

Created by T. Madas

CALCULUS KINEMATICS

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CALCULUS KINEMATICS IN SCALAR FORM

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Question 1 (**)

A particle P is moving on the x axis and its displacement from the origin, x m, t seconds after a given instant, is given by

$$x = \frac{1}{3}t(t^2 - 3t - 24), \quad t \geq 0.$$

Determine the displacement of P when it is instantaneously at rest.

$$\boxed{}, \quad x = -26\frac{2}{3} \text{ m}$$

Differentiating to obtain the velocity

$$x = \frac{1}{3}(t^2 - 3t - 24)$$

$$v = \frac{dx}{dt} = \frac{1}{3}(2t - 3)$$

$$v = \frac{2}{3}t - 1$$

Instantaneously at rest $\Rightarrow v = 0$

$$0 = \frac{2}{3}t - 1$$

$$\Rightarrow (t-1.5) = 0$$

$$\Rightarrow t = 1.5$$

This displacement value can now be found

$$x(1.5) = \frac{1}{3} \times 1.5 (1.5^2 - 3 \times 1.5 - 24) = -26\frac{2}{3}$$

Question 2 ()**

A particle P is moving on the x axis and its acceleration $a \text{ ms}^{-2}$, t seconds after a given instant, is given by

$$a = 6t - 18, t \geq 0.$$

The particle is initially at the origin O , moving with a speed of 15 ms^{-1} in the positive x direction.

- Determine the times when P is instantaneously at rest.
- Find the distance between the points, at which P is instantaneously at rest.

 , $t = 1, t = 5$, $d = 32 \text{ m}$

a) Find an expression for the velocity

$\Rightarrow a = 6t - 18$

$\Rightarrow V = \int (6t - 18) dt$

$\Rightarrow V = 3t^2 - 18t + C$

ANY INITIALS $t=0 \quad V=+15$
 $15 = 0 - 0 + C$
 $C = 15$

USE THE VELOCITY EXPRESSION WHEN $V=0$

$\Rightarrow 0 = 3t^2 - 18t + 15$

$\Rightarrow 0 = 3t^2 - 18t + 15$

$\Rightarrow 0 = t^2 - 6t + 5$

$\Rightarrow 0 = (t-1)(t-5)$

$\Rightarrow t = 1, 5$

b) INTEGRATE THE VELOCITY EXPRESSION, TO OBTAIN A DISPLACEMENT

$\Rightarrow V = 3t^2 - 18t + 15$

$\Rightarrow x = \int (3t^2 - 18t + 15) dt$

$\Rightarrow x = t^3 - 9t^2 + 15t + D$

when $t=0 \quad x=0$ (given)
 $\therefore D = 0$

$\Rightarrow x = t^3 - 9t^2 + 15t$

$x_1 = 1^3 - 9 \times 1^2 + 15 \times 1 = 7$

$x_5 = 5^3 - 9 \times 5^2 + 15 \times 5 = 125 - 225 + 75 = -25$

\therefore TOTAL DISTANCE IS $7 + 25 = 32 \text{ m}$

ALTERNATIVE FOR PART (b) BY SPEED-TIME GRAPH

Sketching: $V = 3t^2 - 18t + 15$
 $V = 3(t^2 - 6t + 5)$
 $V = 3(t-1)(t-5)$

\therefore DISTANCE = $\left| \int_1^5 (3t^2 - 18t + 15) dt \right|$

$= \left| \left[t^3 - 9t^2 + 15t \right]_1^5 \right|$

$= \left| (125 - 225 + 75) - (1 - 9 + 15) \right|$

$= \left| -25 - 7 \right|$

$= \left| -32 \right|$

$= 32 \text{ m}$

\therefore Answer

Question 3 ()**

A particle P is moving on the x axis and its velocity $v \text{ ms}^{-1}$, t seconds after a given instant, is given by

$$v = t^2 - 4t - 12, \quad t \geq 0.$$

When $t = 0$, its displacement x from the origin O is 20 m.

- Find the acceleration of P when $t = 3$.
- Find the acceleration of P , when P is instantaneously at rest.
- Determine the distance of P from O , when P is instantaneously at rest.

 , $a = 2 \text{ ms}^{-2}$, $a = 8 \text{ ms}^{-2}$, $d = 52 \text{ m}$

a) DIFFERENTIATE (w.r.t t , to find an expression for the acceleration)

$$v = t^2 - 4t - 12$$

$$a = \frac{dv}{dt} = 2t - 4$$

$$a \Big|_{t=3} = 2 \times 3 - 4$$

$$a = 2 \text{ ms}^{-2}$$

b) SOLVE $v=0$

$$\Rightarrow t^2 - 4t - 12 = 0$$

$$\Rightarrow (t-6)(t+2) = 0$$

$$\Rightarrow t = 6$$

$$a \Big|_{t=6} = 2 \times 6 - 4$$

$$= 8 \text{ ms}^{-2}$$

c) INTEGRATE THE VELOCITY EXPRESSION, TO OBTAIN A DISPLACEMENT EXPRESSION

$$v = t^2 - 4t - 12$$

$$\Rightarrow x = \int v \, dt = \int t^2 - 4t - 12 \, dt$$

$$\Rightarrow x = \frac{1}{3}t^3 - 2t^2 - 12t + C$$

$t=0 \quad x=20$
 $20 = 0 + 0 + 0 + C$

$$\Rightarrow x = \frac{1}{3}t^3 - 2t^2 - 12t + 20$$

$$x \Big|_0^6 = \frac{1}{3} \times 6^3 - 2 \times 6^2 - 12 \times 6 + 20 = 72 - 72 - 72 + 20 = -52$$

\therefore DISTANCE OF 52 m

c) ALTERNATIVE BY SPEED-TIME GRAPH (FOR PART c)

- Sketch the speed-time graph.
- Express $AM^2 = \int_0^6 t^2 - 4t - 12 \, dt$

$$= \left[\frac{1}{3}t^3 - 2t^2 - 12t \right]_0^6$$

$$= (72 - 72 - 72) - 0$$

$$= -72$$

- Now the particle was +20 (displacement) when $t=0$
- Since the displacement is $-72 + 20 = -52$
- Take the distance as 52 m

Question 4 (***)

A particle P is moving on the x axis and its velocity $v \text{ ms}^{-1}$, t s after a given instant, is given by

$$v = t^2(3-t), \quad t \geq 0.$$

When $t = 2$, P is observed to be 4 m from the origin O , in the positive x direction.

- a) Find the acceleration of P when $t = 2$.

The particle is at instantaneous rest initially, and when $t = T$.

- b) Determine the distance of P from O when $t = T$.

, $a = 0$, $d = 6.75 \text{ m}$

a) DIFFERENTIATE VELOCITY TO OBTAIN ACCELERATION

$$v = 3t^2 - t^3$$

$$a = \frac{dv}{dt} = 6t - 3t^2$$

$$a = (6 \times 2) - (3 \times 2^2) = 12 - 12 = 0$$

\therefore ZERO ACCELeration

b) BY INSPECTION, AT BEST WHEN $v=0$, MEANS $t=0$ OR $t=3$

IE $v = t^2(3-t)$
 $0 = t^2(3-t)$
 $t = 0$ or $t = 3$

INTEGRATE TO OBTAIN DISPLACEMENT

$$v = 3t^2 - t^3$$

$$x = \int (3t^2 - t^3) dt$$

$$x = t^3 - \frac{1}{4}t^4 + C$$

APPLY CONDITION $t=2, x=4$

$$4 = 2^3 - \frac{1}{4}2^4 + C$$

$$4 = 8 - 4 + C$$

$$C = 0$$

$\therefore x = t^3 - \frac{1}{4}t^4$

FINDING LENGTH TO $T=3$

$$x = 3^3 - \frac{1}{4}3^4 = 6.75$$

$\therefore 6.75 \text{ m}$

ALTERNATIVE FOR (b)

WORKING AT THE VELOCITY GRAPH

distance = displacement here as graph is above the x axis

$$= \int_0^3 t^2(3-t) dt = \int_0^3 (3t^2 - t^3) dt$$

$$= \left[t^3 - \frac{1}{4}t^4 \right]_0^3$$

$$= \left(27 - \frac{81}{4} \right) - (0 - 0)$$

$$= \frac{27}{4}$$

$$= 6.75$$

$\therefore 6.75 \text{ m}$

Question 5 (*)**

A particle P is moving on the x axis and its acceleration $a \text{ ms}^{-2}$, t seconds after a given instant, is given by

$$a = 8 - 2t, \quad t \geq 0.$$

Initially, P is on the positive x axis 84 m away from the origin O , and is moving towards O with a speed of 7 ms^{-1} .

