MAE 322
Machine Design
Shafts -1

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Why shafts?

- Motors rotate
- Rotational linkages from motor are the simplist
- Transmit power/kinetics
- Movement/kinematics

- Linear shafts, used for alignment and guides, are not what is being discussing.

Shaft Materials

- Deflection primarily controlled by geometry, not material
- Stress controlled by geometry, not material
- Strength controlled by material property

Shaft Design

- Material Selection (usually steel, unless you have good reasons)
- Geometric Layout (fit power transmission equipment, gears, pulleys)
- Stress and strength
  - Static strength
  - Fatigue strength
- Deflection and rigidity
  - Bending deflection
  - Torsional deflection
  - Slope at bearings and shaft-supported elements
  - Shear deflection due to transverse loading of short shafts
- Vibration due to natural frequency (whirl)

Shaft Materials

- Shafts are commonly made from low carbon, CD or HR steel, such as AISI 1020–1050 steels.
- Fatigue properties don't usually benefit much from high alloy content and heat treatment.
- Surface hardening usually only used when the shaft is being used as a bearing surface.

Shaft Materials

- Cold drawn steel typical for $d < 3$ in.
- HR steel common for larger sizes. Should be machined all over.
- Low production quantities
  - Lathe machining is typical
  - Minimum material removal may be design goal
- High production quantities
  - Forming or casting is common
  - Minimum material may be design goal

Shaft Layout

- Issues to consider for shaft layout
  - Axial layout of components
  - Supporting axial loads (bearings)
  - Providing for torque transmission (gearing/sprockets)
  - Assembly and Disassembly (repair & adjustment)

