# MA 114 Worksheet Calendar – Fall 2016

Thur, Aug 25: Worksheet01 – Review substitution, area between curves

Tues, Aug 28: Worksheet02 – Integration by parts

Thur, Sep 01: Worksheet03 – Partial fractions

Tues, Sep 06: Worksheet04 – Special trig integrals

Thur, Sep 08: Worksheet05 – Numerical integration – trapezoid, midpoint

Tues, Sep 13: Worksheet06 – Simpson's Rule, Improper integrals

Thur, Sep 15: Worksheet07 – Sequences

Tues, Sep 20: Worksheet08 – Review for Exam 01

Thur, Sep 22: Worksheet09 – Recursive sequences, series

Tues, Sep 27: Worksheet10 – Series, Integral Test

Thur, Sep 29: Worksheet11 – Comparison and Limit Comparison Tests

Tues, Oct 04: Worksheet12 – Alternating series, absolute & conditional convergence

Thur, Oct 06: Worksheet13 – Ratio & root tests

Tues, Oct 11: Worksheet14 – Power series

Thur, Oct 13: Worksheet15 – Taylor series

Tues, Oct 18: Worksheet16 – Review for Exam 02

Thur, Oct 20: Worksheet17 – Average value of a function

Tues, Oct 25: Worksheet18 – Volumes – known cross-section and washers

Thur, Oct 27: Worksheet19 – Volumes by washers and shells

Tues, Nov 1: Worksheet20 – Arc length & surface area

Thur, Nov 3: Worksheet21 – Centers of mass & moments

Tues, Nov 8: Worksheet22 – Parametric equations

Thur, Nov 10: Worksheet23 – Calculus with parametric equations

Tues, Nov 15: Worksheet24 – Review for Exam 03

Thur, Nov 17: Worksheet25 – Polar coordinates

Tues, Nov 22: Worksheet26 – Calculus with polar coordinates

Tues, Nov 29: Worksheet27 – Differential equations, direction fields

Thur, Dec 01: Worksheet28 – Separable equations

Tues, Dec 06: Worksheet29 – Conic sections

Thur, Dec 08: Worksheet30 – Review for Final Exam

# MA 114 Worksheet #01: Substitution Review

1. Evaluate the following indefinite integrals and indicate the substitutions that you use.

   (a) $\displaystyle\int \frac{4}{(1+2x)^3}dx$

   (b) $\displaystyle\int x^2\sqrt{x^3+1}$

   (c) $\displaystyle\int \cos^4 x \sin x \, dx$

   (d) $\displaystyle\int \sec^3 x \tan x \, dx$

   (e) $e^x \sin(e^x)$

   (f) $\dfrac{2x+3}{x^2+3x}$

2. Evaluate the following definite integrals and indicate the substitutions that you use.

   (a) $\displaystyle\int_0^7 \sqrt{4+3x}\, dx$

   (b) $\displaystyle\int_0^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\, dx$

   (c) $\displaystyle\int_0^3 \frac{dx}{6x+1}$

   (d) $\displaystyle\int_{-\pi/3}^{\pi/3} x^4 \sin x \, dx$

   (e) $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\, dz$

   (f) $\displaystyle\int_0^4 \frac{x}{\sqrt{1+2x}}\, dx$

3. If $f$ is continuous and $\int_0^6 f(x) = 8$, find $\int_0^2 f(3x)$.

4. If $f$ is continuous and $\int_0^{25} f(x)dx = 16$, find $\int_0^5 xf(x^2)\, dx$.

5. Find the area of the region between the graphs of $y = x^2$ and $y = x^4$.

6. Find the area of the regions enclosed by the graphs of $y = \sqrt{x}$ and $y = \frac{1}{4}x + \frac{3}{4}$ in two ways.

   (a) Write this as an integral in $x$.

   (b) Solve each equation to express $x$ in terms of $y$ and write an integral with respect to $y$.

7. Find the area of the region enclosed by the graphs of $y = x+1$ and $y = x^3 + x^2 - x + 1$.

8. What is the area of the region bounded by $f(x) = \dfrac{1}{x}$, $x = e^2$, $x = e^8$ and $x$-axis? Sketching the region might be helpful.

9. If $f$ is continuous on $[0, 1]$, show that

$$\int_0^1 f(x)dx = \int_0^1 f(1-x)dx.$$

10. Find the area of the region bounded by the parabola $y = x^2$, the tangent line to the parabola at $(1, 1)$ and the $x$-axis.

# MA 114 Worksheet #02: Integration by parts

1. Which of the following integrals should be solved using substitution and which should be solved using integration by parts?

   (a) $\int x\cos(x^2)\,dx$,

   (b) $\int e^x\sin(x)\,dx$,

   (c) $\int \frac{\ln(\arctan(x))}{1+x^2}dx$,

   (d) $\int xe^{x^2}\,dx$

2. Solve the following integrals using integration by parts:

   (a) $\int x^2\sin(x)\,dx$,

   (b) $\int (2x+1)e^x\,dx$,

   (c) $\int x\sin(3-x)\;dx$,

   (d) $\int 2x\arctan(x)\,dx$,

   (e) $\int \ln(x)\,dx$

   (f) $\int x^4\ln(x)\,dx$

   (g) $\int e^x\sin x\,dx$

   (h) $\int x\ln(1+x)\,dx$ Hint: Make a substitution first, then try integration by parts.

3. Let $f(x)$ be a twice differentiable function with $f(1)=2$, $f(4)=7$, $f'(1)=5$ and $f'(4)=3$. Evaluate $\int_1^4 xf''(x)\,dx$

4. If $f(0)=g(0)=0$ and $f''$ and $g''$ are continuous, show that

$$\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx.$$

# MA 114 Worksheet #03: Integration by Partial Fractions

1. Write out the general form for the partial fraction decomposition but do not determine the numerical value of the coefficients.
   (a) $\dfrac{1}{x^2+3x+2}$
   (b) $\dfrac{x+1}{x^2+4x+4}$
   (c) $\dfrac{x}{(x^2+1)(x+1)(x+2)}$
   (d) $\dfrac{2x+5}{(x^2+1)^3(2x+1)}$
2. Compute the following integrals.
   (a) $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$
   (b) $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$
   (c) $\displaystyle\int \frac{x^3-2x^2+1}{x^3-2x^2}\,dx$
   (d) $\displaystyle\int \frac{x^3+4}{x^2+4}\,dx$
   (e) $\displaystyle\int \frac{1}{x(x^2+1)}\,dx$
3. Compute
$$\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$$
by first making the substitution $u=\sqrt[6]{x}$.

