \frac{1}{n}$ (b) $\sum \frac{n-1}{n}$ (c) $\sum \frac{1}{\ln n}$

3. can be shown to **converge** using either Comparison Test.

   (a) $\sum \frac{1}{n \ln n}$ (b) $\sum \frac{(-1)^n}{n^2 + 1}$ (c) $\sum \frac{\ln n}{n}$

4. can be shown to **converge** using the Alternating Series Test.

   (a) $\sum \frac{(-1)^n}{\sqrt{n}}$ (b) $\sum \frac{\cos(\pi n)}{n}$ (c) $\sum \frac{(-1)^n n}{n+1}$

5. can be shown to **diverge** using the Alternating Series Test.

   (a) $\sum \frac{\cos n}{n}$ (b) $\sum \frac{(-1)^n}{n}$ (c) $\sum \frac{(-1)^n n}{n+1}$

6. can be shown to **converge** using the Ratio or Root Test.

   (a) $\sum \frac{n}{n^3 + 5n}$ (b) $\sum \frac{3^n}{2^{2n+1} + 1}$ (c) $\sum \frac{2^n}{n!}$

7. can be shown to **diverge** using the Ratio or Root Test.

   (a) $\sum \frac{n^2}{n^3 + 5n}$ (b) $\sum \frac{3^n}{2^{n+1} + 1}$ (c) $\sum \frac{n^n}{n!}$