MA 131 Lecture Notes
Exponential Functions, Inverse Functions, and Logarithmic Functions

Exponential Functions

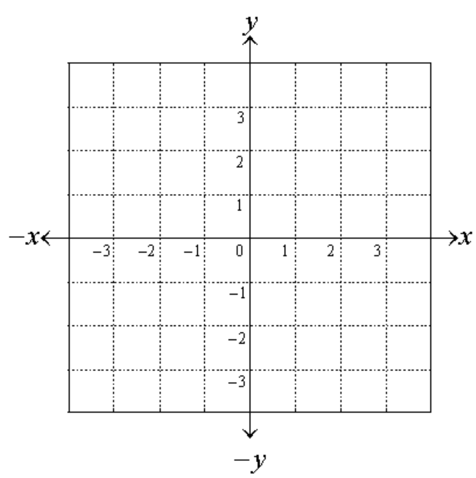
We say that a function is an **algebraic function** if it is created by a combination of algebraic processes such as addition, subtraction, multiplication, division, roots, etc. Functions that are not algebraic are called **transcendental functions**. Examples of algebraic functions include polynomials and rational functions and examples of transcendental functions include exponential and logarithmic functions.

Definition

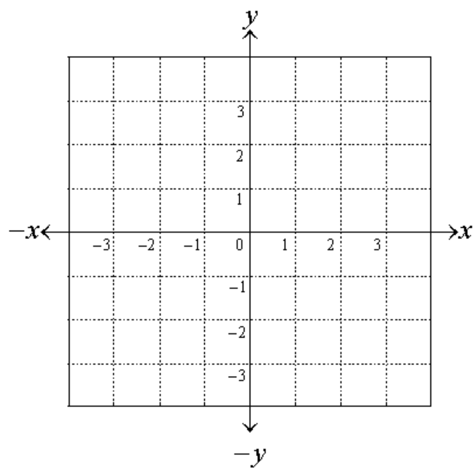
The **exponential function *f* with base *a*** is denoted by $f(x) = a^x$ where $a > 0$, $a \neq 1$, and x is any real number.

Note that when *a=1* the expression is a constant function. Also, *a* is non-negative since the function would not be defined for any even root.

The graph of all exponential functions follows the same pattern and shape. Graph the following by finding coordinates on the graph.



Graph $f(x) = a^x$ where *a* is greater than one.



Graph $f(x) = a^{-x}$ where *a* is greater than one.

Characteristics of Exponential Functions	
Graph of $y = a^{x}$ ***, a>1***	**Graph of** $y = a^{-x}$ ***, a>1***
Domain: $(-\infty, \infty)$	Domain: $(-\infty, \infty)$
Range: $(0, \infty)$	Range: $(0, \infty)$
Intercept: $(0,1)$	Intercept: $(0,1)$
Increasing	Decreasing
x-axis is a horizontal asymptote	*x*-axis is a horizontal asymptote
Continuous	Continuous
	Reflection of graph of $y = a^{x}$ about the *y*-axis

There exists a very special irrational number that is often used as a base for exponential functions. This base is $e = 2.71828...$ and we call it the **natural base**. The function given by $f(x) = e^{x}$ is called the **natural exponential function**. Note that in exponential functions the variable is the exponent and the base stays the same.

Exercise

Sketch the graph of $f(x) = e^{x}$.

We use the natural exponential function to determine the investment earnings of **continuously compounded interest.** Previously we discussed the formula to find compound interest. We evaluated it based on different compounding times such as yearly, monthly, and daily. But if we are interested in doing the compounding continuously we can modify the formula that we used. (This is due to one definition of the value of *e* as $e = \lim_{n \to \infty}\left(1+\frac{1}{n}\right)^{n}$.)

Formulas for Compound Interest

After *t* years, the balance *A* in an account with principal *P* and annual interest rate *r* (in decimal form) is given by the following formulas.

I.) For *n* compounding per year: $A = P\left(1+\frac{r}{n}\right)^{nt}$

II.) For continuous compounding: $A = Pe^{rt}$

Example:

Suppose you invested $5000 into an account with an interest rate of 8% and left it there for five years. If the amount was compounded continuously, how much money would you have in the account at the end of the fifth year?

(Answer: $7459.12)

Example:

Suppose you wanted a balance of $100,000 after ten years for an interest bearing account paying 9% compounded continuously. How much would your initial investment need to be?

(Answer: $40,656.97)

The formula for continuously compounded interest can be used in other examples by modifying what each variable stands for. Consider the following:

The number of a certain type of bacteria increases according to the model $P(t) = 100e^{0.01896t}$. Where *t* is the time (in hours.)

Find *P(0)* and interpret.

What is the rate? How do we interpret it?

What is *P(10)*? And what does it represent?