# MA 114 Worksheet #04: Special Trig Integrals

1. Compute the following integrals:

   (a) $\displaystyle\int_0^2 \frac{u^3}{\sqrt{16-u^2}}du$

   (b) $\displaystyle\int \frac{1}{x^2\sqrt{25-x^2}}dx$

   (c) $\displaystyle\int_0^{\pi/2} \cos^2(x)\,dx$

   (d) $\displaystyle\int \sqrt{\cos x}\sin^3 x\,dx$

   (e) $\displaystyle\int_0^{2\pi} \sin^2\left(\tfrac{1}{3}\theta\right)\,d\theta$

   (f) $\displaystyle\int_0^{\pi/2} (2-\sin\theta)^2\,d\theta$

   (g) $\displaystyle\int \frac{\sqrt{1-x^2}}{x^4}dx$

   (h) $\displaystyle\int_0^3 \frac{x}{\sqrt{36-x^2}}dx$. Hint: Use the substitution $x = 6u$.

   (i) $\displaystyle\int_0^{1/2} x\sqrt{1-4x^2}\,dx$. Hint: Substitute $x = u/2$.

2. Let $r > 0$. Consider the identity

$$\int_0^s \sqrt{r^2-x^2}\,dx = \frac{1}{2}r^2\arcsin(s/r) + \frac{1}{2}s\sqrt{r^2-s^2}$$

   where $0 \leq s \leq r$.

   (a) Plot the curves $y = \sqrt{r^2-x^2}$, $x = s$, and $y = \dfrac{x}{s}\sqrt{r^2-x^2}$.

   (b) Using part (a), verify the identity geometrically.

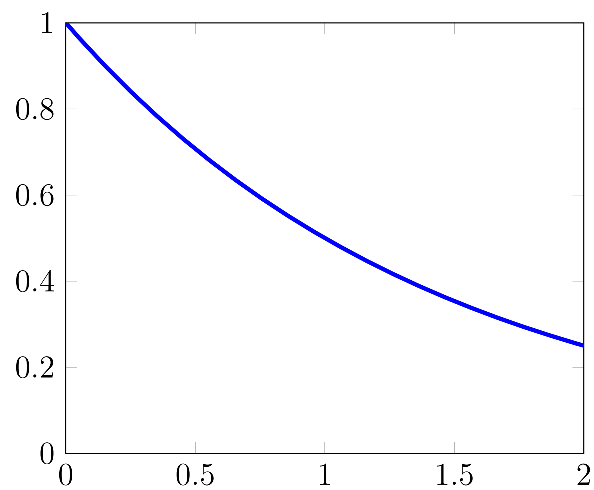
   (c) Verify the identity using trigonometric substitution.

# MA 114 Worksheet #05: Numerical Integration

1. (a) Write down the Midpoint rule and illustrate how it works with a sketch.
   (b) Write down the Trapezoid rule and the error bound associated with it.
   (c) How large should $n$ be in the Midpoint rule so that you can approximate
   $$\int_0^1 \sin x\, dx$$
   with an error less than $10^{-7}$?

2. Use the Midpoint rule to approximate the value of $\displaystyle\int_{-1}^{1} e^{-x^2}\, dx$ with $n = 4$. Draw a sketch to determine if the approximation is an overestimate or an underestimate of the integral.

3. The left, right, Trapezoidal, and Midpoint Rule approximations were used to estimate $\int_0^2 f(x)\, dx$, where $f$ is the function whose graph is shown. The estimates were 0.7811, 0.8675, 0.8632, and 0.9540, and the same number of sub- intervals were used in each case.
   (a) Which rule produced which estimate?
   (b) Between which two approximations does the true value of $\int_0^2 f(x)\, dx$ lie?




4. Draw the graph of $f(x) = \sin\left(\frac{1}{2}x^2\right)$ in the viewing rectangle $[0, 1]$ by $[0, 0.5]$ and let $I = \int_0^1 f(x)\, dx$.
   (a) Use the graph to decide whether $L_2$, $R_2$, $M_2$, and $T_2$ underestimate or overestimate $I$.
   (b) For any value of $n$, list the numbers $L_n$, $R_n$, $M_n$, $T_n$, and $I$ in increasing order.
   (c) Compute $L_5$, $R_5$, $M_5$, and $T_5$. From the graph, which do you think gives the best estimate of $I$?

5. The velocity in meters per second for a particle traveling along the axis is given in the table below. Use the Midpoint rule and Trapezoid rule to approximate the total distance the particle traveled from $t = 0$ to $t = 6$.

| $t$ | $v(t)$ |
|---|---|
| 0 | 0.75 |
| 1 | 1.34 |
| 2 | 1.5 |
| 3 | 1.9 |
| 4 | 2.5 |
| 5 | 3.2 |
| 6 | 3.0 |

# MA 114 Worksheet #06: Simpson's Rule & Improper Integrals

1. (a) Write down Simpson's rule and illustrate how it works with a sketch.
   (b) How large should $n$ be in the Simpson's rule so that you can approximate
   $$\int_0^1 \sin x \, dx$$
   with an error less than $10^{-7}$?

2. Approximate the integral $\int_1^2 \frac{1}{x}\, dx$ using Simpson's rule. Choose $n$ so that your error is certain to be less than $10^{-3}$. Compute the exact value of the integral and compare to your approximation.

3. Simpson's Rule turns out to exactly integrate polynomials of degree three or less. Show that Simpson's rule gives the *exact* value of $\int_0^h p(x)\, dx$ where $h > 0$ and $p(x) = ax^3 + bx^2 + cx + d$. [Hint: First compute the exact value of the integral by direct integration. Then apply Simpson's rule with n = 2 and observe that the approximation and the exact value are the same.]

4. For each of the following, determine if the integral is proper or improper. If it is improper, explain why. Do ***not*** evaluate any of the integrals.
   (a) $\displaystyle\int_0^2 \frac{x}{x^2 - 5x + 6}\, dx$
   (b) $\displaystyle\int_1^2 \frac{1}{2x - 1}\, dx$
   (c) $\displaystyle\int_1^2 \ln(x-1)\, dx$
   (d) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin x}{1 + x^2}\, dx$
   (e) $\displaystyle\int_0^{\pi/2} \sec x \, dx$

5. For the integrals below, determine if the integral is convergent or divergent. Evaluate the convergent integrals.
   (a) $\displaystyle\int_{-\infty}^{0} \frac{1}{2x - 1}\, dx$
   (b) $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\, dx$
   (c) $\displaystyle\int_0^2 \frac{x - 3}{2x - 3}\, dx$
   (d) $\displaystyle\int_0^{\infty} \sin\theta \, d\theta$

6. Consider the improper integral
   $$\int_1^{\infty} \frac{1}{x^p}\, dx.$$
   Integrate using the generic parameter $p$ to prove the integral converges for $p > 1$ and diverges for $p \leq 1$. You will have to distinguish between the cases when $p = 1$ and $p \neq 1$ when you integrate.