- Find an expression for the velocity of P .
- Calculate the maximum velocity of P .
- Determine the times when P is instantaneously at rest.
- Show that when $t = 12$, P is passing through O .

$$v = -t^2 + 8t - 7, \quad v_{\max} = 9 \text{ ms}^{-1}, \quad t = 1, 7$$

Handwritten solution for Question 5:

(a) $a = 8 - 2t, t \geq 0, x = 84, v = -7$

$v = \int a \, dt$
 $v = \int (8 - 2t) \, dt$
 $v = 8t - t^2 + C$
 When $t = 0, v = -7$
 $-7 = 0 + C$
 $C = -7$
 $v = 8t - t^2 - 7$

(b) $v = f(t) \Rightarrow \leftarrow \text{max}$
 $\Rightarrow \frac{dv}{dt} = 0$
 $\Rightarrow 8 - 2t = 0$
 $\Rightarrow t = 4$
 $\therefore v_{\max} = 8(4) - 4^2 - 7 = 9 \text{ ms}^{-1}$

(c) $x = \int v \, dt$
 $\Rightarrow x = \int (8t - t^2 - 7) \, dt$
 $\Rightarrow x = 4t^2 - \frac{1}{3}t^3 - 7t + k$
 When $t = 0, x = 84$
 $84 = 0 + k$
 $k = 84$
 $\Rightarrow x = 4t^2 - \frac{1}{3}t^3 - 7t + 84$
 When $t = 12$
 $x = 576 - 576 - 84 + 84$
 $x = 0$
 Hence A.F.O.

(d) $v = 0$
 $\Rightarrow 8t - t^2 - 7 = 0$
 $\Rightarrow 0 = t^2 - 8t + 7$
 $\Rightarrow 0 = (t - 7)(t - 1)$
 $\Rightarrow t = 1, 7$

Question 6 (***)

A particle is moving in a straight line.

At time t s, the particle has displacement x m from a fixed origin O and is moving with velocity v ms⁻¹.

When $t=1$, $x=-5$ and $v=1$.

The acceleration a of the particle is given by

$$a = (16 - 6t) \text{ ms}^{-2}, t \geq 0.$$

The particle passes through O with speed U when $t=T$, $T > 0$.

Find the possible values of U .

, $U = 8, 24$

USE INTEGRATION TO OBTAIN A VELOCITY EXPRESSION

$$a = \frac{dv}{dt} = 16 - 6t$$

$$v = \int 16 - 6t \, dt$$

$$v = 16t - 3t^2 + A$$

USE $t=1, v=1$

$$\Rightarrow 1 = 16 - 3 + A$$

$$\Rightarrow A = -12$$

$$\Rightarrow v = -3t^2 + 16t - 12$$

INTEGRATE AGAIN TO GET THE DISPLACEMENT

$$x = \int -3t^2 + 16t - 12 \, dt$$

$$x = -t^3 + 8t^2 - 12t + B$$

USE $t=1, x=-5$

$$\Rightarrow -5 = -1 + 8 - 12 + B$$

$$\Rightarrow -5 = -5 + B$$

$$\Rightarrow B = 0$$

$$\Rightarrow x = -t^3 + 8t^2 - 12t$$

NOW SOLVING $x=0$ (PASSES THROUGH THE ORIGIN)

$$\Rightarrow 0 = -t^3 + 8t^2 - 12t$$

$$\Rightarrow t^3 - 8t^2 + 12t = 0$$

$$\Rightarrow t(t^2 - 8t + 12) = 0$$

$$\Rightarrow t(t-2)(t-6) = 0$$

$$t = \begin{cases} 0 \\ 2 \\ 6 \end{cases}$$

FINALLY WE CAN FIND THE VELOCITY $v = -3t^2 + 16t - 12$

- $t=2$
 $v_1 = -3(2)^2 + 16(2) - 12$
 $v_1 = -12 + 32 - 12$
 $v_1 = 8$
- $t=6$
 $v_2 = -3(6)^2 + 16(6) - 12$
 $v_2 = -108 + 96 - 12$
 $v_2 = -24$

\therefore THE POSSIBLE SPEEDS ARE 8 ms^{-1} & 24 ms^{-1}

Question 7 (***)

A particle P is moving on the x axis and its displacement from the origin, x m, t seconds after a given instant, is given by

$$x = 2t^3 - 3t^2 + At + B, \quad t \geq 0,$$

where A and B are constants.

- a) Find the value of t when the acceleration of P is zero.

When $t = 1.5$ s, P is passing through the origin O , and is moving in the negative x direction with speed 7.5 ms^{-1} .

- b) Determine the value of A and the value of B .
- c) Determine the time when P is instantaneously at rest.
- d) Calculate as an exact surd the value of t , when P is passing through O again.

$$t = \frac{1}{2}, \quad A = -12, \quad B = 18, \quad t = 2, \quad t = \sqrt{6}$$

Handwritten solution for Question 7:

(a) $x = 2t^3 - 3t^2 + At + B$
 $v = \frac{dx}{dt} = 6t^2 - 6t + A$
 $a = \frac{dv}{dt} = 12t - 6$
 $a = 0$
 $12t - 6 = 0$
 $t = \frac{1}{2}$

(b) $t = 1.5$, $x = 0$
 $0 = 2(1.5)^3 - 3(1.5)^2 + A(1.5) + B$
 $0 = \frac{27}{4} - \frac{27}{4} + \frac{3}{2}A + B$
 $0 = \frac{3}{2}A + B$
 $v = 6t^2 - 6t + A$
 $-7.5 = 6(1.5)^2 - 6(1.5) + A$
 $-7.5 = 9 - 9 + A$
 $A = -12$
 $\therefore 0 = \frac{3}{2}(-12) + B$
 $B = 18$

(c) $v = 6t^2 - 6t - 12$
 $v = 0$
 $0 = 6t^2 - 6t - 12$
 $0 = t^2 - t - 2$
 $0 = (t-2)(t+1)$
 $t = 2$ (since $t \geq 0$)

(d) $x = 2t^3 - 3t^2 - 12t + 18$
 $x = 0$
 $0 = 2t^3 - 3t^2 - 12t + 18$
 $0 = (2t-3)(t^2 - 4t + 6)$
 $0 = (2t-3)(t-2)(t-\sqrt{6})$
 $t = 2$ (already found)
 $t = \sqrt{6}$ (1.5 is already known)

Question 8 (***)

A car is travelling on a straight horizontal road with constant velocity of 37.5 ms^{-1} .

The driver applies the brakes and the car decelerates at $(9.25 - t) \text{ ms}^{-2}$, where t s is the time since the instant when the brakes were first applied.

- a) Show that while the car is decelerating its velocity is given by

$$\frac{1}{4}(2t^2 - 37t + 150) \text{ ms}^{-1}.$$

- b) Hence find the time taken to bring the car to rest.
 c) Determine the distance covered while the car was decelerating.

$t = 6 \text{ s}$, $d = 94.5 \text{ m}$

Handwritten solution for Question 8:

(a) DECELERATION of $(9.25 - t) \Rightarrow$ ACCELERATION of $(t - 9.25)$
 Then $\frac{dv}{dt} = t - 9.25$
 $v = \int (t - 9.25) dt$
 $v = \frac{1}{2}t^2 - 9.25t + C$
 When $t=0$, $v = 37.5 \Rightarrow C = 37.5$
 $v = \frac{1}{2}t^2 - 9.25t + 37.5$
 $v = \frac{1}{4}(2t^2 - 37t + 150)$ As required

(b) $v = 0 \Rightarrow \frac{1}{4}(2t^2 - 37t + 150) = 0$
 $2t^2 - 37t + 150 = 0$ BY QUADRATIC FORMULA OR FACTORISATION
 $(t - 6)(2t - 25) = 0$
 $t = \frac{6}{2} = 3$ ← ONE OF THE SOLUTIONS STUMPED !!

(c) $s = \int_0^6 \frac{1}{4}(2t^2 - 37t + 150) dt = \frac{1}{4} \left[\frac{2}{3}t^3 - \frac{37}{2}t^2 + 150t \right]_0^6$
 $= \frac{1}{4} \left[(144 - 666 + 900) - (0) \right]$
 $= 94.5 \text{ m}$

Question 9 (***)

A particle P is moving on the x axis and its velocity $v \text{ ms}^{-1}$ in the positive x direction, t seconds after a given instant, is given by

$$v = t^2 - 2t - 24, \quad t \geq 0.$$

When $t = 3$, P is observed passing through the origin.

- Find the acceleration of P when $t = 3$.
- Determine the distance of P from O when it is instantaneously at rest.
- Find the time at which P is passing through O again.