Is the number of bacteria growing? Can we determine how the function would be different if we wanted the number to be decreasing?

(Answer: about 121)

Inverse Functions

Since a function can be represented by the collection of ordered pairs satisfying the equation, we can reverse the ordered pairs to create another relation called the inverse relation. Only when that inverse relation is a function do we say the function has an **inverse function**. We may arrive at this by restricting the domain of the inverse relation to create a function. We use the following notation, for a function *f*, we say the inverse function is f^{-1} .

It is true then that if we apply a function followed by its inverse function, we arrive back to the value we started with. We can think of an inverse function as un-doing what the function did.

Thus $\left(f \circ f^{-1}\right)(x) = \left(f^{-1} \circ f\right)(x) = x$.

Some common examples of inverse functions would be the following: if $f(x) = x + 5$ then $f^{-1}(x) = x - 5$ since the way to undo *adding* five is by *subtracting* five. Another example would be $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$ but only if we restrict the domain on *f* to be the set of non-negative real numbers.

Definition of Inverse Functions
Let *f* and *g* be two functions such that $f(g(x)) = x$ for every *x* in the domain of *g* and $g(f(x)) = x$ for every *x* in the domain of *f.* Under these conditions, the function *g* is the **inverse function** of the function *f*. The function *g* is denoted by f^{-1} (read "*f* inverse"). So $\left(f \circ f^{-1}\right)(x) = x$ and $\left(f^{-1} \circ f\right)(x) = x$. The domain of *f* must be equal to the range of f^{-1}, and the range of *f* must be equal to the domain of f^{-1}.
Note: f^{-1} means the inverse function, it does not mean the reciprocal (like negative exponents mean in other parts of algebra.)

Finding Inverse Functions
1. In the equation $f(x)$, replace $f(x)$ by *y*.
2. Interchange the roles of *x* and *y*.
3. Solve the new equation for *y*. If the new equation does not represent *y* as a function of *x*, the function *f* does not have an inverse function. If the new equation does represent *y* as a function of *x*, continue to step 4.
4. Replace *y* by $f^{-1}(x)$ in the new equation.
5. Verify that *f* and f^{-1} are inverse functions of each other by sowing that the domain of *f* is equal to the range of f^{-1}, the range of *f* is equal to the domain of f^{-1}, and $f^{-1}(f(x)) = x = f(f^{-1}(x))$.

Examples: Find f^{-1}.

$f(x) = \frac{3x-5}{7}$

$f(x) = \sqrt[3]{x+2}$

$f(x) = \frac{5x-1}{2x+3}$

The graph of an inverse function is going to be symmetric with respect to the line *y=x*.

Horizontal Line Test for Inverse Functions

A function *f* has an inverse function if and only if no *horizontal* line intersects the graph of *f* at more than one point.

Consider the function $f(x) = x^2$. What does the graph say about this function having an inverse function? Is there a way to restrict the domain to remedy this?

Exercises:

Verify that the following are inverse functions of one another. $f(x) = \frac{1}{x+1}$ and $f^{-1}(x) = \frac{1-x}{x}$
Find the inverse function for $f(x) = (x-5)^2$

Logarithmic Functions

The function $f(x) = a^x$ passes the horizontal line test and therefore has an inverse function. Since it is a transcendental function, we cannot define the inverse algebraically. Instead we use a new type of function called the logarithmic functions (or "log" functions.) The log function is actually defined to be the inverse of the exponential function. It undoes raising numbers to a power.

Definition

For $x > 0$, $a > 0$, and $a \neq 1$,then $y = \log_a x$ if and only if $x = a^y$.

The function given by $f(x) = \log_a x$ is called the **logarithmic function with base *a***.

We say $y = \log_a x$ is the logarithmic form of the same equation $x = a^y$ which is called the exponential form. We think of a logarithm as being an exponent. It answers the question, "a raised to what power is x?" Then *y* is the power.

Examples:

$\log_2 8 = 3$ since $8 = 2^3$

Find the following:

$\log_2 32 =$	$\log_2\left(\frac{1}{4}\right) =$	$\log_2 1 =$
$\log_3 9 =$	$\log_9 3 =$	$\log_4 64 =$
$\log_{374859}(374859) =$	$\log_{398} 1 =$	$\log_{10}(0.10) =$

Since the log function is the inverse function of the exponential function, it follows that the domain of the log function is the range of an exponential function. We learned in the last section that the range of an exponential function is $(0, \infty)$. Therefore the domain of any log function is $(0, \infty)$. We call *x* the **argument** of the log function for $\log_a(x)$.

Evaluate $\log_{10}(500)$.