7. Use the Comparison Theorem to determine whether the following integrals are convergent or divergent.
   (a) $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$
   (b) $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^6+x}}\,dx$

8. Explain why the following computation is wrong and determine the correct answer. (Try sketching or graphing the integrand to see where the problem lies.)

$$\begin{aligned}\int_2^{10} \frac{1}{2x-8}dx &= \frac{1}{2}\int_{-4}^{12} \frac{1}{u}\,du \\ &= \frac{1}{2}\ln|x|\Big|_{-4}^{12} \\ &= \frac{1}{2}(\ln 12 - \ln 4)\end{aligned}$$

   where we used the substitution

$$\begin{cases} u(x) = 2x-8 & \\ u(2) = -4 & u(10) = 12 \\ \dfrac{du}{dx} = 2 & \end{cases}$$

9. A manufacturer of light bulbs wants to produce bulbs that last about 700 hours but, of course, some bulbs burn out faster than others. Let $F(t)$ be the fraction of the companys bulbs that burn out before $t$ hours, so $F(t)$ always lies between 0 and 1.
   (a) Make a rough sketch of what you think the graph of $F$ might look like.
   (b) What is the meaning of the derivative $r(t) = F'(t)$?
   (c) What is the value of $\int_0^\infty r(t)\,dt$? Why?

# MA 114 Worksheet #07: Sequences

1. (a) Give the precise definition of a **sequence**.

   (b) What does it mean to say that $\lim_{x\to a} f(x) = L$ when $a = \infty$? Does this differ from $\lim_{n\to\infty} f(n) = L$? Why or why not?

   (c) What does it means for a sequence to converge? Explain your idea, not just the definition in the book.

   (d) Sequences can diverge in different ways. Describe two distinct ways that a sequence can diverge.

   (e) Give two examples of sequences which converge to 0 and two examples of sequences which converges to a given number $L \neq 0$.

2. Write the first four terms of the sequences with the following general terms:

   (a) $\dfrac{n!}{2^n}$

   (b) $\dfrac{n}{n+1}$

   (c) $(-1)^{n+1}$

   (d) $\{a_n\}_{n=1}^{\infty}$ where $a_n = \dfrac{3}{n}$.

   (e) $\{a_n\}_{n=1}^{\infty}$ where $a_n = 2^{-n} + 2$.

   (f) $\{b_k\}_{k=1}^{\infty}$ where $b_k = \dfrac{(-1)^k)}{k^2}$.

3. Find a formula for the $n$th term of each sequence.

   (a) $\left\{\frac{1}{1}, -\frac{1}{8}, \frac{1}{27}, -\frac{1}{64}, \dots\right\}$

   (b) $\left\{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right\}$

   (c) $\{1, 0, 1, 0, 1, 0, \dots\}$

   (d) $\{-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}, \dots, \}$

4. Suppose that a sequence $\{a_n\}$ is bounded above and below. Does it converge? If not, find a counterexample.

5. The limit laws for sequences are the same as the limit laws for functions. Suppose you have sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ with $\lim_{n\to\infty} a_n = 15$, $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} c_n = 1$. Use the limit laws of sequences to answer the following questions.

   (a) Does the sequence $\left\{\dfrac{a_n \cdot c_n}{b_n + 1}\right\}_{n=1}^{\infty}$ converge? If so, what is its limit?

   (b) Does the sequence $\left\{\dfrac{a_n + 3 \cdot c_n}{2 \cdot b_n + 2}\right\}_{n=1}^{\infty}$ converge? If so, what is its limit?

# MA 114 Worksheet #09: Recursive sequences & Series

1. Write out the first five terms of Let
   (a) $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = 3a_{n-1} + {a_n}^2$.
   (b) $a_1 = 6$, $a_{n+1} = \dfrac{a_n}{n}$.
   (c) $a_1 = 2$, $a_{n+1} = \dfrac{a_n}{a_n + 1}$.
   (d) $a_1 = 1$, $a_{n+1} = \sqrt{\left(\dfrac{2}{a_n}\right)^2 + 1}$.
   (e) $a_1 = 2$, $a_2 = 1$, and $a_{n+1} = a_n - a_{n-1}$.
2. (a) For what values of $x$ does the sequence $\{x^n\}_{n=1}^{\infty}$ converge?
   (b) For what values of $x$ does the sequence $\{n^x\}_{n=1}^{\infty}$ converge?
   (c) If $\lim\limits_{n\to\infty} b_n = \sqrt{2}$, find $\lim\limits_{n\to\infty} b_{n-3}$.
3. (a) Determine whether the sequence defined as follows is convergent or divergent:
   $$a_1 = 1 \qquad a_{n+1} = 4 - a_n \qquad \text{for } n > 1.$$
   (b) What happens if the first term is $a_1 = 2$?
4. A fish farmer has 5000 catfish in his pond. The number of catfish increases by 8% per month and the farmer harvests 300 catfish per month.
   (a) Show that the catfish population $P_n$ after $n$ months is given recursively by
   $$P_n = 1.08P_{n-1} - 300 \qquad P_0 = 5000.$$
   (b) How many catfish are in the pond after six months?

# MA 114 Worksheet #10: Series & The Integral Test

1. Identify the following statements as true or false and explain your answers.
   (a) If the sequence of partial sums of an in nite series is bounded the series converges.
   (b) $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} a_n$ if the series converges.
   (c) $\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} a_{n+1}$ if both series converge.
   (d) If $c$ is a nonzero constant and if $\sum_{n=1}^{\infty} ca_n$ converges then so does $\sum_{n=1}^{\infty} a_n$.
   (e) A finite number of terms of an infinite series may be changed without affecting whether or not the series converges.
   (f) Every infinite series with only finitely many nonzero terms converges.

2. Write the following in summation notation:
   (a) $\frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \ldots$
   (b) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

3. Calculate $S_3$, $S_4$, and $S_5$ and then find the sum of the telescoping series $S = \sum_{n=1}^{\infty} \left( \frac{1}{n+1} - \frac{1}{n+2} \right)$.