, $a = 4 \text{ ms}^{-2}$, $d = 36 \text{ m}$, $t = \sqrt{72} \approx 8.49$

Left Page Solution:

$v = t^2 - 2t - 24, \quad t \geq 0$ subject to $t=3, x=0$

a) Obtain the acceleration by differentiating the velocity

$$a = \frac{dv}{dt} = \frac{d}{dt}(t^2 - 2t - 24)$$

$$a = 2t - 2$$

$$a|_{t=3} = 2 \times 3 - 2 = 4 \text{ ms}^{-2}$$

b) Find the times when $v=0$

$$0 = t^2 - 2t - 24$$

$$0 = (t+4)(t-6)$$

$$t = -4 \text{ or } 6$$

Using integration to obtain an expression for the displacement x

$$x = \int v \, dt = \int (t^2 - 2t - 24) \, dt$$

$$x = \frac{1}{3}t^3 - t^2 - 24t + C$$

Apply condition $t=3, x=0$

$$0 = \frac{1}{3}(3)^3 - (3)^2 - 24(3) + C$$

$$0 = 9 - 9 - 72 + C$$

$$C = 72$$

$$x = \frac{1}{3}t^3 - t^2 - 24t + 72$$

Right Page Solution:

Finally when $t=6$

$$x(6) = \frac{1}{3}(6)^3 - (6)^2 - 24(6) + 72$$

$$= 72 - 36 - 144 + 72 = -36$$

\therefore A distance of 36 m from O

Alternative for part (b) using velocity time graph

$v = t^2 - 2t - 24$

$y = (t+4)(t-6)$

Displacement = Area under graph

$$= \int_{-4}^6 (t^2 - 2t - 24) \, dt$$

$$= \left[\frac{1}{3}t^3 - t^2 - 24t \right]_{-4}^6$$

$$= \left(\frac{1}{3}(6)^3 - (6)^2 - 24(6) \right) - \left(\frac{1}{3}(-4)^3 - (-4)^2 - 24(-4) \right)$$

$$= -36 \text{ as above}$$

c) Using $x = \frac{1}{3}t^3 - t^2 - 24t + 72$ with $x=0$ & noting $t=3$ is a known solution (given as condition)

$$\frac{1}{3}t^3 - t^2 - 24t + 72 = 0$$

$$t^3 - 3t^2 - 72t + 216 = 0$$

$$t^2(t-3) - 72(t-3) = 0 \quad (\text{Use algebraic division})$$

$$(t-3)(t^2 - 72) = 0$$

$$t = 3 \text{ or } t^2 = 72$$

$$\therefore t = \sqrt{72} \approx 8.49 \text{ s}$$

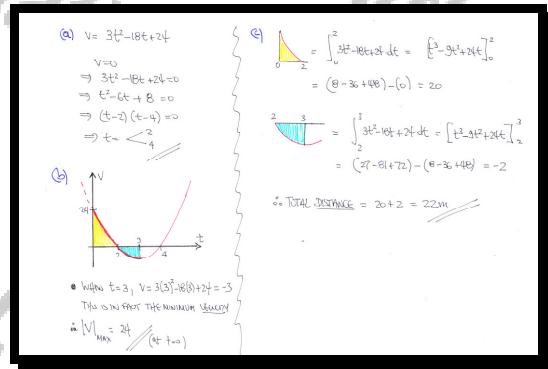
Question 10 (***)

A particle P is moving on the x axis and its velocity $v \text{ ms}^{-1}$ in the positive x direction, t seconds after a given instant, is given by

$$v = 3t^2 - 18t + 24, \quad t \geq 0.$$

- Find the times when P is instantaneously at rest.
- Determine the greatest speed of P in the interval $0 \leq t \leq 3$.
- Calculate the total distance covered by P in the interval $0 \leq t \leq 3$.

$$t = 2, 4, \quad |v|_{\max} = 24 \text{ ms}^{-1}, \quad d = 22 \text{ m}$$



Question 11 (***)

A particle P is moving on a straight line.

At time t seconds, the distance of P from a fixed origin O is x metres and its acceleration is

$$(8 - 2t) \text{ ms}^{-2}$$

in the direction of x increasing.

It is further given that when $t = 0$, P was moving towards O with speed 7 ms^{-1} .

Determine the total distance covered by P in the first 7 seconds.

, $d = 39\frac{1}{3} \text{ m}$

● START BY OBTAINING AN EXPRESSION FOR THE VELOCITY.
 $\Rightarrow a = 8 - 2t$
 $\Rightarrow \frac{dv}{dt} = 8 - 2t$
 $\Rightarrow \int dv = \int (8 - 2t) dt$
 $\Rightarrow \int_{v=7}^v dv = \int_{t=0}^t (8 - 2t) dt$
 $\Rightarrow [v]_{v=7}^v = [8t - t^2]_{t=0}^t$
 $\Rightarrow v - 7 = (8t - t^2) - 0$
 $\Rightarrow v = -7 + 8t - t^2$

● SKETCHING THE GRAPHS $v = f(t)$

DISTANCE = $\int_{t=0}^1 (-7 + 8t - t^2) dt + \int_{t=1}^7 (-7 + 8t - t^2) dt$
 $= \int_{t=0}^1 (-7 + 8t - t^2) dt + \int_{t=1}^7 (-7 + 8t - t^2) dt$
 $= [-7t + 4t^2 - \frac{1}{3}t^3]_{t=0}^1 + [-7t + 4t^2 - \frac{1}{3}t^3]_{t=1}^7$
 $= 0 - (-7 + 14 - \frac{1}{3}) + (-49 + 196 - \frac{343}{3}) - (-7 + 28 - 7)$
 $= \frac{118}{3}$
 $= 39\frac{1}{3} \text{ m}$

Question 12 (***)

A particle is moving in a straight line in an electromagnetic field.

Its velocity, $v \text{ ms}^{-1}$, at time $t \text{ s}$, $t \geq 0$, is given by

$$v = t^2 + kt + 3.2,$$

where k is a non zero constant.

- Given that the particle achieves its minimum velocity when $t = 2.4 \text{ s}$, show that $k = -4.8$.
- Determine the values of t when the particle is instantaneously at rest.
- Calculate the total distance covered by the particle for $0 \leq t \leq 6$.

$$\boxed{}, \quad \boxed{t = 0.8}, \quad \boxed{t = 4}, \quad \boxed{d = \frac{5896}{375} \approx 15.72 \text{ m}}$$

$v = t^2 + kt + 3.2; t \geq 0$

a) THE MINIMUM VELOCITY SATISFIES $\frac{dv}{dt} = 0$
 $\Rightarrow 2t + k = 0$
 $\Rightarrow 2 \times 2.4 + k = 0$
 $\Rightarrow k = -4.8$

b) SOLVING $v = 0$
 $\Rightarrow 0 = t^2 - 4.8t + 3.2$
 $\Rightarrow 0 = 5t^2 - 24t + 16$
 $\Rightarrow 0 = (5t - 4)(t - 4)$
 $\Rightarrow t = \frac{4}{5} = 0.8$
 $t = 4$

c) SKETCHING THE VELOCITY TIME GRAPH

$\int v dt = \int t^2 - 4.8t + 3.2 dt$
 $= \frac{1}{3}t^3 - 2.4t^2 + 3.2t + C$

$A_1 = \left[\frac{1}{3}t^3 - 2.4t^2 + 3.2t \right]_0^{0.8}$
 $A_2 = \left(\frac{64}{15} - \frac{15.36}{5} + \frac{25.6}{5} \right) - \left(\frac{0}{3} - 0 + 0 \right)$
 $A_3 = \left(72 - \frac{480}{5} + \frac{96}{5} \right) - \left(\frac{64}{3} - \frac{48}{5} + \frac{32}{5} \right)$
 $A = \frac{448}{375}$

$A_1 = \left[\frac{1}{3}t^3 - 2.4t^2 + 3.2t \right]_0^{0.8}$
 $A_2 = \left(\frac{64}{15} - \frac{15.36}{5} + \frac{25.6}{5} \right) - \left(\frac{0}{3} - 0 + 0 \right)$
 $A_3 = \left(72 - \frac{480}{5} + \frac{96}{5} \right) - \left(\frac{64}{3} - \frac{48}{5} + \frac{32}{5} \right)$
 $A_3 = \frac{136}{15}$

$\therefore \text{TOTAL DISTANCE} = \frac{448}{375} + \frac{2048}{375} + \frac{136}{15}$
 $= \frac{5896}{375}$
 $\approx 15.72 \text{ m}$

Question 13 (****)

A particle P is moving on the x axis and its acceleration $a \text{ ms}^{-2}$, t seconds after a given instant, is given by

$$a = 4t - 9, \quad t \geq 0.$$

When $t = 1$, P is moving with a velocity of -3 ms^{-1} .

- Find the minimum velocity of P .
- Determine the times when P is instantaneously at rest.
- Find the distance travelled by P in the first $4\frac{1}{2}$ seconds of its motion.

$$\boxed{}, \quad \boxed{v_{\min} = -6.125 \text{ ms}^{-1}}, \quad \boxed{t = \frac{1}{2}, 4}, \quad \boxed{d = \frac{389}{24} \approx 16.21 \text{ m}}$$

Q) INTEGRATING BACKWARD TO GET A VELOCITY EXPRESSION

$\rightarrow a = 4t - 9$
 $\rightarrow v = \int (4t - 9) dt$
 $\rightarrow v = 2t^2 - 9t + C$

APPLY $t=1, v=-3$
 $\rightarrow -3 = 2 - 9 + C$
 $\rightarrow C = 4$

$\rightarrow v = 2t^2 - 9t + 4$

NEED v IS MINIMUM WHEN THE ACCELERATION IS ZERO, IF $\frac{dv}{dt} = 0$

$4t - 9 = 0$ (OR MAY ALSO COMPLET THE SQUARE HERE)
 $4t = 9$
 $t = \frac{9}{4}$

$\therefore v\left(\frac{9}{4}\right) = 2\left(\frac{9}{4}\right)^2 - 9\left(\frac{9}{4}\right) + 4 = -\frac{49}{8}$
 \therefore **MINIMUM VELOCITY IS -6.125 ms^{-1}**

b) SKETCH $v=0$ BY FACTORISING

$2t^2 - 9t + 4 = 0$
 $(2t - 1)(t - 4) = 0$
 $t = \frac{1}{2}$
 $t = 4$

c) OBTAIN AN EXPRESSION FOR THE DISPLACEMENT SUBJECT TO THE USUAL CONSIDERATIONS

$v = 2t^2 - 9t + 4$
 $s = \int (2t^2 - 9t + 4) dt$
 $s = \frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t + D$
 TWO $s=0 \Rightarrow D=0$