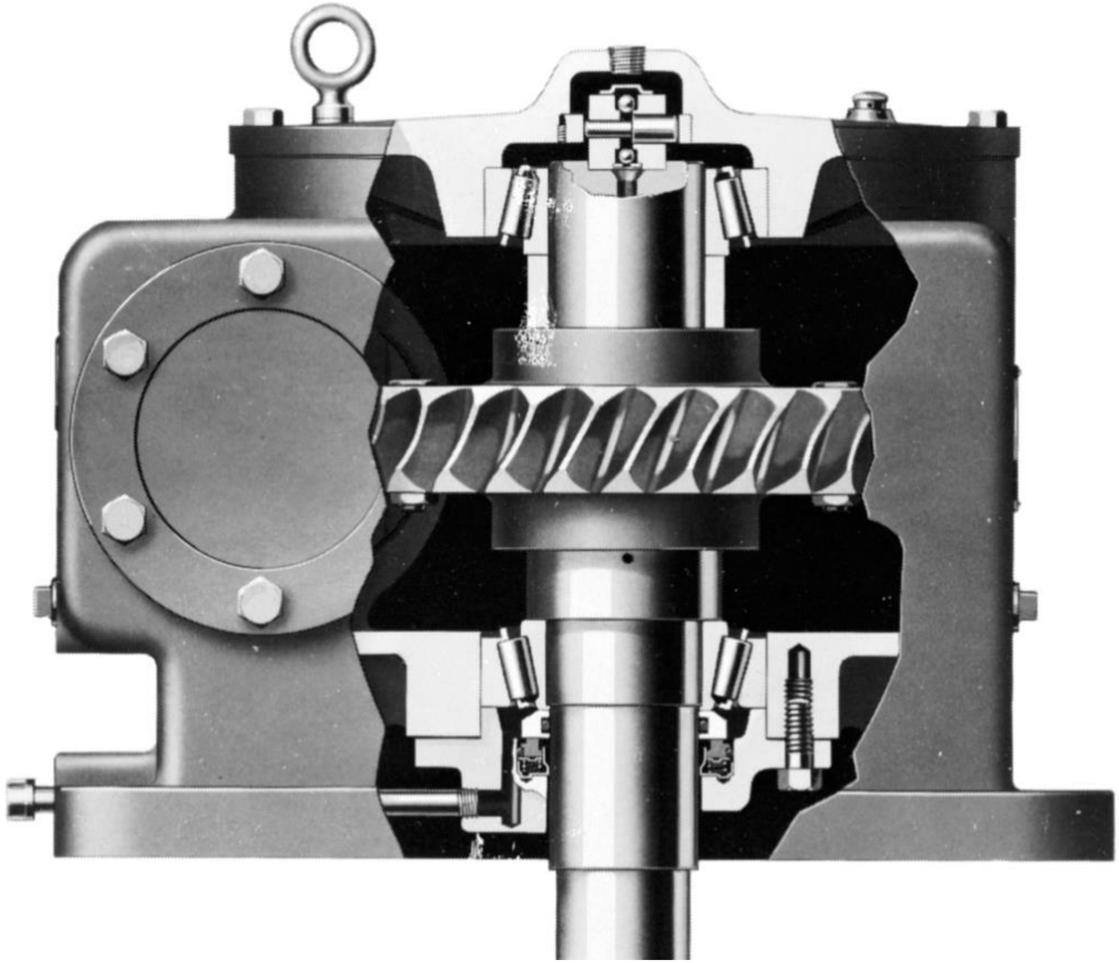


Fig. 7–1

Axial Layout of Components

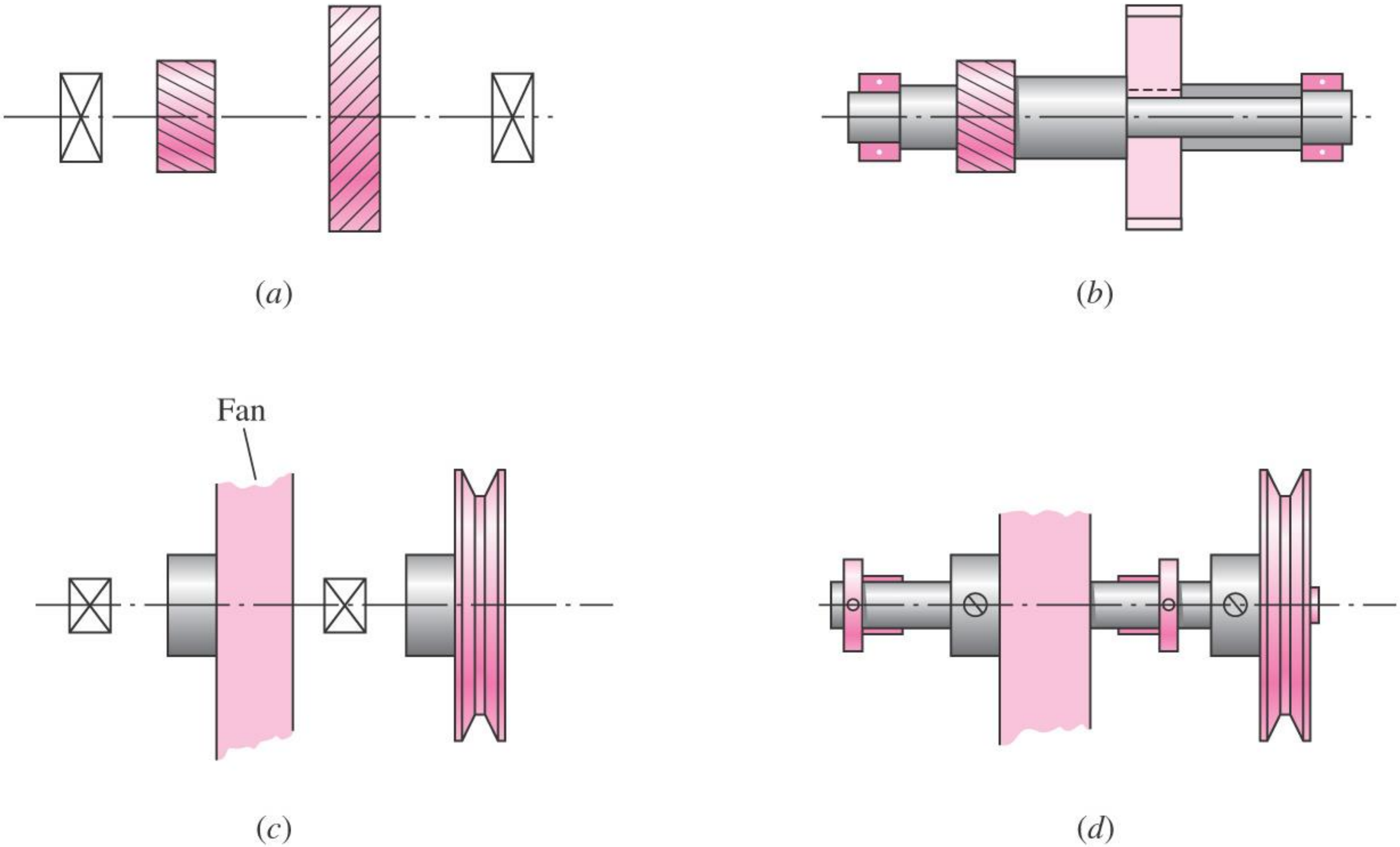


Fig. 7–2

Supporting Axial Loads

- Axial loads must be supported through a bearing to the frame.
- Generally best for only one bearing to carry axial load to shoulder
- Allows greater tolerances and prevents binding