4. Use the formula for the sum of a geometric series to find the sum or state that the series diverges and why:
   (a) $\frac{1}{1} + \frac{1}{8} + \frac{1}{8^2} + \ldots$
   (b) $\sum_{n=0}^{\infty} \left( \frac{\pi}{e} \right)^n$

5. Use the Integral Test to determine if the following series converge or diverge:
   (a) $\sum_{n=0}^{\infty} \frac{1}{1+n^2}$
   (b) $\sum_{n=1}^{\infty} n^2 e^{-n^3}$
   (c) $\sum_{n=2}^{\infty} \frac{n}{(n^2+2)^{3/2}}$

6. Show that the infinite series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges otherwise by Integral Test.

# MA 114 Worksheet #11: Comparison & Limit Comparison Tests

1. (a) Explain the test for divergence. Why should you never use this test to prove that a series converges?

   (b) State the comparison test for series. Explain the idea behind this test.

   (c) Suppose that the sequences $\{x_n\}$ and $\{y_n\}$ satisfy $0 \leq x_n \leq y_n$ for all $n$ and that $\sum_{n=1}^{\infty} y_n$ is convergent. What can you conclude? What can you conclude if instead $\sum_{n=1}^{\infty} y_n$ diverges?

   (d) State the limit comparison test. Explain how you apply this test.

2. Use the appropriate test — Divergence Test, Comparison Test or Limit Comparison Test — to determine whether the infinite series is convergent or divergent.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}+1}$

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\sqrt{n^2+2}}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{2+5^n}$

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{4^n+2}{3^n+1}$

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{n^4}$

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{(n+1)!}$

   (g) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{10}{n}\right)^{10}$

   (h) $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{n^2\sqrt{n}}$

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{\sqrt{n^2+2}}$

   (j) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+n+1}{3n^2+14n+7}$

   (k) $\displaystyle\sum_{n=0}^{\infty} \frac{1+2^n}{2+5^n}$

   (l) $\displaystyle\sum_{n=0}^{\infty} \frac{2}{n^2+5n+2}$

   (m) $\displaystyle\sum_{n=0}^{\infty} \frac{e^{1/n}}{n}$

   (n) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2-\cos^2 n}$

# MA 114 Worksheet #12: Alternating Series & Absolute/Conditional Convergence

1. (a) Let $a_n = \dfrac{n}{3n+1}$. Does $\{a_n\}$ converge? Does $\sum_{n=1}^{\infty} a_n$ converge?

   (b) Give an example of a divergent series $\sum_{n=1}^{\infty} a_n$ where $\lim\limits_{n\to\infty} a_n = 0$.

   (c) Does there exist a convergent series $\sum_{n=1}^{\infty} a_n$ which satisfies $\lim\limits_{n\to\infty} a_n \neq 0$? Explain.

   (d) When does a series converge absolutely? When does a series converge conditionally?

   (e) State the alternating series test.

   (f) Prove that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.

   (g) State the Alternating Series Estimation Theorem.

2. Test the following series for convergence or divergence.

   (a) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2n}$

   (b) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{\ln n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{2/3}}$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{4^n + 5^n}$

   (e) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-5}{18}\right)$

3. Use the Alternating Series Estimation Theorem to estimate the sum correct to four decimal places.

   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-0.8)^n}{n!}$

   (b) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{8^n}$

4. Approximate the sum of the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{(2n)!}$ correct to four decimal places; *i. e.* so that $|\text{error}| < 0.00005$.

# MA 114 Worksheet #13: Ratio & Root Tests

1. (a) State the Root Test.
   (b) State the Ratio Test.

2. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.
   (a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[5]{n}}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n}}{\ln(n+1)}$
   (c) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{3n^3+2n}{4n^3+1}\right)^n$
   (d) $\displaystyle\sum_{n=1}^{\infty} 13\cos(5)^{n-1}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n^2}{n!}$
   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{n!}$
   (g) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n}{(11-\cos^2(n))^n}$

3. Identify the following statements as true or false. If the statement is true, cite evidence from the text to support it. If the statement is false, correct it so that it is a true statement from the text.
   (a) To prove that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ converges you should compute the limit $\displaystyle\lim_{n\to\infty} a_n$. If this limit is 0 then the series converges.
   (b) To apply the Ratio Test to the series $\displaystyle\sum_{n=1}^{\infty} a_n$ you should compute $\displaystyle\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. If this limit is less than 1 then the series converges absolutely.
   (c) To apply the Root Test to the series $\displaystyle\sum_{n=1}^{\infty} a_n$ you should compute $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|}$. If this limit is 1 or larger then the series diverges.
   (d) One way to prove that a series is convergent is to prove that it is absolutely convergent.
   (e) An infinite series converges when the limit of the sequence of partial sums converges.

# MA 114 Worksheet #14: Power Series

1. (a) Give the definition of the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$

   (b) For what values of $x$ does the series $\sum_{n=1}^{\infty} 2\big(\cos(x)\big)^{n-1}$ converge?

   (c) Find a formula for the coefficients $c_k$ of the power series $\frac{1}{0!}+\frac{2}{1!}x+\frac{3}{2!}x^2+\frac{4}{3!}x^3+\cdots$.

   (d) Find a formula for the coefficients $c_n$ of the power series $1+2x+x^2+2x^3+x^4+2x^5+x^6+\cdots$.

   (e) Suppose $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$ where $c \neq 0$. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$.

   (f) Consider the function $f(x) = \frac{5}{1-x}$. Find a power series that is equal to $f(x)$ for every $x$ satisfying $|x| < 1$.

   (g) Define the terms *power series*, *radius of convergence*, and *interval of convergence.*

2. Find the radius and interval of convergence for

   (a) $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(x-3)^n$.

   (b) $4\sum_{n=0}^{\infty} \frac{2^n}{n}(4x-8)^n$.

   (c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$.

   (d) $\sum_{n=0}^{\infty} n!(x-2)^n$.

   (e) $\sum_{n=0}^{\infty} (5x)^n$

   (f) $\sum_{n=0}^{\infty} \sqrt{n}x^n$

   (g) $\sum_{n=0}^{\infty} \frac{x^n}{\sqrt{n}}$

   (h) $\sum_{n=0}^{\infty} \frac{x^n}{3^n \ln n}$

   (i) $\sum_{n=0}^{\infty} \frac{(x=2)^n}{n^n}$

   (j) $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n^4}$

   (k) $\sum_{n=0}^{\infty} \frac{(5x)^n}{n^3}$

3. Use term by term integration and the fact that $\int \frac{1}{1+x^2}\,dx = \arctan(x)$ to derive a power series centered at $x=0$ for the arctangent function. HINT: $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$.