$s = \frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t$
 $s = \frac{2}{3}(4.5)^3 - \frac{9}{2}(4.5)^2 + 4(4.5)$
 $s(4.5) = \frac{389}{24} = 16.208\bar{3}$
 $s(4) = -\frac{49}{8} = -6.125$
 $s(4.5) = -6.125 + 16.208\bar{3} = 10.083\bar{3}$

DRAWING A DIAGRAM

By symmetry $A_1 = A_2 = \int v dt$

$A_1 + A_2 = 2 \int_{1/2}^4 (2t^2 - 9t + 4) dt$
 $= 2 \left[\frac{2}{3}t^3 - \frac{9}{2}t^2 + 4t \right]_{1/2}^4$
 $= 2 \left[\left(\frac{16}{3} - \frac{144}{2} + 16 \right) - \left(\frac{1}{24} - \frac{9}{8} + 2 \right) \right]$
 $= \frac{389}{12} - \left(\frac{49}{6} \right)$
 $= \frac{389}{24} = 16.208\bar{3}$

$\therefore d = \frac{23}{12} + \frac{389}{24} = \frac{389}{24} \approx 16.21 \text{ m}$

Question 14 (****)

A particle is moving in a straight line, so that its velocity, $v \text{ ms}^{-1}$, at time t s satisfies

$$v = 2t + kt^2, \quad 0 \leq t \leq 10,$$

where k is a non zero constant.

When $t = 10$, the particle reaches an acceleration of 1.8 ms^{-2} , which it maintains for a further 10 s.

- Show that $k = -0.01$.
- Sketch a detailed velocity time graph, which describes the motion of this particle, for $0 \leq t \leq 20$.
- Find the distance travelled by the particle for $0 \leq t \leq 20$.

 , $d = 376\frac{1}{3} \text{ m}$

a) Obtain and express the acceleration by differentiation

$$v = 2t + kt^2$$

$$a = \frac{dv}{dt} = 2 + 2kt$$

3rd March 2019, a = 1.8

$$\Rightarrow 1.8 = 2 + 20k$$

$$\Rightarrow -0.2 = 20k$$

$$\Rightarrow k = -0.01$$

As expected

b) Now find t or 20 seconds, v is a quadratic

$$v = 2t - 0.01t^2 = 0.01(200 - t^2)$$

Use the graph

<ul style="list-style-type: none"> • When $t=0$ $v = 2(-0.01)^2$ $v = 2(0) - 0.01(0)^2$ $v = 2(0) - 1$ $v = 19$ 	<ul style="list-style-type: none"> • For $t > 10$ $a = 1.8$ $a = 1.8$ $s = 10$ $v = 19$
---	--

d) For the first 10 seconds

$$\int_0^{10} (2t - 0.01t^2) dt = \left[t^2 - \frac{0.01}{3}t^3 \right]_0^{10}$$

$$= \left(100 - \frac{10}{3} \right) - (0)$$

$$= \frac{290}{3}$$

For the next 10 seconds

OR

$$s = ut + \frac{1}{2}at^2$$

$$s = 19 \times 10 + \frac{1}{2}(1.8)(10)^2$$

$$s = 280$$

∴ Total distance = 290 + 280 = 570 = 376 2/3 m

Question 15 (****)

Russel is driving through the countryside, along a straight horizontal road at a constant speed of 22.5 ms^{-1} .

He sees a fallen tree blocking the road ahead, at a distance of 75 m ahead, so he immediately applies the brakes trying to stop his car before it hits the fallen tree.

The way he applies the brakes is such so that the **deceleration** of his car is given by $(3 + \frac{1}{4}t) \text{ ms}^{-2}$, where t is measured since the instant he first applied the brakes.

Russel's car stops D m **before** he hits the tree.

Determine the value of D .

, $D = 3$

STATE BY OBTAINING EXPRESSIONS FOR VELOCITY & DISPLACEMENT?

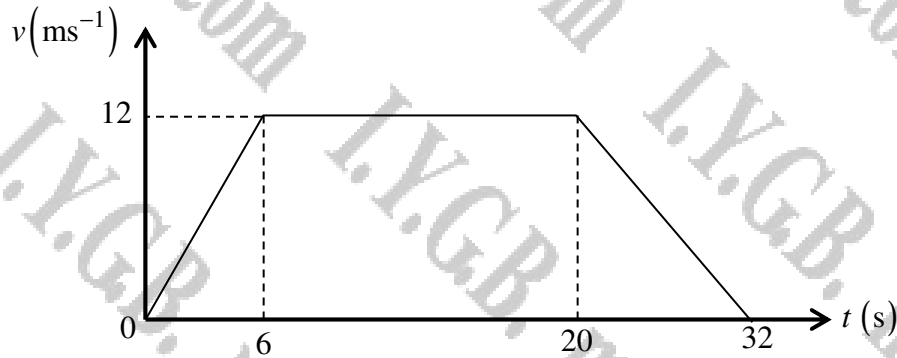
Deceleration of $3 + \frac{1}{4}t \Rightarrow a = 3 + \frac{1}{4}t$
 $\rightarrow v = 3t + \frac{1}{8}t^2 + 22.5$ (SPEED) \leftarrow
 $\Rightarrow s = \frac{3}{2}t^2 + \frac{1}{24}t^3 + 22.5t + C$ (DISPLACEMENT) \leftarrow

LEADER TO A STOP: $v = 0$
 $\Rightarrow 0 = 3t + \frac{1}{8}t^2 + 22.5$
 $\Rightarrow \frac{1}{8}t^2 + 3t + 22.5 = 0$
 $\Rightarrow t^2 + 24t + 180 = 0$
 $\Rightarrow (-6)(-6)(t+30) = 0$
 $\Rightarrow t = -6$ (crossed out)

(USING THE DISPLACEMENT EXPRESSION) NOW $t = 6$
 $\Rightarrow s = \frac{3}{2}(6)^2 + \frac{1}{24}(6)^3 + 22.5(6) + C$
 $\Rightarrow s = -54 - 9 + 135$
 $\Rightarrow s = 72$

\therefore AS THE TREE WAS 75m AWAY
 $D = 3$

Question 16 (****)



The figure above shows the speed time graph (t, v) of a car travelling along a straight horizontal road between two sets of traffic lights.

The car starts from rest at the first set of lights and accelerates uniformly for 6 s, reaching a speed of 12 ms^{-1} .

This speed is maintained for 14 s, before the car decelerates uniformly for 12 s, coming to rest as it reaches the second set of lights.

The distance of the car, $s(t)$, measured from the first set of traffic lights is given by

$$s(t) = \begin{cases} f_1(t) & 0 \leq t < 6 \\ f_2(t) & 6 \leq t < 20 \\ f_3(t) & 20 \leq t < 32 \end{cases}$$

where $f_1(t)$, $f_2(t)$ and $f_3(t)$ are functions of t .

Determine simplified expressions for $f_1(t)$, $f_2(t)$ and $f_3(t)$.

, $f_1(t) = t^2$, $f_2(t) = 12t - 36$, $f_3(t) = \frac{1}{2}t^2 + 32t - 236$

LOOKING AT THE GRAPH OPPOSITE

- GRAD $v_1 = \frac{12}{6} = 2$
- $v_1(t) = 2t$
- GRAD $v_2 = 0$
- $v_2 = 12$
- GRAD $v_3 = -\frac{12}{12} = -1$
- $v_3 = 0 = -1(t - 32)$
- $v_3 = 32 - t$

NOW WE CAN IDENTIFY THE AS REQUESTED

- $s_1(t) = \int_0^t 2t \, dt = [t^2]_0^t = t^2 - 0 = t^2$
- $s_2(t) = 6^2 = 36$
- $s_2(t) = 36 + \int_6^t 12 \, dt = 36 + [12t]_6^t = 36 + (12t - 72) = 12t - 36$
- $s_3(32) = 12 \times 20 - 36 = 204$

THENCE WE FINALLY OBTAIN

$$s_3(t) = 204 + \int_{20}^t (32 - t) \, dt = 204 + \left[32t - \frac{1}{2}t^2 \right]_{20}^t$$

$$= 204 + \left[(32t - \frac{1}{2}t^2) - (640 - 200) \right]$$

$$= \frac{1}{2}t^2 + 32t - 236$$

$$s(t) = \begin{cases} t^2 & 0 \leq t < 6 \\ 12t - 36 & 6 \leq t < 20 \\ \frac{1}{2}t^2 + 32t - 236 & 20 \leq t < 32 \end{cases}$$

Question 17 (****+)

A particle P is moving on the x axis and its velocity $v \text{ ms}^{-1}$, t seconds after a given instant, is given by

$$v = \begin{cases} 6t - t^2 & 0 \leq t \leq 5 \\ 25 - 4t & t > 5 \end{cases}$$

The particle is initially at the origin O .