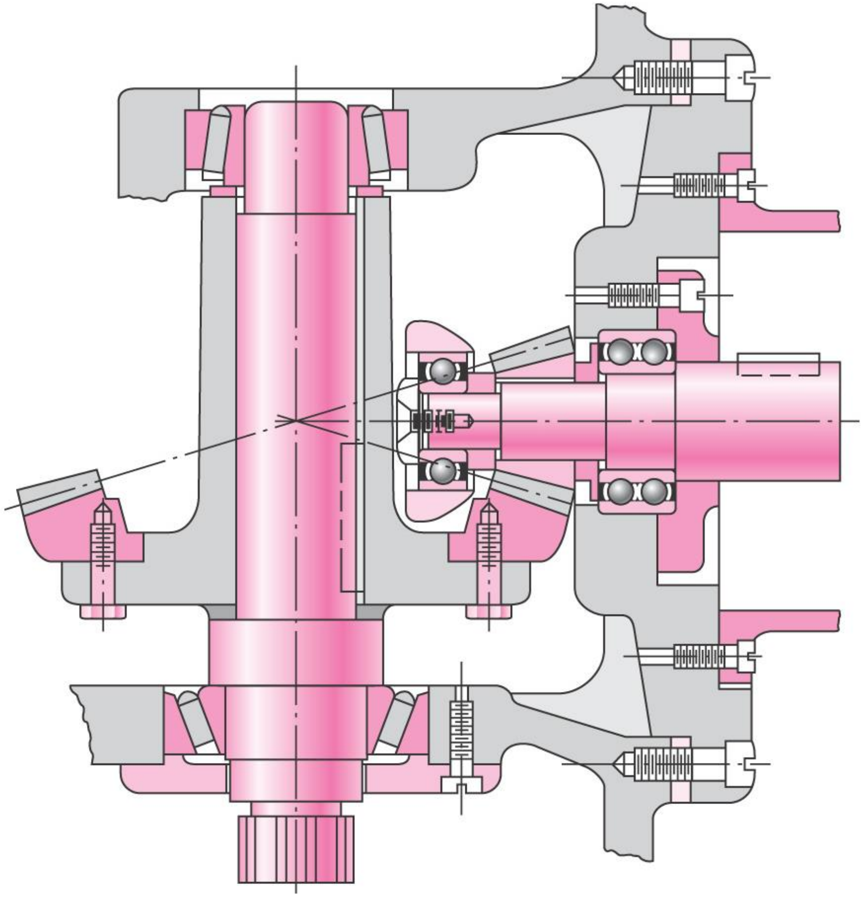


Fig. 7–4

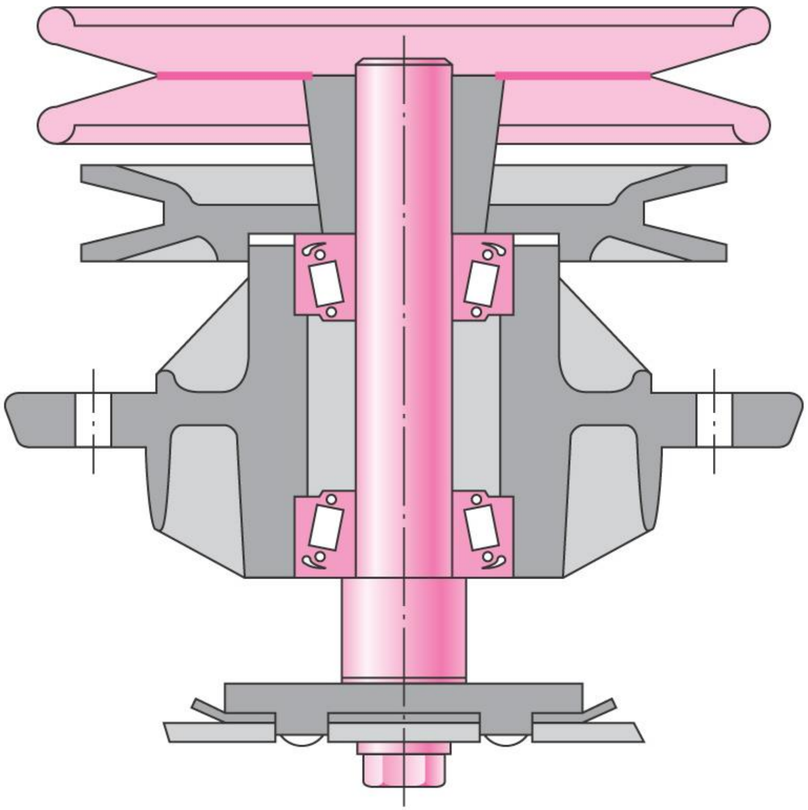


Fig. 7–3

Providing for Torque Transmission

- Common means of transferring torque to shaft
  - Keys
  - Splines
  - Setscrews
  - Pins
  - Press or shrink fits
  - Tapered fits
- Keys are one of the most effective
  - Slip fit of component onto shaft for easy assembly
  - Positive angular orientation of component
  - Can design key to be weakest link to fail in case of overload