4. Use the same idea as above to give a series expression for $\ln(1+x)$, given that $\frac{dx}{1+x} = \ln(1+x)$. You will again want to manipulate the fraction $\frac{1}{1+x} = \frac{1}{1-(-x)}$ as above.

5. Write $(1+x^2)^{-2}$ as a power series. HINT: use term by term differentiation.

# MA 114 Worksheet #15: Taylor & Maclaurin Series

1. (a) Suppose that $f(x)$ has a power series representation for $|x| < R$. What is the general formula for the Maclaurin series for $f$?
   (b) Suppose that $f(x)$ has a power series representation for $|x - a| < R$. What is the general formula for the Taylor series for $f$ about $a$?
   (c) Let $f(x) = 1 + 2x + 3x^2 + 4x^3 + 5x^4$. Find the Maclaurin series for $f$.
   (d) Let $f(x) = 1 + 2x + 3x^2 + 4x^3$. Find the Taylor series for $f(x)$ centered at $x = 1$.

2. Assume that each of the following functions has a power series expansion. Find the Maclaurin series for each. Be sure to provide the domain on which the expansion is valid.
   (a) $f(x) = \ln(1 + x)$
   (b) $f(x) = xe^{2x}$

3. Use a known Maclaurin series to obtain the Maclaurin series for the given function. Specify the radius of convergence for the series.
   (a) $f(x) = \dfrac{x^2}{1 - 3x}$
   (b) $f(x) = e^x + e^{-x}$
   (c) $f(x) = e^{-x^2}$
   (d) $f(x) = x^5 \sin(3x^2)$
   (e) $f(x) = \sin^2 x$.
   Hint: $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$

4. Find the following Taylor expansions about $x = a$ for each of the following functions and their associated radii of convergence.
   (a) $f(x) = e^{5x}$, $a = 0$.
   (b) $f(x) = \sin(\pi x)$, $a = 1$.

5. Differentiate the series in 1(b) to find a Taylor series for $\cos(x)$.

6. Use Maclaurin series to find the following limit: $\displaystyle\lim_{x \to 0} \frac{x - \tan^{-1}(x)}{x^3}$.

7. Approximate the following integral using a 6th order Taylor polynomial for $\cos(x)$: $\displaystyle\int_0^1 x \cos(x^3)\, dx$

8. Use power series multiplication to find the first three terms of the Maclaurin series for $f(x) = e^x \ln(1 - x)$.

# MA 114 Worksheet #17: Average value of a function

1. Write down the equation for the average value of an integrable function $f(x)$ on $[a, b]$.

2. Find the average value of the following functions over the given interval.

   (a) $f(x) = x^3,\ [0, 4]$

   (b) $f(x) = x^3,\ [-1, 1]$

   (c) $f(x) = \cos(x),\ \left[0, \frac{|pi}{6}\right]$

   (d) $f(x) = \frac{1}{x^2+1},\ [-1, 1]$

   (e) $f(x) = \frac{\sin \pi/x}{x^2},\ [1, 2]$

   (f) $f(x) = e^{-nx},\ [-1, 1]$

   (g) $f(x) = 2x^3 - 6x^2,\ [-1, 3]$

   (h) $f(x) = x^n$ for $n \geq 0$, $[0, 1]$

3. In a certain city the temperature (in $^\circ F$) $t$ hours after 9 am was modeled by the function $T(t) = 50 = 14 \sin \frac{\pi t}{12}$. Find the average temperature during the period from 9 am to 9 pm.

4. The velocity $v$ of blood that flows in a blood vessel with radius $R$ and length $l$ at a distance $r$ from the central axis is

$$v(r) = \frac{P}{4\eta l}(R^2 - r^2)$$

   where $P$ is the pressure difference between the ends of the vessel and $\eta$ is the viscosity of the blood. Find the average velocity (with respect to $r$) over the interval $0 < r < R$. Compare the average velocity with the maximum velocity.

5. Breathing is cyclic and a full respiratory cycle from the beginning of inhalation to the end of exhalation takes about 5 s. The maximum rate of air flow into the lungs is about 0.5 L/s. This explains, in part, why the function $f(t) = \frac{1}{2}\sin(2\pi t/5)$ has often been used to model the rate of air flow into the lungs. Use this model to find the volume of inhaled air in the lungs at time t. Then use this new function to compute the average volume of inhaled air in the lungs in one respiratory cycle.

# MA 114 Worksheet #18: Volumes I

1. If a solid has a cross-sectional area given by the function $A(x)$, what integral should be evaluated to find the volume of the solid?

2. Calculate the volume of the following solid. The base is a square, one of whose sides is the interval $[0, l]$ along the $x$-axis. The cross sections perpendicular to the $x$-axis are rectangles of height $f(x) = x^2$.

3. Calculate the volume of the following solid. The base is the region enclosed by $y = x^2$ and $y = 3$. The cross sections perpendicular to the $y$-axis are squares.

4. The base of a certain solid is the triangle with vertices at $(-10, 5)$, $(5, 5)$, and the origin. Cross-sections perpendicular to the y-axis are squares. Find the volume of the solid.

5. Calculate the volume of the following solid. The base is a circle of radius $r$ centered at the origin. The cross sections perpendicular to the $x$-axis are squares.

6. Calculate the volume of the following solid. The base is the parabolic region $\{(x, y) \mid x^2 \le y \le 4\}$. The cross sections perpendicular to the $y$-axis are right isosceles triangles whose hypotenuse lies in the region.

7. Sketch the solid given by the integral
$$\pi \int_0^1 (y^2 + 1)^2 - 1\, dy.$$

8. For each of the following, use disks or washers to find the an integral expression for the volume of the region. Evaluate the integrals for parts (a) and (d).
   (a) $R$ is region bounded by $y = 1 - x^2$ and $y = 0$; about the $x$-axis.
   (b) $R$ is region bounded by $y = \dfrac{1}{x}$, $x = 1$, $x = 2$, and $y = 0$; about the $x$-axis.
   (c) $R$ is region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$; about the $y$-axis.
   (d) $R$ is region bounded by $y = 1 - x^2$ and $y = 0$; about the line $y = -1$.
   (e) Between the regions in part (a) and part (d), which volume is bigger? Why?
   (f) $R$ is region bounded by $y = e^{-x}$, $y = 1$, and $x = 2$; about the line $y = 2$.
   (g) $R$ is region bounded by $y = x$ and $y = \sqrt{x}$; about the line $x = 2$.