- Find the greatest speed of P for $0 \leq t \leq 5$.
- Show that the distance of P from O when $t = 5$ is $33\frac{1}{3} \text{ m}$.
- State the time at which P is instantaneously at rest for $t > 5$.
- Hence determine the **total distance** travelled by P during the first 10 seconds of its motion.

$$v_{\max} = 9 \text{ ms}^{-1}, \quad t = \frac{25}{4} = 6.25 \text{ s}, \quad d = \frac{775}{12} \approx 64.58 \text{ m}$$

Handwritten solution for Question 17:

$v = \begin{cases} 6t - t^2 & 0 \leq t \leq 5 \\ 25 - 4t & t > 5 \end{cases}$ $t=0, a=0$

(a) $v = 6t - t^2$
 $\frac{dv}{dt} = 6 - 2t$
 Solve for zero gives
 $t = 3$ $\therefore v_{\max} = 9$

(b) $v = 6t - t^2$
 $a = \int (6t - t^2) dt$
 $a = 3t^2 - \frac{1}{3}t^3 + C$
 when $t=0$ $a=0$ $C=0$
 $a = 3t^2 - \frac{1}{3}t^3$
 when $t=5$
 $a = 3 \times 5^2 - \frac{1}{3} \times 5^3 = \frac{100}{3}$
 \therefore distance $33\frac{1}{3} \text{ m}$

(c) $v = 25 - 4t$
 $0 = 25 - 4t$
 $t = \frac{25}{4} = 6.25$

(d) FIND AN EXPRESSION FOR THE DISPLACEMENT OF x vs t
 $a_1 = \int (6t - t^2) dt$
 $a_1 = 3t^2 - \frac{1}{3}t^3 + D$
 when $t=0$ $a=0$ $D=0$
 $a_1 = 3t^2 - \frac{1}{3}t^3$ $0 \leq t \leq 5$
 $a_2 = \int (25 - 4t) dt$
 $a_2 = 25t - 2t^2 + D$
 when $t=5$ $a_1 = a_2$
 $3 \times 5^2 - \frac{1}{3} \times 5^3 = 25 \times 5 - 2 \times 5^2 + D$
 $D = -\frac{100}{3}$
 $a_2 = \begin{cases} 3t^2 - \frac{1}{3}t^3 & 0 \leq t \leq 5 \\ 25t - 2t^2 - \frac{100}{3} & t > 5 \end{cases}$
 when $t = 6.25$ $v = 0$
 \therefore PARTICLE REVERSES
 when $t = 6.25$ $a = \frac{875}{4}$
 $t = 10$ $a = \frac{25}{4}$
 DISTANCE = $2 \times \frac{875}{4} - \frac{25}{4} \approx 64.58 \text{ m}$

Handwritten solution for Question 17:

DISPLACEMENT BY SPLITTING A NEIGHBY TRIANGLE

$v = \begin{cases} 6t - t^2 & 0 \leq t \leq 5 \\ 25 - 4t & t > 5 \end{cases}$ $v = t(6-t)$

v vs t graph showing area A_1 (under $v = 6t - t^2$) and A_2 (under $v = 25 - 4t$).

(a) $v = 6t - t^2$
 $\frac{dv}{dt} = 6 - 2t$
 when $t=3$ $v = 9$

(b) $A = \int_0^5 (6t - t^2) dt = [3t^2 - \frac{1}{3}t^3]_0^5 = (75 - \frac{125}{3}) - 0 = \frac{100}{3}$

(c) $v = 0$ ON THE STRAIGHT LINE $\Rightarrow 25 - 4t = 0$
 $t = 6.25$

(d) TOTAL DISTANCE IS GIVEN BY
 $A_1 + \frac{1}{2} \times \frac{25}{4} \times \frac{25}{4} = \frac{100}{3} + \frac{25 \times 25}{8} = \frac{100}{3} + \frac{625}{8} = \frac{800 + 625 \times 3}{24} = \frac{2775}{24} = \frac{775}{8} \approx 64.58 \text{ m}$

Question 18 (****+)

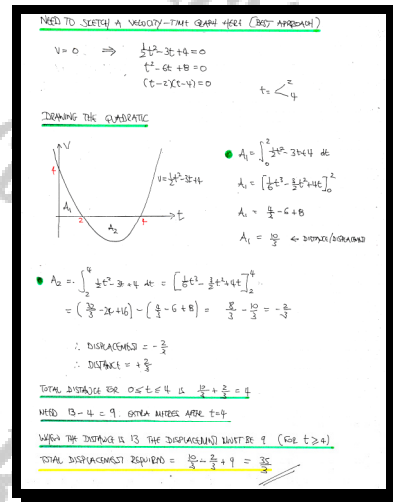
A particle P is moving on the x axis and its velocity $v \text{ ms}^{-1}$ in the positive x direction, t seconds after a given instant, is given by

$$v = \frac{1}{2}t^2 - 3t + 4, \quad t \geq 0.$$

The particle is passing through the origin when $t = 0$

Determine the displacement of the particle from the origin, when it has covered a **total distance** of 13 m.

, $x = \frac{35}{3}$



Question 19 (****+)

A car moving on a straight road is modelled as a particle moving on the x axis, and its acceleration $a \text{ ms}^{-2}$, t seconds after a given instant, is given by

$$a = \begin{cases} 4 - \frac{1}{2}t & 0 \leq t \leq 8 \\ 0 & t > 8 \end{cases}$$

The car starts from rest at the origin O .

- Find a similar expression for the velocity of the car, as that of its acceleration.
- State the time it takes for the car to reach its maximum speed.
- Show that the displacement of P from O is given by

$$x = \begin{cases} 2t^2 - \frac{1}{12}t^3 & 0 \leq t \leq 8 \\ 16t - \frac{128}{3} & t > 8 \end{cases}$$

- Calculate the time it takes the car to cover the first 1000 m.

, $v = \begin{cases} 4t - \frac{1}{4}t^2 & 0 \leq t \leq 8 \\ 16 & t > 8 \end{cases}$, $t = 8 \text{ s}$, $t = 65\frac{1}{6} \text{ s}$

a) Integrate the acceleration section by section

$\rightarrow a_1 = 4 - \frac{1}{2}t \quad 0 \leq t \leq 8$ $\Rightarrow a_2 = 0$
 $\rightarrow v_1 = \int 4 - \frac{1}{2}t \quad \Rightarrow v_2 = \text{constant, say } D$
 $\rightarrow v_1 = 4t - \frac{1}{4}t^2 + C$ using v_1 with $t=8$
 $t=0, v=0 \Rightarrow C=0$ $v_1(8) = 4(8) - \frac{1}{4}(8)^2$
 $\therefore v_1 = 4t - \frac{1}{4}t^2, 0 \leq t \leq 8$ $\therefore v_2 = 16, t > 8$

b) THE TIME IS 8 SECONDS
 (SEE SPEED TIME GRAPH)

c) Repeat the process for displacement

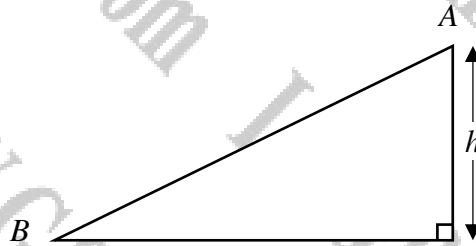
$x_1 = \int 4t - \frac{1}{4}t^2 dt \quad (0 \leq t \leq 8)$ $x_2 = \int 16 dt$
 $x_1 = 2t^2 - \frac{1}{12}t^3 + E$ $x_2 = 16t + F$
 $t=0, x=0, E=0$ using x_1 with $t=8$
 $x_1(8) = 2(8)^2 - \frac{1}{12}(8)^3$
 $\therefore x_1 = 2t^2 - \frac{1}{12}t^3 \quad 0 \leq t \leq 8$ $x_2(8) = 2(8)^2 - \frac{1}{12}(8)^3$
 $\therefore x_2(8) = \frac{256}{3}$
 $16(8) + F = \frac{256}{3}$
 $F = -\frac{128}{3}$

$\therefore a_1 = 4t - \frac{1}{2}t$
 $\therefore a_2 = \begin{cases} 2t^2 - \frac{1}{12}t^3 & 0 \leq t \leq 8 \\ 16t - \frac{128}{3} & t > 8 \end{cases}$

d) Firstly note that $x(t) = 2t^2 - \frac{1}{12}t^3 < 1000$

SET $x_2 = 1000$
 $\Rightarrow 16t - \frac{128}{3} = 1000$
 $\Rightarrow 16t = \frac{3008}{3}$
 $\Rightarrow t = \frac{376}{3}$
 $\therefore t = 65\frac{1}{6}$

Question 20 (***)



A particle is sliding down the line of greatest slope of a **smooth** plane inclined at a fixed angle to the horizontal. The particle experiences no other resistances.

The particle is released from rest from a point A at the top of the plane and takes 12 seconds to slide down to a point B on the plane. Point A lies at a vertical distance of h above the level of B , as shown in the figure above.

The particle slides down by 1 cm during the first second of its motion, and in each subsequent second it slides down by an extra 3 cm than in the previous second.

Show that $h = 6\frac{3}{7}$, measured in millimetres.