Assembly and Disassembly

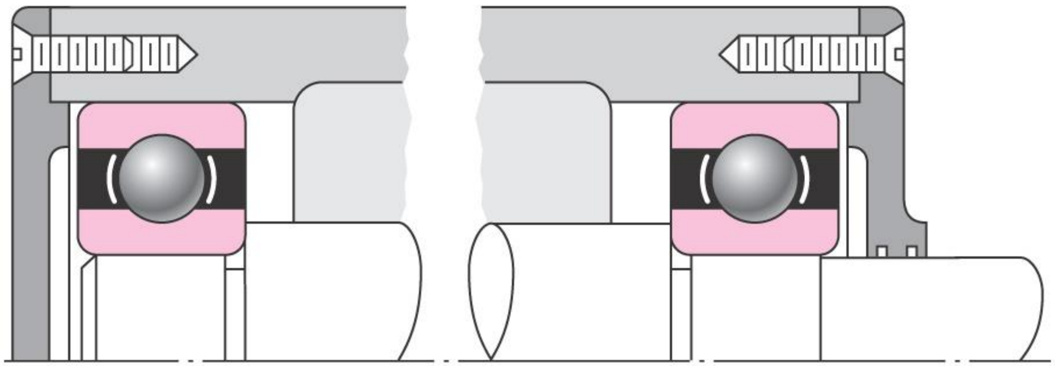


Fig. 7–5

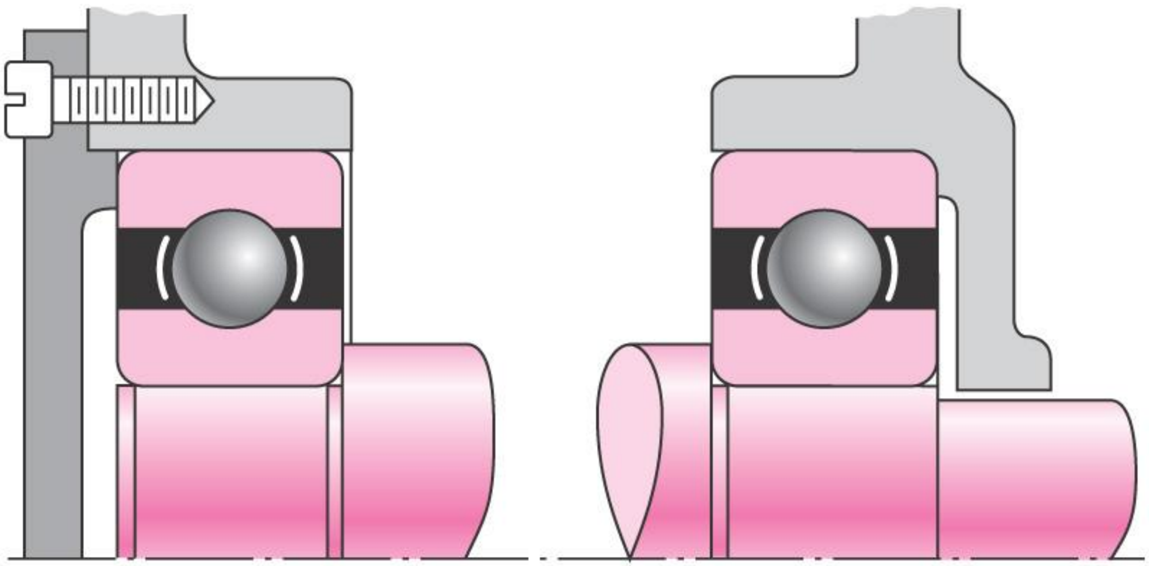


Fig. 7–6

Assembly and Disassembly

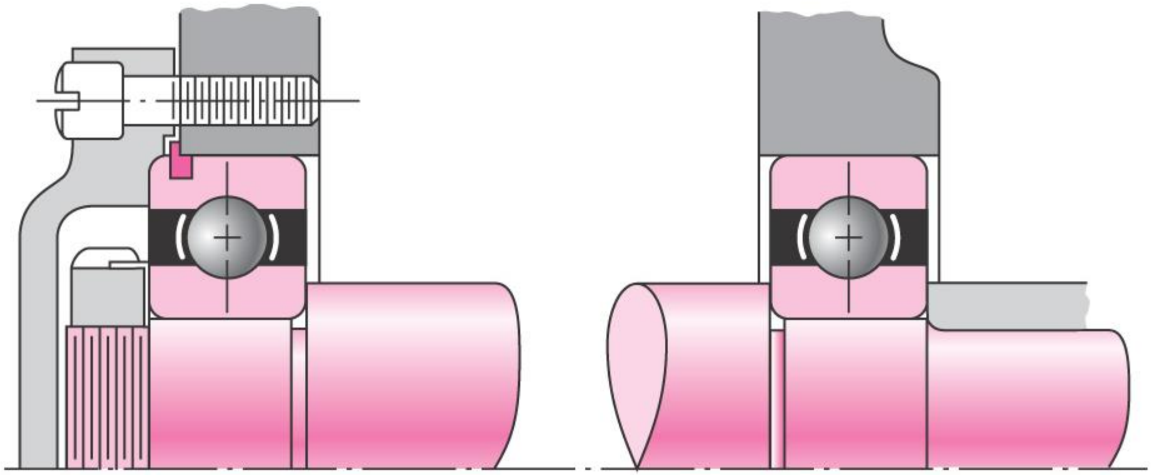


Fig. 7–7

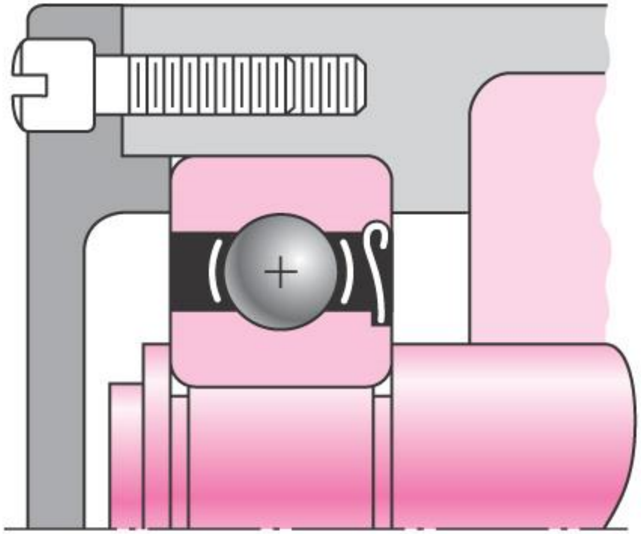


Fig. 7–8

Shaft Design for Stress

- Stresses are only evaluated at critical locations
- Critical locations are usually
  - On the outer surface
  - Where the bending moment is large
  - Where the torque is present
  - Where stress concentrations exist

Shaft Stresses

- Standard stress equations can be customized for shafts for convenience
- Axial loads are generally small and constant, so will be ignored in this section
- Standard alternating and midrange stresses

$$\sigma_a = K_f \frac{M_a c}{I} \qquad \sigma_m = K_f \frac{M_m c}{I} \qquad (7\text{–}1)$$

$$\tau_a = K_{fs} \frac{T_a r}{J} \qquad \tau_m = K_{fs} \frac{T_m r}{J} \qquad (7\text{–}2)$$