9. Find the volume of the cone obtained by rotating the region under the segment joining $(0, h)$ and $(r, 0)$ about the $y$-axis.

10. The torus is the solid obtained by rotating the circle $(x - a)^2 + y^2 = b^2$ around the $y$-axis (assume that $a > b$). Show that it has volume $2\pi^2 ab^2$.
[Hint: Draw a picture, set up the problem and evaluate the integral by interpreting it as the area of a circle.]

# MA 114 Worksheet #19: Volumes II

1. (a) Write a general integral to compute the volume of a solid obtained by rotating the region under $y = f(x)$ over the interval $[a, b]$ about the $y$-axis using the method of cylindrical shells.

   (b) If you use the disk method to compute the same volume, are you integrating with respect to $x$ or $y$? Why?

2. Sketch the enclosed region and use the Shell Method to calculate the volume of rotation about the $y$-axis.

   (a) $y = 3x - 2$, $y = 6 - x$, $x = 0$

   (b) $y = x^2$, $y = 8 - x^2$, $x = 0$, for $x \geq 0$

   (c) $y = 8 - x^3$, $y = 8 - 4x$, for $x \geq 0$

3. For each of the following, use disks or washers to find the an integral expression for the volume of the region. Evaluate the integrals for parts (a) and (d).

   (a) $R$ is region bounded by $y = 1 - x^2$ and $y = 0$; about the $x$-axis.

   (b) $R$ is region bounded by $y = \frac{1}{x}$, $x = 1$, $x = 2$, and $y = 0$; about the $x$-axis.

   (c) $R$ is region bounded by $x = 2\sqrt{y}$, $x = 0$, and $y = 9$; about the $y$-axis.

   (d) $R$ is region bounded by $y = 1 - x^2$ and $y = 0$; about the line $y = -1$.

   (e) Between the regions in part (a) and part (d), which volume is bigger? Why?

   (f) $R$ is region bounded by $y = e^{-x}$, $y = 1$, and $x = 2$; about the line $y = 2$.

   (g) $R$ is region bounded by $y = x$ and $y = \sqrt{x}$; about the line $x = 2$.

4. A soda glass has the shape of the surface generated by revolving the graph of $y = 6x^2$ for $0 \leq x \leq 1$ about the $y$-axis. Soda is extracted from the glass through a straw at the rate of $1/2$ cubic inch per second. How fast is the soda level in the glass dropping when the level is 2 inches? (Answer should be implicitly in units of inches per second.)

5. The torus is the solid obtained by rotating the circle $(x-a)^2 + y^2 = b^2$ around the $y$-axis (assume that $a > b$). Show that it has volume $2\pi^2 ab^2$.
   [Hint: Draw a picture, set up the problem and evaluate the integral by interpreting it as the area of a circle.]

# MA 114 Worksheet #20: Arc length & Surface area

1. (a) Write down the formula for the arc length of a function $f(x)$ over the interval $[a, b]$ including the required conditions on $f(x)$.
   (b) Write down the formula for the surface area of a solid of revolution generated by rotating a function $f(x)$ over the interval $[a, b]$ around the $x$-axis. Include the required conditions on $f(x)$.
   (c) Write down the formula for the surface area of a solid of revolution generated by rotating a function $f(x)$ over the interval $[a, b]$ around the $y$-axis. Include the required conditions on $f(x)$.

2. Find an integral expression for the arc length of the following curves. Do ***not*** evaluate the integrals.
   (a) $f(x) = \sin(x)$ from $x = 0$ to $x = 2$.
   (b) $f(x) = x^4$ from $x = 2$ to $x = 6$.
   (c) $x^2 + y^2 = 1$

3. Find the arc length of the following curves.
   (a) $f(x) = x^{3/2}$ from $x = 0$ to $x = 2$.
   (b) $f(x) = \ln(\cos(x))$ from $x = 0$ to $x = \pi/3$.
   (c) $f(x) = e^x$ from $x = 0$ to $x = 1$.

4. Set up a function $s(t)$ that gives the arc length of the curve $f(x) = 2x + 1$ from $x = 0$ to $x = t$. Find $s(4)$.

5. Compute the surface areas of revolution about the $x$-axis over the given interval for the following functions.
   (a) $y = x$, $[0, 4]$
   (b) $y = x^3$, $[0, 2]$
   (c) $y = \left(4 - x^{2/3}\right)^{3/2}$, $[0, 8]$
   (d) $y = e^{-x}$, $[0, 1]$
   (e) $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$, $[1, e]$
   (f) $y = \sin x$, $[0, \pi]$
   (g) Find the surface area of the torus obtained by rotating the circle $x^2 + (y - b)^2 = r^2$ about the $x$-axis.
   (h) Show that the surface area of a right circular cone of radius $r$ and height $h$ is $\pi r\sqrt{r^2 + h^2}$.
   Hint: Rotate a line $y = mx$ about the $x$-axis for $0 \leq x \leq h$, where $m$ is determined by the radius $r$.

# MA 114 Worksheet #21: Centers of Mass

1. Find the center of mass for the system of particles of masses 4, 2, 5, and 1 located at the coordinates $(1, 2)$, $(-3, 2)$, $(2, -1)$, and $(4, 0)$.

2. Point masses of equal size are placed at the vertices of the triangle with coordinates $(3, 0)$, $(b, 0)$, and $(0, 6)$, where $b > 3$. Find the center of mass.

3. Find the centroid of the region under the graph of $y = 1 - x^2$ for $0 \le x \le 1$.

4. Find the centroid of the region under the graph of $f(x) = \sqrt{x}$ for $1 \le x \le 4$.

5. Find the centroid of the region between $f(x) = x - 1$ and $g(x) = 2 - x$ for $1 \le x \le 2$.

6. Let $m > n \ge 0$. Find the centroid of the region between $x_m$ and $x_n$ for $0 \le x \le 1$. Find values for $m$ and $n$ that force the centroid to lie outside of the region.

# MA 114 Worksheet #22: Parametric Curves

1. (a) How is a curve different from a parametrization of the curve?

   (b) Suppose a curve is parameterized by $(x(t), y(t))$ and that there is a time $t_0$ with $x'(t_0) = 0$, $x''(t_0) > 0$, and $y'(t_0) > 0$. What can you say about the curve near $(x(t_0), y(t_0))$?