Since 500 is not an exact power of ten we cannot find an exact value for this log. It is considered simplified as written (and therefore an exact value.) However, if we needed a decimal approximation for this we would plug it into our calculator. There should be a "log" button on the calculator. After plugging it in, you should get about 2.7. (This answer makes sense since we know 10 to the power of 2 is 100 and to the power of 3 is 1000, so the answer is somewhere in between. Note it is not exactly half way.)

Properties of Logarithms
1. $\log_a 1 = 0$ because $a^0 = 1$
2. $\log_a a = 1$ because $a^1 = a$
3. $\log_a a^x = x$ and $a^{\log_a x} = x$ (Inverse Property)
4. If $\log_a x = \log_a y$, then $x = y$ (One-to-One Property)

Examples: Solve

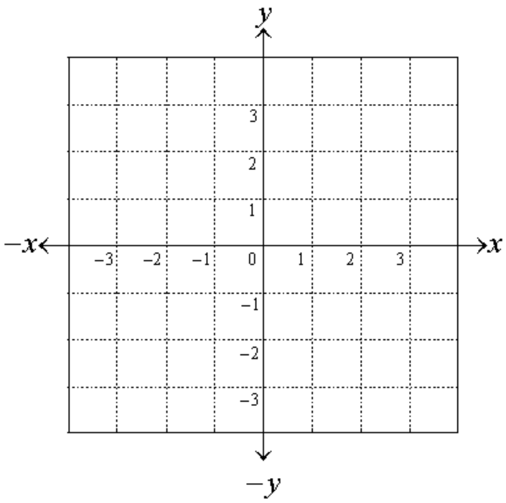
$\log_2 x = \log_2 7$

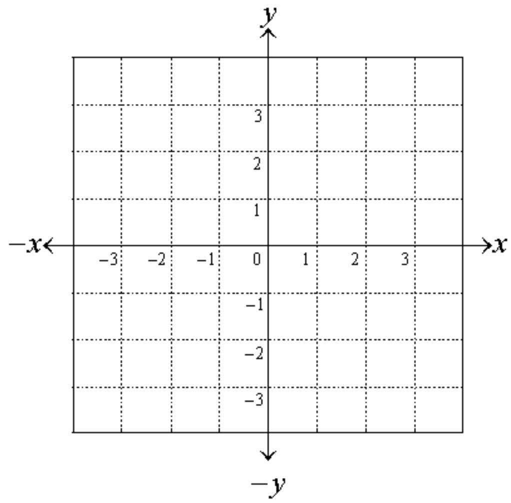
$\log_4 x = 2$

$\log_6 6^{52} = x$

We find the graph of a logarithmic function with base *a* by sketching the appropriate exponential function with base *a* and reflecting it across the line $y = x$.

Sketch both $f(x) = a^x$ and $g(x) = \log_a x$, *a>1*





Characteristics of Logarithmic Functions
Domain $(0, \infty)$
Range $(-\infty, \infty)$
Intercept $(1,0)$
Increasing
One-to-one, therefore has an inverse
y-axis is a vertical asymptote
Continuous
Reflection of graph of $f(x) = a^x$ about the line $y = x$

Definition

The function defined by $f(x) = \log_e(x) = \ln(x)$, $x > o$ is called the **natural logarithmic function.**

Properties of the Natural Logarithm
1. $\ln 1 = 0$ because $e^0 = 1$
2. $\ln e = 1$ because $e^1 = e$
3. $\ln e^x = x$ and $e^{\ln x} = x$ (Inverse Property)
4. If $\ln x = \ln y$, then $x = y$ (One-to-One Property)

Evaluate the following:

$\ln \frac{1}{e^2} =$	$3\ln e =$	$e^{\ln 6} =$

Earlier we said that the domain of a log function is restricted to positive real numbers. This makes for a third consideration when finding the domain of a function (the first two being no division by zero and no negatives under radicals.)

Find the domain for the following.

$f(x) = \ln(x+5)$
$f(x) = \ln(x^2 - 4)$
$f(x) = \ln(x^4)$

) Properties of Logarithms

Usually you will only find two options for logarithms on a calculator, either "log" or "ln." "Log" without any subscript implies the base is ten. We've already noted that "ln" is the natural base, or base *e*. Sometimes we wish to evaluate a log that does not have either of those bases, in that case we need the change of base formula to convert the log as written into something we can plug into the calculator.

Change-of-Base Formula

Let *a, b,* and *x* be positive numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base as follows:

Base *b*	Base *10*	Base *e*
$\log_a x = \frac{\log_b x}{\log_b a}$	$\log_a x = \frac{\log x}{\log a}$	$\log_a x = \frac{\ln x}{\ln a}$

Example:

Rewrite the following in base ten and then in the natural base.