- Customized for round shafts

$$\sigma_a = K_f \frac{32 M_a}{\pi d^3} \qquad \sigma_m = K_f \frac{32 M_m}{\pi d^3} \qquad (7\text{–}3)$$

$$\tau_a = K_{fs} \frac{16 T_a}{\pi d^3} \qquad \tau_m = K_{fs} \frac{16 T_m}{\pi d^3} \qquad (7\text{–}4)$$

Shaft Stresses

- Combine stresses into von Mises stresses
  - Alternating (amplitude) von Mises stress
  - Mean (static) part von Mises stress

$$\sigma'_a = (\sigma_a^2 + 3\tau_a^2)^{1/2} = \left[\left(\frac{32K_f M_a}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs} T_a}{\pi d^3}\right)^2\right]^{1/2} \qquad (7\text{–}5)$$

$$\sigma'_m = (\sigma_m^2 + 3\tau_m^2)^{1/2} = \left[\left(\frac{32K_f M_m}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs} T_m}{\pi d^3}\right)^2\right]^{1/2} \qquad (7\text{–}6)$$

Shaft Stresses

- Substitute von Mises stresses into failure criteria equation. For example, using modified Goodman line,

$$\frac{1}{n} = \frac{\sigma_a'}{S_e} + \frac{\sigma_m'}{S_{ut}}$$

$$\frac{1}{n} = \frac{16}{\pi d^3}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_{ut}}\left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}\right\} \quad (7\text{–}7)$$

- Solving for *d* is convenient for design purposes

$$d = \left(\frac{16n}{\pi}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_{ut}}\left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}\right\}\right)^{1/3} \quad (7\text{–}8)$$

Shaft Stresses

- Similar approach can be taken with any of the fatigue failure criteria
- Equations are referred to by referencing both the Distortion Energy method of combining stresses and the fatigue failure locus name. For example, *DE-Goodman*, *DE-Gerber*, etc.
- In analysis situation, can either use these customized equations for factor of safety, or can use standard approach from Ch. 6.
- In design situation, customized equations for d are much more convenient.

Shaft Stresses

- *DE-Gerber*

$$\frac{1}{n} = \frac{8A}{\pi d^3 S_e}\left\{1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}}\right)^2\right]^{1/2}\right\} \qquad (7\text{–}9)$$

$$d = \left(\frac{8nA}{\pi S_e}\left\{1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}}\right)^2\right]^{1/2}\right\}\right)^{1/3} \qquad (7\text{–}10)$$

where

$$A = \sqrt{4(K_f M_a)^2 + 3(K_{fs} T_a)^2}$$

$$B = \sqrt{4(K_f M_m)^2 + 3(K_{fs} T_m)^2}$$

Shaft Stresses

- *DE-ASME Elliptic*

$$\frac{1}{n} = \frac{16}{\pi d^3}\left[4\left(\frac{K_f M_a}{S_e}\right)^2 + 3\left(\frac{K_{fs} T_a}{S_e}\right)^2 + 4\left(\frac{K_f M_m}{S_y}\right)^2 + 3\left(\frac{K_{fs} T_m}{S_y}\right)^2\right]^{1/2} \qquad (7\text{–}11)$$

$$d = \left\{\frac{16n}{\pi}\left[4\left(\frac{K_f M_a}{S_e}\right)^2 + 3\left(\frac{K_{fs} T_a}{S_e}\right)^2 + 4\left(\frac{K_f M_m}{S_y}\right)^2 + 3\left(\frac{K_{fs} T_m}{S_y}\right)^2\right]^{1/2}\right\}^{1/3} \qquad (7\text{–}12)$$

- *DE-Soderberg*

$$\frac{1}{n} = \frac{16}{\pi d^3}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_y}\left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}\right\} \qquad (7\text{–}13)$$