   (c) What parametric equations represent the circle of radius 5 with center $(2, 4)$?

   (d) Represent the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{c^2} = 1$ with parametric equations.

   (e) Do the two sets of parametric equations

   $$y_1(t) = 5\sin(t),\ x_1(t) = 5\cos(t),\ 0 \le t \le 2\pi$$

   and

   $$y_2(t) = 5\sin(t),\ x_2(t) = 5\cos(t),\ 0 \le t \le 20\pi$$

   represent the same parametric curve? Discuss.

2. Consider the curve parametrized by $c(t) = (\sin(t) + \frac{t}{\pi}, \left(\frac{t}{\pi}\right)^2)$, for $0 \le t \le 2\pi$.

   (a) Plot the points given by $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, 2\pi$.

   (b) Consider the derivatives of $x(t)$ and $y(t)$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. What does this tell you about the curve near these points?

   (c) Use the above information to plot the curve.

3. Find a Cartesian equation for the following parametric curves. Sketch the curves to see if you solved them correctly.

   (a) $x = \sqrt{t}$, $y = 1 - t$.

   (b) $x = 3t - 5$, $y = 2t + 1$.

   (c) $x = \cos(t)$, $y = \sin(t)$.

4. Represent each of the following curves as parametric equations traced just once on the indicated interval.

   (a) $y = x^3$ from $x = 0$ to $x = 2$.

   (b) $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$.

5. A particle travels from the point $(2, 3)$ to $(-1, -1)$ along a straight line over the course of 5 seconds. Write down a set of parametric equations which describe the position of the particle for any time between 0 and 5 seconds.

# MA 114 Worksheet #23: Calculus with Parametric Curves

1. For the following parametric curves, find an equation for the tangent to the curve at the specified value of the parameter.
   (a) $x = e^{\sqrt{t}}$, $y = t - \ln(t^2)$ at $t = 1$.
   (b) $x = \cos(\theta) + \sin(2\theta)$, $y = \cos(\theta)$, at $\theta = \pi/2$.
2. For the following parametric curve, find $dy/dx$.
   (a) $x = e^{\sqrt{t}}$, $y = t + e^{-t}$.
   (b) $x = t^3 - 12t$, $y = t^2 - 1$.
   (c) $x = 4\cos(t)$, $y = \sin(2t)$.
3. Find $d^2y/dx^2$ for the curve $x = 7 + t^2 + e^t$, $y = \cos(t) + \dfrac{1}{t}$, $0 < t \leq \pi$.
4. Find the arc length of the following curves.
   (a) $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leq t \leq 1$.
   (b) $x = 4\cos(t)$, $y = 4\sin(t)$, $0 \leq t \leq 2\pi$.
   (c) $x = 3t^2$, $y = 4t^3$, $1 \leq t \leq 3$.
5. What is the speed of the curve $c(t) = (x(t), y(t))$? Use this to find the minimum speed of a particle with trajectory $c(t) = (t^2, 2\ln(t))$, for $t > 0$.
6. Suppose you wrap a string around a circle. If you unwind the string from the circle while holding it taut, the end of the string traces out a curve called the *involute* of the circle. Suppose you have a circle of radius $r$ centered at the origin, with the end of the string all the way wrapped up resting at the point $(r, 0)$. As you unwrap the string, define $\theta$ to be the angle formed by the $x$-axis and the line segment from the center of the circle to the point up to which you have unwrapped the string.
   (a) Draw a picture and label $\theta$.
   (b) Show that the parametric equations of the involute are given by $x = r(\cos\theta + \theta\sin\theta)$, $y = r(\sin\theta - \theta\cos\theta)$.
   (c) Find the length of the involute for $0 \leq \theta \leq 2\pi$.

# MA 114 Worksheet #25: Polar coordinates

1. Convert from rectangular to polar coordinates:
   (a) $(1, \sqrt{3})$
   (b) $(-1, 0)$
   (c) $(2, -2)$
2. Convert from polar to rectangular coordinates:
   (a) $\left(2, \frac{\pi}{6}\right)$
   (b) $\left(-1, \frac{\pi}{2}\right)$
   (c) $\left(1, -\frac{\pi}{4}\right)$
3. Sketch the graph of the polar curves:
   (a) $\theta = \frac{3\pi}{4}$
   (b) $r = \pi$
   (c) $r = \cos\theta$
   (d) $r = \cos(2\theta)$
   (e) $r = 1 + \cos\theta$
   (f) $r = 2 - 5\sin\theta$
4. Find the equation in polar coordinates of the line through the origin with slope $\frac{1}{3}$.
5. Find the polar equation for:
   (a) $x^2 + y^2 = 9$
   (b) $x = 4$
   (c) $y = 4$
   (d) $xy = 4$
6. Convert the equation of the circle $r = 2\sin\theta$ to rectangular coordinates and find the center and radius of the circle.
7. Find the distance between the polar points $(3, \pi/3)$ and $(6, 7\pi/6)$.

# MA 114 Worksheet #26: Calculus with polar coordinates

1. Find $dy/dx$ for the following polar curves.

   (a) $r = 2\cos\theta + 1$ (b) $r = 1/\theta$ (c) $r = 2e^{-\theta}$

2. In each of the following, compute the slope of the tangent line at the given point. Then sketch the curve and the tangent line.

   (a) $r = \sin\theta$ at $\theta = \pi/3$. (b) $r = 1/\theta$ at $\theta = \pi/2$.

3. (a) Give the formula for the area of region bounded by the polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$. Give a geometric explanation of this formula.
   (b) Give the formula for the length of the polar curve $r = f(\theta)$ from $\theta = a$ to $\theta = b$.
   (c) Use these formulas to establish the formulas for the area and circumference of a circle.

4. Find the slope of the tangent line to the polar curve $r = \theta^2$ at $\theta = \pi$.

5. Find the point(s) where the tangent line to the polar curve $r = 2 + \sin\theta$ is horizontal.

6. Find the area enclosed by one leaf of the curve $r = \sin 2\theta$.

7. Find the arc length of one leaf of the curve $r = \sin 2\theta$.

8. Find the area of the region bounded by $r = \cos\theta$ for $\theta = 0$ to $\theta = \pi/4$.

9. Find the area of the region that lies inside both the curves $r = \sqrt{3}\sin\theta$ and $r = \cos\theta$.

10. Find the area in the first quadrant that lies inside the curve $r = 2\cos\theta$ and outside the curve $r = 1$.

11. Find the length of the curve $r = \theta^2$ for $0 \le \theta \le 2\pi$.

12. Write down an integral expression for the length of the curve $r = \sin\theta + \theta$ for $0 \le \theta \le \pi$ but do not compute the integral.