A, proof

USE THE SUMMATION FORMULA FOR ARITHMETIC PROGRESSIONS WITH $a = 0.01$ AND $d = 0.03$ (IN CM)

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

$$S_{12} = \frac{12}{2} [2(0.01) + (12-1)(0.03)] \quad \leftarrow \text{DISPLACEMENT AT TIME } t \text{ SECONDS}$$

$$S_{12} = \frac{12}{2} [0.02 + 0.03(11)]$$

$$S_{12} = \frac{12}{2} [0.35 - 0.01]$$

$$S_{12} = \frac{12}{2} (34 - 1)$$

$$S_{12} = \frac{12}{2} (33^2 - 1)$$

DIFFERENTIATE TO OBTAIN EXPRESSIONS FOR VELOCITY AND ACCELERATION

$$v_t = \frac{1}{200} (3t - 1)$$

$$a_t = \frac{3}{200} = 0.015$$

Now look at A DIAGRAM

$\sin \alpha = \frac{1}{3.464}$
 $\sin \beta = \frac{3}{3.464}$
 $\sin \theta = \frac{3}{3.464}$

NOW THE DISTANCE COVERED ON THE PLANE AFTER 12 SECONDS

$$S_{12} = \frac{1}{200} (3t^2 - t)$$

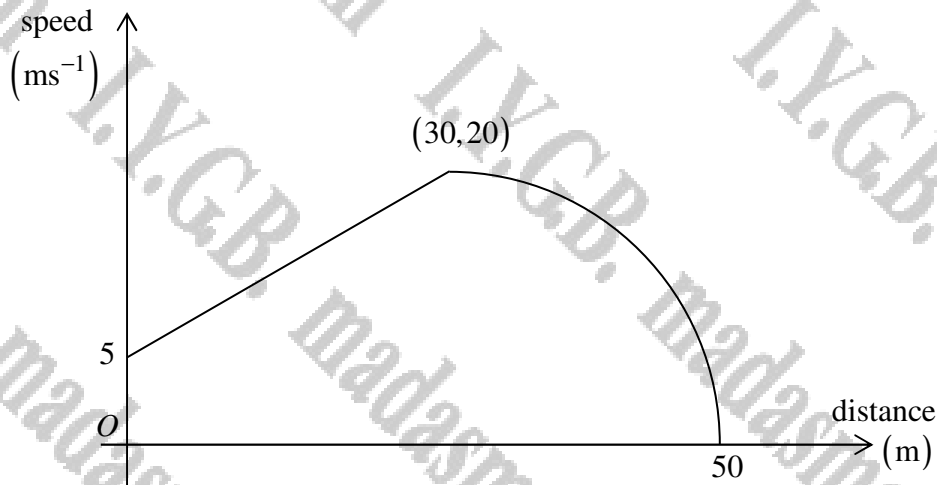
$$S_{12} = \frac{1}{200} (3(12)^2 - 12)$$

$$S_{12} = 2.1$$

FINALLY WE HAVE

$\frac{1}{2.1} = \sin \theta$
 $\frac{1}{2.1} = \frac{3}{3.464}$
 $h = \frac{1}{1.05}$ (CHECK)
 $h = \frac{1}{1.05}$ (cm)
 $h = \frac{100}{105}$ (cm)
 $h = 6\frac{3}{7}$ mm

Question 21 (****)



The speed distance graph of the journey of a particle is shown above.

It consists of a straight line segment joining the point $(0, 5)$ to $(30, 20)$, joined to a quarter circle of radius 20. The total distance covered by the particle is 50 m.

Determine in exact form the total journey time of the particle.

You may assume without proof that

$$\int \frac{1}{\sqrt{a^2 - (u-b)^2}} du = \arcsin\left(\frac{u-b}{a}\right) + \text{constant}$$

, $t = \left(\frac{1}{2}\pi + 4\ln 2\right) \text{ s}$

<p><u>SEPARATING WITH THE 'DISTANCE - SPEED' GRAPH</u></p> <ul style="list-style-type: none"> Gradient of line = $\frac{\Delta v}{\Delta x} = \frac{15}{30} = \frac{1}{2}$ Equation of line $v = \frac{1}{2}x + 5$ Equation of the circle is given by $(x-30)^2 + v^2 = 20^2$ $v^2 = 400 - (x-30)^2$ $v = \sqrt{400 - (x-30)^2}$ <p><u>THE EQUATION OF THE VELOCITY IS GIVEN BY</u></p> $v = \begin{cases} \frac{1}{2}x + 5 & 0 \leq x \leq 30 \\ \sqrt{400 - (x-30)^2} & 30 < x \leq 50 \end{cases}$	<p><u>INTEGRATING THE FIRST SECTION</u></p> $v = \frac{dx}{dt} = \frac{1}{2}x + 5$ $\Rightarrow \frac{1}{\frac{1}{2}x + 5} dx = 1 dt$ $\Rightarrow \int_{30}^{50} \frac{1}{\frac{1}{2}x + 5} dx = \int_{t_0}^t 1 dt$ $\Rightarrow \int_{30}^{50} \frac{2}{x+10} dx = \int_{t_0}^t 1 dt$ $\Rightarrow [2\ln(x+10)]_{30}^{50} = [t]_{t_0}^t$ $\Rightarrow 2[\ln 40 - \ln 10] = t - t_0$ $\Rightarrow t = 2[\ln 40 - \ln 10]$ $\Rightarrow t = 2\ln 4$	<p><u>MOVING INTO THE SECOND SECTION OF THE GRAPH</u></p> $v = \frac{dx}{dt} = \sqrt{400 - (x-30)^2}$ $\Rightarrow \frac{1}{\sqrt{400 - (x-30)^2}} dx = 1 dt$ $\Rightarrow \int_{30}^{50} \frac{1}{\sqrt{20^2 - (x-30)^2}} dx = \int_{t_0}^t 1 dt$ <p><u>USING THE REVERSE FORM</u></p> $\Rightarrow \left[\arcsin\left(\frac{x-30}{20}\right)\right]_{30}^{50} = [t]_{t_0}^t$ $\Rightarrow \arcsin 1 - \arcsin 0 = t - t_0$ $\Rightarrow t = \frac{\pi}{2} + 4\ln 2$ $\Rightarrow t = \frac{1}{2}\pi + 4\ln 2$
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Created by T. Madas

CALCULUS KINEMATICS IN VECTOR FORM

Created by T. Madas

Question 1 ()**

The position vector, \mathbf{r} m, of a particle, t seconds after a given instant is given by

$$\mathbf{r} = (2t^2 - 1)\mathbf{i} + (6t - 5t^2)\mathbf{j}, \quad t \geq 0,$$

where \mathbf{i} and \mathbf{j} are unit vectors pointing due east and due north, respectively.

Given that the mass of the particle is 0.5 kg, determine the magnitude of the resultant force acting on the particle.

$$F = \sqrt{29} \approx 5.39 \text{ N}$$

$$\begin{aligned} \mathbf{r} &= (2t^2 - 1)\mathbf{i} + (6t - 5t^2)\mathbf{j} \\ \mathbf{v} &= \frac{d\mathbf{r}}{dt} = (4t)\mathbf{i} + (6 - 10t)\mathbf{j} \\ \mathbf{a} &= \frac{d\mathbf{v}}{dt} = 4\mathbf{i} - 10\mathbf{j} \\ |\mathbf{a}| &= \sqrt{4^2 + (-10)^2} = \sqrt{116} \\ F &= ma \\ F &= \frac{1}{2} \times \sqrt{116} \\ F &= \frac{1}{2} \times 2\sqrt{29} \\ F &= \sqrt{29} \\ &\approx 5.39 \text{ N} \end{aligned}$$

Question 2 ()**

The position vector, \mathbf{r} m, of a particle P , t s after a given instant is given by

$$\mathbf{r} = (t^3 - 2t)\mathbf{i} + (4t^2 + t)\mathbf{j}, \quad t \geq 0,$$

where \mathbf{i} and \mathbf{j} are unit vectors pointing due east and due north, respectively.

- Find the magnitude of the acceleration of the particle, when $t = 1$.
- Determine the value of t when P is moving parallel to the vector $\mathbf{i} + \mathbf{j}$.

$$a = 10 \text{ ms}^{-2}, \quad t = 3$$

$$\begin{aligned} \text{(a)} \quad \mathbf{r} &= (t^3 - 2t)\mathbf{i} + (4t^2 + t)\mathbf{j} \\ \mathbf{v} &= \frac{d\mathbf{r}}{dt} = (3t^2 - 2)\mathbf{i} + (8t + 1)\mathbf{j} \\ \mathbf{a} &= \frac{d\mathbf{v}}{dt} = 6t\mathbf{i} + 8\mathbf{j} \\ \text{when } t=1 \\ \mathbf{a} &= 6\mathbf{i} + 8\mathbf{j} \\ |\mathbf{a}| &= \sqrt{6^2 + 8^2} = 10 \text{ ms}^{-2} \\ \text{(b)} \quad \mathbf{v} &= (3t^2 - 2)\mathbf{i} + (8t + 1)\mathbf{j} \\ &\text{IF MOVING IN THE DIRECTION} \\ &\text{of } \mathbf{i} + \mathbf{j} \text{ then} \\ &\Rightarrow 3t^2 - 2 = 8t + 1 \\ &\Rightarrow 3t^2 - 8t - 3 = 0 \\ &\Rightarrow (3t + 1)(t - 3) = 0 \\ &t = 3 \end{aligned}$$

Question 3 (***)

The velocity, $v \text{ ms}^{-1}$, of a particle P , t seconds after a given instant is given by

$$\mathbf{v} = (4t - 3)\mathbf{i} + (2t + 3)\mathbf{j}, \quad t \geq 0,$$

where \mathbf{i} and \mathbf{j} are unit vectors pointing due east and due north, respectively.