$\log_5 8$

$\log_{17} 100$

Properties of Logs

Since the log function is the inverse function of the exponential function there are many properties of logs that are corresponding properties to exponentials. For example, we learned early on that an exponent to an exponent means we multiply them. Or division meant subtraction of exponents. The following are rules that we need to use to work with logs.

Properties of Logarithms		
Let *a* be a positive number not equal to one, and let *n* be a real number. If *u* and *v* are positive real numbers, then the following properties are true.		
Logarithm with base *a*	Natural logarithm	
$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$	Product Rule
$\log_a\left(\frac{u}{v}\right) = \log_a u - \log_a v$	$\ln\left(\frac{u}{v}\right) = \ln u - \ln v$	Quotient Rule
$\log_a(u)^{n`} = n\log_a u$	$\ln(u)^{n`} = n\ln u$	Power Rule

Examples

Express the following logs in terms of *ln3* and *ln4*.

$\ln 12$

$\ln \frac{9}{4}$

We are much more interested in using these properties to solve equations and therefore it is more important to be able to apply these to logs with algebraic expressions as arguments.

Example:

Expand the following

$\ln\left(\frac{3x^5}{y+1}\right)$

$\log_3\left(\sqrt{3x(y-2)}\right)$

Condense the following:

$4\log x - \log y$

$\log_7(x^2+2) - 5\log_7 x$

$\ln 5 - \ln x + \ln 8$

Optional

Solving Exponential and Logarithmic Equations

We will use the inverse properties rule and the one-to-one property to solve exponential and logarithmic equations.

Recall the following:

5.	$\log_a a^x = x$ and $a^{\log_a x} = x$ (Inverse Property)
6.	If $\log_a x = \log_a y$, then $x = y$ (One-to-One Property)

And the following is true for the natural log.

5.	$\ln e^x = x$ and $e^{\ln x} = x$ (Inverse Property)
6.	If $\ln x = \ln y$, then $x = y$ (One-to-One Property)

We can also conclude the following:

If $a^x = y$, if we "apply the log" to both sides, we get $\log_a a^x = \log_a y$ and thus $x = \log_a y$, which is actually our original definition of logarithms.

Strategies for Solving Exponential and Logarithmic Equations
1. Rewrite the original equation in a form that allows the use of the One-to-One Property of exponential or logarithmic functions.
2. Rewrite an *exponential* equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a *logarithmic* equation in exponential form and apply the Inverse Property of exponential functions.

Examples: Solve for *x*

Using the One-to-One Property-look for a common base and rewrite.

$2^x = 16$

$3^x = \frac{1}{27}$

Using the One-to-One Property-get a log on both sides.

$\ln 5 - \ln x = 0$

Using the inverse property-take the log of both sides

$e^x = 15$

$5^x = 3$

Using the inverse property-raise both sides as a power of a base (Sometimes you can think of this one as applying the definition of a log)

$\ln x = 5$

$\log_2 x = 3$

Solving Exponential Equations (with more complicated expressions)

If the two sides cannot be expressed as the same base, then we should isolate the exponent (with the variable) and apply the log to both sides.

Example:

First solve by taking the $\log_2$ of both sides. Then solve by taking $\ln$ of both sides. Determine why these are equivalent answers.

$2^x = 11$

Solve the following by taking the log of both sides using any base you choose. Determine why every base will give an equivalent answer. (Remember to isolate the exponent first.)

$3 \cdot 4^{x+5} = 6$

Solve

$3^{x^2} = 5$

Recall quadratic type to solve the next equation.

$e^{2x} - 5e^x + 4 = 0$

Solving Logarithmic Equations

The process we use is to raise both sides of the equation as a power to a base. We call this process **exponentiating** each side of the equation.

Examples: Solve

$\log x = 3$

$3\log_4 2x = 7$

Recall the One-to-One Property to solve the following

$\log_5(3x-2) = \log_5(5x+2)$

Solve the following using techniques we have already learned to solve equations.

$3 - 2\ln(x+1) = 13$

Sometimes we need to use the properties of log to combine logs into a single log before we can exponentiate.

Since the argument of a log function must be positive, it is possible to have extraneous solutions. You should always check your answer.

Solve (and check)

$$\log(5x) + \log(x-1) = 2$$

Exercises: Solve for *x*.

$5^{-\frac{x}{2}} = \frac{1}{5}$
$e^{2x} = 50$
$6(2^{3x-1}) - 7 = 9$

$e^{2x} - 3e^{x} = 4$
$\dfrac{119}{e^{6x} - 14} = 7$
$\ln 2x = 7$
$\ln \sqrt{x+2} = 1$
$7 + 3\ln x = 5$
$\ln x + \ln(x+3) = 1$

$\log(x+4)-\log x=\log(x+2)$
$\log 4x-\log(12+\sqrt{x})=2$