13. Consider the sequence of circles, $C_n$, defined by the equations $x^2 + \left(y + \dfrac{1}{\sqrt{n}}\right)^2 = \dfrac{1}{n}$. Define $a_n$ as the area of circle $C_n$ and $b_n$ as the area between circles $C_n$ and $C_{n+1}$.
    (a) Sketch the picture of this infinite sequence of circles.
    (b) Does $\sum_{n=1}^{\infty} a_n$ converge?
    (c) Does $\sum_{n=1}^{\infty} b_n$ converge?
    (d) Define the circles $D_n$ by the equations $x^2 + \left(y + \dfrac{1}{n}\right)^2 = \dfrac{1}{n^2}$ with $d_n$ as the area of $D_n$. Does $\sum_{n=1}^{\infty} d_n$ converge?

# MA 114 Worksheet #27: Differential equations & Direction fields

1. (a) Is $y = \sin(3x) + 2e^{4x}$ a solution to the differential equation $y'' + 9y = 50e^{4x}$? Explain why or why not.

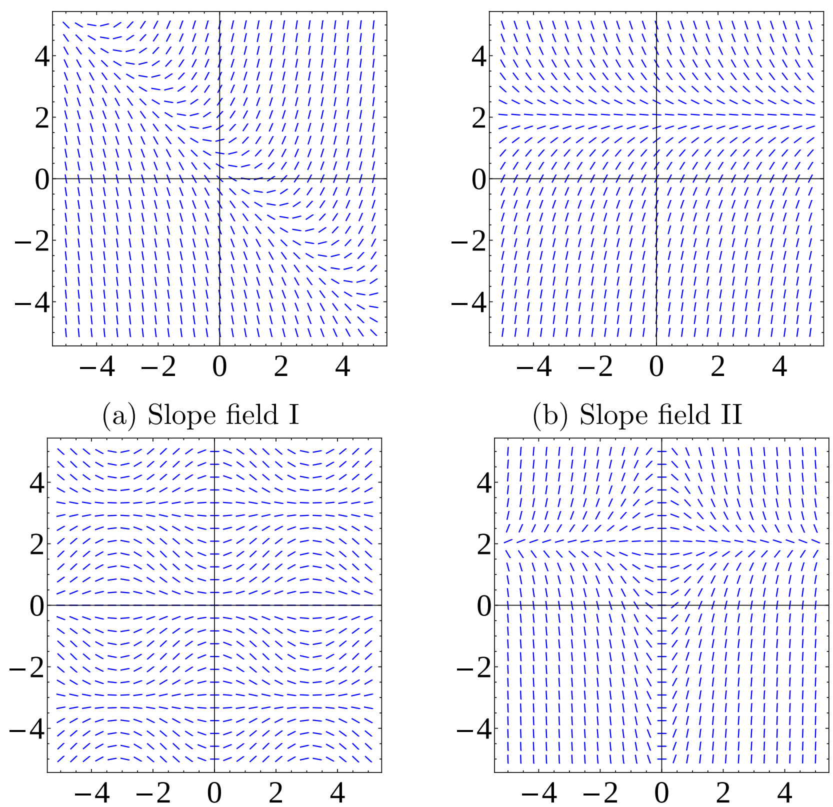
   (b) Explain why every solution of $dy/dx = y^2 + 6$ must be an increasing function.

   (c) What does is mean to say that a differential equation is linear or nonlinear.

2. Find all values of $\alpha$ so that $y(x) = e^{\alpha x}$ is a solution of the differential equation $y'' + y' - 12y = 0$.

3. Match the differential equation with its slope field. Give reasons for your answer.

$$y' = 2 - y \qquad y' = x(2 - y) \qquad y' = x + y - 1 \qquad y' = \sin(x)\sin(y)$$



(a) Slope field I

(b) Slope field II

(c) Slope Field III

(d) Slope field IV

Figure 1: Slope fields for Problem 3

4. Use slope field labeled IV to sketch the graphs of the solutions that satisfy the given initial conditions

$$y(0) = -1, \quad y(0) = 0, \quad y(0) = 1.$$

5. Sketch the slope field of the differential equation. Then use it to sketch a solution curve that passes through the given point
   (a) $y' = y - 2x$, $(1, 0)$
   (b) $y' = xy - x^2$, $(0, 1)$
6. Use Eulers method with step size 0.5 to compute the approximate $y$-values, $y_1$, $y_2$, $y_3$, and $y_4$ of the solution of the initial-value problem $y' = y - 2x$, $y(1) = 0$.

# MA 114 Worksheet #28: Separable equations

1. Use Separation of Variables to find the general solutions to the following differential equations.
   (a) $y' + 4xy^2 = 0$
   (b) $\sqrt{1 - x^2}y' = xy$
   (c) $(1 + x^2)y' = x^3y$
   (d) $\sqrt{1 + y^2}\, y' + \sec x = 0$
2. (Extra) A tank has the shape of the parabola $y = x^2$ revolved about the $x$-axis. water leaks from a hole of area $B = 0.0005\ m^2$ at the bottom of the tank. Let $y(t)$ be the water level at time $t$. How long does it take for the tank to empty if the initial water level is $y(0) = 1\ m$?

# MA 114 Worksheet #29: Conic sections

1. The point in a lunar orbit nearest the surface of the moon is called perilune and the point farthest from the surface is called apolune. The Apollo 11 spacecraft was placed in an elliptical lunar orbit with perilune altitude 110 km and apolune altitude 314 km (above the moon). Find an equation of this ellipse if the radius of the moon is 1728 km and the center of the moon is at one focus.

2. Find an equation for the ellipse with foci $(1, 1)$ and $(-1, -1)$ and major axis of length 4.

3. Use parametric equations and Simpsons Rule with $n = 12$ to estimate the circumference of the ellipse $9x^2 + 4y^2 = 36$.

4. Find the area of the region enclosed by the hyperbola $4x^2 - 25y^2 = 100$ and the vertical line through a focus.

5. If an ellipse is rotated about its major axis, find the volume of the resulting solid.

6. Find the centroid of the region enclosed by the $x$-axis and the top half of the ellipse $9x^2 + 4y^2 = 36$.

7. Calculate the surface area of the ellipsoid that is generated by rotating an ellipse about its major axis.