- a) Find the magnitude of the acceleration of P .

When $t = 1$, the position vector of P is $8\mathbf{j} \text{ m}$.

- b) Determine the **initial distance** of P from the origin O .

$$a = \sqrt{20} \approx 4.47 \text{ ms}^{-2}, \quad d = \sqrt{17} \approx 4.12 \text{ m}$$

Handwritten solution for Question 3:

$\mathbf{v} = (4t-3)\mathbf{i} + (2t+3)\mathbf{j}$ $t=1 \quad \mathbf{r} = 8\mathbf{j}$

(a) $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 4\mathbf{i} + 2\mathbf{j}$
 $|\mathbf{a}| = \sqrt{4^2 + 2^2} = \sqrt{20} \approx 4.47 \text{ ms}^{-2}$

(b) $\mathbf{y} = (4t-3)\mathbf{i} + (2t+3)\mathbf{j}$
 $\mathbf{r} = \int (4t-3)\mathbf{i} + (2t+3)\mathbf{j} \, dt$
 $\mathbf{r} = (2t^2 - 3t + C)\mathbf{i} + (t^2 + 3t + D)\mathbf{j}$
 when $t=1 \quad \mathbf{r} = 8\mathbf{j}$
 $8\mathbf{j} = (-1+C)\mathbf{i} + (4+D)\mathbf{j} \quad \therefore C=1$
 $\mathbf{r} = (2t^2 - 3t + 1)\mathbf{i} + (t^2 + 3t + 4)\mathbf{j}$
 when $t=0, \quad \mathbf{r} = \mathbf{i} + 4\mathbf{j}$
 DISTANCE FROM O IS $\sqrt{1^2 + 4^2} = \sqrt{17} \approx 4.12 \text{ m}$

Question 4 (***)

The velocity, $v \text{ ms}^{-1}$, of a particle of mass 2 kg, t s after a given instant is given by

$$\mathbf{v} = 6t^2\mathbf{i} - 6t^{\frac{3}{2}}\mathbf{j}, \quad t \geq 0,$$

where \mathbf{i} and \mathbf{j} are unit vectors pointing due east and due north, respectively.

- a) Find the magnitude of the resultant force acting on the particle, when $t = 1$.

When $t = 0$, the particle is at the point A whose position vector is $(2\mathbf{i} + \mathbf{j})$ m and when $t = 1$ the particle is at the point B .

- b) Determine the distance AB .

$$F = 30 \text{ N}, \quad |AB| \approx 3.12 \text{ m}$$

(a) $\mathbf{v} = 6t^2\mathbf{i} - 6t^{\frac{3}{2}}\mathbf{j}$
 $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 12t\mathbf{i} - 9t^{\frac{1}{2}}\mathbf{j}$
 At $t=1$
 $\mathbf{a} = 12\mathbf{i} - 9\mathbf{j}$
 $|\mathbf{a}| = \sqrt{12^2 + 9^2} = 15 \text{ m/s}^2$
 $F = ma = 30 \text{ N}$

(b) $\mathbf{v} = 6t^2\mathbf{i} - 6t^{\frac{3}{2}}\mathbf{j}$
 $\mathbf{r} = \int 6t^2\mathbf{i} - 6t^{\frac{3}{2}}\mathbf{j} dt$
 $\mathbf{r} = (2t^3 + C_1)\mathbf{i} + (-4t^{\frac{3}{2}} + D_1)\mathbf{j}$
 At $t=0$, $\mathbf{r}_0 = 2\mathbf{i} + \mathbf{j}$
 $2 = C_1$
 $1 = D_1$
 $\mathbf{r} = (2t^3 + 2)\mathbf{i} + (-4t^{\frac{3}{2}} + 1)\mathbf{j}$
 At $t=1$
 $\mathbf{r}_1 = 4\mathbf{i} - 3\mathbf{j}$
 $\mathbf{r}_0 = 2\mathbf{i} + \mathbf{j}$
 $|AB| = \sqrt{(4-2)^2 + (-3-1)^2} \approx 3.12$

Question 5 (***)

The velocity, $v \text{ ms}^{-1}$, of a particle of mass 5 kg, $t \text{ s}$ after a given instant is given by

$$\mathbf{v} = (12t^2 - 2)\mathbf{i} + (2t - 3t^2)\mathbf{j}, \quad t \geq 0,$$

where \mathbf{i} and \mathbf{j} are unit vectors pointing due east and due north, respectively.

a) Find the magnitude of the resultant force acting on the particle, when $t = 2$.

b) Find the value of t when the particle's acceleration is parallel to the x axis.

When $t = 0$, the particle is at the point A with position vector $(\mathbf{i} + 6\mathbf{j}) \text{ m}$ and when $t = 1$, the particle is at the point B .

c) Determine the distance AB .

$$F \approx 245 \text{ N}, \quad t = \frac{1}{3}, \quad |AB| = 2 \text{ m}$$

(a) $\mathbf{v} = (12t^2 - 2)\mathbf{i} + (2t - 3t^2)\mathbf{j}$
 $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 24t\mathbf{i} + (2 - 6t)\mathbf{j}$
 when $t = 2$
 $\mathbf{a} = 48\mathbf{i} - 10\mathbf{j}$
 $|\mathbf{a}| = \sqrt{(48)^2 + (-10)^2} = \sqrt{2404}$
 $F = ma$
 $F = 5 \times \sqrt{2404}$
 $F \approx 245 \text{ N}$

(b) ACCELERATION PARALLEL TO x AXIS
 $\mathbf{a} = k\mathbf{i} + 0\mathbf{j}$
 $\mathbf{a} = 24t\mathbf{i} + (2 - 6t)\mathbf{j}$
 $2 - 6t = 0$
 $2 = 6t$
 $t = \frac{1}{3}$

(c) $\mathbf{r} = \int \mathbf{v} dt$
 $\mathbf{r} = \int (12t^2 - 2)\mathbf{i} + (2t - 3t^2)\mathbf{j} dt$
 $\mathbf{r} = (4t^3 - 2t)\mathbf{i} + (t^2 - t^3)\mathbf{j}$
 when $t = 0$, $\mathbf{r}_0 = \mathbf{i} + 6\mathbf{j}$
 $\mathbf{r} + \mathbf{r}_0 = C\mathbf{i} + D\mathbf{j}$
 $\mathbf{r} = (4t^3 - 2t + 1)\mathbf{i} + (t^2 - t^3 + 6)\mathbf{j}$
 when $t = 1$
 $\mathbf{r}_1 = 3\mathbf{i} + 6\mathbf{j}$
 $\therefore A(1, 6)$
 $B(3, 6)$
 $\therefore |AB| = 2$
 (SINCE THEY ARE AT THE SAME HEIGHT)

Question 6 (***)

The position vector, \mathbf{r} m, of a particle of mass 0.5 kg, t s after a given instant satisfies

$$\mathbf{r} = (3t^2 - 7t + 2)\mathbf{i} + (2t^2 - 5t + 2)\mathbf{j}, \quad t \geq 0,$$

where \mathbf{i} and \mathbf{j} are unit vectors pointing due east and due north, respectively.

- Find the value of t when the particle is at the origin.
- Determine the magnitude of the resultant force acting on the particle.
- Find the value of t when the particle is moving parallel to the vector $2\mathbf{i} + \mathbf{j}$.

$$t = 2, \quad F = \sqrt{13} \approx 3.61 \text{ N}, \quad t = 1.5$$

Handwritten solution for Question 6:

(a) $\mathbf{r} = (3t^2 - 7t + 2)\mathbf{i} + (2t^2 - 5t + 2)\mathbf{j}$
 $\mathbf{r} = (3t-1)(t-2)\mathbf{i} + (2t-1)(t-2)\mathbf{j}$
 By inspection when $t=2$ both components are zero, if particle is at the origin
 $\therefore t=2$

(b) $\mathbf{v} = \frac{d\mathbf{r}}{dt} = (6t-7)\mathbf{i} + (4t-5)\mathbf{j}$
 $\mathbf{a} = \frac{d\mathbf{v}}{dt} = 6\mathbf{i} + 4\mathbf{j}$
 $|\mathbf{a}| = \sqrt{6^2 + 4^2} = \sqrt{52}$
 $F = ma$
 $F = 0.5 \times \sqrt{52}$
 $F = \sqrt{13} \approx 3.61 \text{ N}$

(c) $\mathbf{v} = (6t-7)\mathbf{i} + (4t-5)\mathbf{j}$
 "Parallel to $2\mathbf{i} + \mathbf{j}$ "
 $\Rightarrow \frac{6t-7}{4t-5} = \frac{2}{1}$
 $\Rightarrow 6t-7 = 8t-10$
 $3 = 2t$
 $t = \frac{3}{2} = 1.5$

Question 7 (***)

The acceleration \mathbf{a} ms^{-2} of a particle P of mass 0.2 kg, t s after a given instant is given by

$$\mathbf{a} = (2t - 4)\mathbf{i} + 3\mathbf{j}, \quad t \geq 0,$$

where \mathbf{i} and \mathbf{j} are unit vectors pointing along the positive x axis and along the positive y axis, respectively.

- a) Find the magnitude of the resultant force acting on P , when $t = 4$.

It is further given that when $t = 0$, P is at the point A with position vector $(-18\mathbf{i} - 24\mathbf{j})$ m and has velocity $(3\mathbf{i} - 9\mathbf{j})$ ms^{-1} .

- b) Find the value of t when the particle is at rest.
 c) Show that when $t = 6$, P is on the y axis and state its distance from A .
 d) Determine the value of t when the particle is on the x axis.

, $F = 1\text{ N}$, $t = 3$, 18 m , $t = 8$

a) $\mathbf{a} = (2t-4)\mathbf{i} + 3\mathbf{j}$
 $\mathbf{a}_x = (2t-4)\mathbf{i} + 3\mathbf{j}$
 $\mathbf{a}_x = 4\mathbf{i} + 3\mathbf{j}$
 $|\mathbf{a}_x| = \sqrt{4^2 + 3^2}$
 $|\mathbf{a}_x| = 5 \text{ ms}^{-2}$

using $\mathbf{F} = m\mathbf{a}$
 $F = 0.2 \times 5$
 $F = 1\text{ N}$

b) INTEGRATE THE ACCELERATION VECTOR TO OBTAIN VELOCITY VECTOR
 $\Rightarrow \mathbf{v} = \int (2t-4)\mathbf{i} + 3\mathbf{j} \, dt$
 $\Rightarrow \mathbf{v} = (t^2 - 4t + A)\mathbf{i} + (3t + B)\mathbf{j}$

when $t=0$ $\mathbf{v} = 3\mathbf{i} - 9\mathbf{j}$
 $\therefore 0 - 0 = A + 0$
 $A = 3$
 $0 + B = -9$
 $B = -9$

$\therefore \mathbf{v} = (t^2 - 4t + 3)\mathbf{i} + (3t - 9)\mathbf{j}$
 $\mathbf{v} = (t-3)(t-1)\mathbf{i} + 3(t-3)\mathbf{j}$
 \therefore BY INSPECTION $\mathbf{v} = 0$ when $t = 3$

c) INTEGRATE AGAIN TO OBTAIN THE POSITION VECTOR
 $\mathbf{r} = \int (t^2 - 4t + 3)\mathbf{i} + (3t - 9)\mathbf{j} \, dt$
 $\mathbf{r} = (\frac{1}{3}t^3 - 2t^2 + 3t + C)\mathbf{i} + (\frac{3}{2}t^2 - 9t + D)\mathbf{j}$

when $t=0$ $\mathbf{r} = -18\mathbf{i} - 24\mathbf{j}$
 $\Rightarrow -0 - 0 = C + 0$
 $C = -18$
 $0 - 0 = D - 24$
 $D = -24$

$\therefore \mathbf{r} = (\frac{1}{3}t^3 - 2t^2 + 3t - 18)\mathbf{i} + (\frac{3}{2}t^2 - 9t - 24)\mathbf{j}$

when $t = 6$
 $\mathbf{r} = (\frac{1}{3}(6^3 - 2 \cdot 6^2 + 3 \cdot 6 - 18))\mathbf{i} + (\frac{3}{2}(6^2 - 9 \cdot 6 - 24))\mathbf{j}$
 $= (2 - 2 + 18 - 18)\mathbf{i} + (54 - 54 - 24)\mathbf{j} = -24\mathbf{j}$
INTEGRATE ON THE Y AXIS
 \therefore DISTANCE FROM A IS 18 m

d) when $\mathbf{v} = 0$, it is COMPACTED ZERO IN \mathbf{r}
 $\Rightarrow \frac{1}{3}t^3 - 4t + 3 = 0$
 $\Rightarrow t^3 - 12t + 9 = 0$
 $\Rightarrow (t-3)(t+2) = 0$
 $\therefore t = 3$

Question 8 (****)

The position vector, velocity and acceleration of a particle P , t s after a given instant are denoted by \mathbf{r} m, \mathbf{v} ms⁻¹ and \mathbf{a} ms⁻².

When $t = 1$, $\mathbf{r} = 9\mathbf{i} + 2\mathbf{j}$ and $\mathbf{v} = 13\mathbf{i} + \mathbf{j}$, where \mathbf{i} and \mathbf{j} are unit vectors pointing due east and due north, respectively.

It is further given that P has a constant acceleration of $6\mathbf{i}$ ms⁻².

- a) Determine the distance of P from the origin O , when $t = 3$.
- b) Show that P is moving on the curve with equation

$$x = 3y^2 + y - 5.$$

, ≈ 47.17 m

a) INTEGRATE BACK FROM ACCELERATION TO VELOCITY

$\Rightarrow \mathbf{a} = 6\mathbf{i} + 0\mathbf{j}$

$\Rightarrow \mathbf{v} = \int (6\mathbf{i} + 0\mathbf{j}) dt$

$\Rightarrow \mathbf{v} = (6t + A)\mathbf{i} + B\mathbf{j}$

When $t=1$, $\mathbf{v} = 13\mathbf{i} + \mathbf{j}$

$\Rightarrow 13\mathbf{i} + \mathbf{j} = (6(1) + A)\mathbf{i} + B\mathbf{j}$

$\Rightarrow 13 = 6 + A$ and $1 = B$

$\Rightarrow A = 7$ and $B = 1$

$\Rightarrow \mathbf{v} = (6t + 7)\mathbf{i} + \mathbf{j}$

INTEGRATE AGAIN TO OBTAIN THE POSITION VECTOR

$\Rightarrow \mathbf{r} = \int ((6t + 7)\mathbf{i} + \mathbf{j}) dt$

$\Rightarrow \mathbf{r} = (3t^2 + 7t + C)\mathbf{i} + (t + D)\mathbf{j}$

When $t=1$, $\mathbf{r} = 9\mathbf{i} + 2\mathbf{j}$

$\Rightarrow 9\mathbf{i} + 2\mathbf{j} = (3(1)^2 + 7(1) + C)\mathbf{i} + (1 + D)\mathbf{j}$

$\Rightarrow 9 = 10 + C$ and $2 = 1 + D$

$\Rightarrow C = -1$ and $D = 1$

$\Rightarrow \mathbf{r} = (3t^2 + 7t - 1)\mathbf{i} + (t + 1)\mathbf{j}$

NOW WHEN $t=3$

$\Rightarrow \mathbf{r} = (3(3^2) + 7(3) - 1)\mathbf{i} + (3 + 1)\mathbf{j} = 49\mathbf{i} + 4\mathbf{j}$

$\Rightarrow |\mathbf{r}| = \sqrt{49^2 + 4^2} = \sqrt{2425} \approx 49.17 \text{ m}$

b) $\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3t^2 + 7t - 1 \\ t + 1 \end{pmatrix}$

$t = y - 1$

$\Rightarrow x = 3(y - 1)^2 + 7(y - 1) - 1$

$\Rightarrow x = 3(y^2 - 2y + 1) + 7y - 7 - 1$

$\Rightarrow x = 3y^2 - 6y + 3 + 7y - 8$

$\Rightarrow x = 3y^2 + y - 5$

ALTERNATIVE FOR PART (a) - REMEMBER AS $t=1 \rightarrow t=0$

$\mathbf{a} = 6\mathbf{i}$ CONSTANT

$t=1$ (NORMAL)

$\mathbf{v} = 13\mathbf{i} + \mathbf{j}$

$\mathbf{r} = 9\mathbf{i} + 2\mathbf{j}$

$\mathbf{a} = 6\mathbf{i}$

$\mathbf{v} = \int 6\mathbf{i} dt = 6t\mathbf{i} + \mathbf{v}_0$

$13\mathbf{i} + \mathbf{j} = 6(1)\mathbf{i} + \mathbf{v}_0$

$\mathbf{v}_0 = 7\mathbf{i} + \mathbf{j}$

$\mathbf{v} = 6t\mathbf{i} + 7\mathbf{i} + \mathbf{j} = (6t + 7)\mathbf{i} + \mathbf{j}$

$\mathbf{r} = \int ((6t + 7)\mathbf{i} + \mathbf{j}) dt = (3t^2 + 7t)\mathbf{i} + t\mathbf{j} + \mathbf{r}_0$

$9\mathbf{i} + 2\mathbf{j} = (3(1)^2 + 7(1))\mathbf{i} + 1\mathbf{j} + \mathbf{r}_0$

$9\mathbf{i} + 2\mathbf{j} = 10\mathbf{i} + \mathbf{j} + \mathbf{r}_0$

$\mathbf{r}_0 = -\mathbf{i} + \mathbf{j}$

$\mathbf{r} = (3t^2 + 7t - 1)\mathbf{i} + (t + 1)\mathbf{j}$

$t=3$, $\mathbf{r} = 49\mathbf{i} + 4\mathbf{j}$

$|\mathbf{r}| = \sqrt{49^2 + 4^2} = \sqrt{2425} \approx 49.17 \text{ m}$