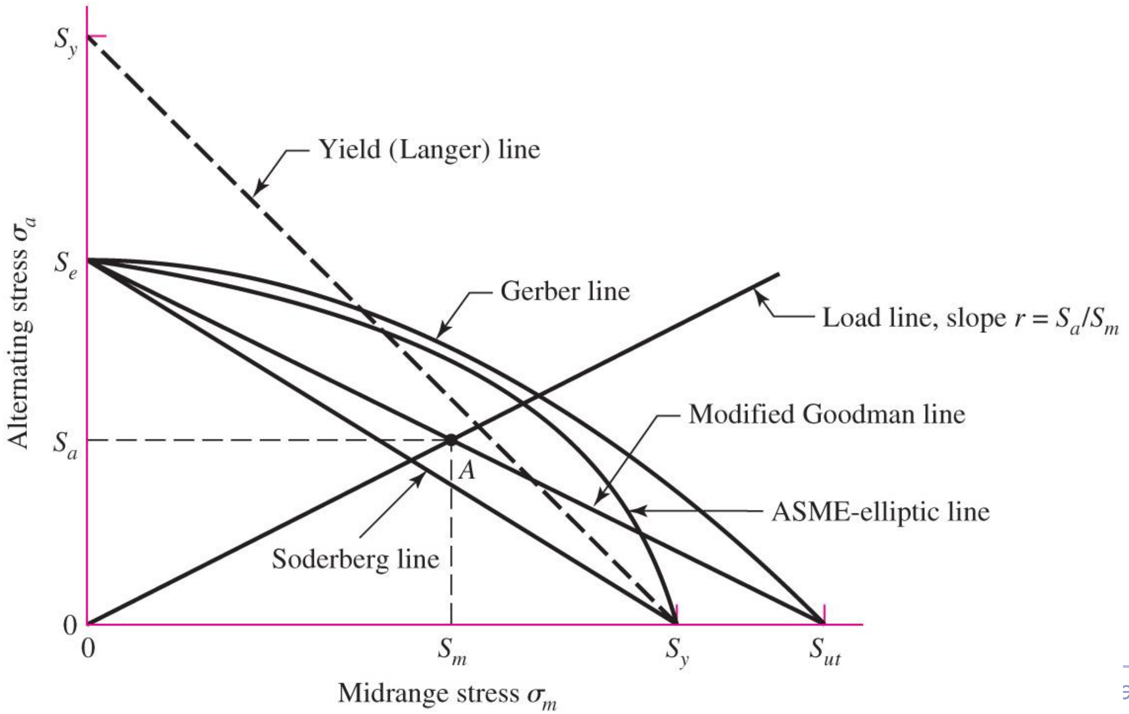
$$d = \left(\frac{16n}{\pi}\left\{\frac{1}{S_e}\left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} + \frac{1}{S_y}\left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}\right\}\right)^{1/3} \qquad (7\text{–}14)$$

Shaft Stresses for Rotating Shaft

- For rotating shaft with steady, alternating bending and torsion
  - Bending stress is completely reversed (alternating), since a stress element on the surface cycles from equal tension to compression during each rotation
  - Torsional stress is steady (constant or static)
  - Previous equations simplify with M_m and T_a equal to 0

Checking for Yielding in Shafts

- Always necessary to consider static failure, even in fatigue situation
- Soderberg criteria inherently guards against yielding
- ASME-Elliptic criteria takes yielding into account, but is not entirely conservative
- Gerber and modified Goodman criteria require specific check for yielding



Checking for Yielding in Shafts

- Use von Mises maximum stress to check for yielding,

$$\sigma'_{\max} = \left[(\sigma_m + \sigma_a)^2 + 3(\tau_m + \tau_a)^2\right]^{1/2}$$

$$= \left[\left(\frac{32K_f(M_m + M_a)}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs}(T_m + T_a)}{\pi d^3}\right)^2\right]^{1/2}$$

(7–15)

$$n_y = \frac{S_y}{\sigma'_{\max}} \qquad (7\text{–}16)$$

- Alternate simple check is to obtain conservative estimate of $\sigma'_{\max}$ by summing σ'_a and σ'_m

$$\sigma'_{\max} \approx \sigma'_a + \sigma'_m$$

Example 7–1

At a machined shaft shoulder the small diameter d is 1.100 in, the large diameter D is 1.65 in, and the fillet radius is 0.11 in. The bending moment is 1260 lbf · in and the steady torsion moment is 1100 lbf · in. The heat-treated steel shaft has an ultimate strength of $S_{ut} = 105$ kpsi and a yield strength of $S_y = 82$ kpsi. The reliability goal for the endurance limit is 0.99.

(*a*) Determine the fatigue factor of safety of the design using each of the fatigue failure criteria described in this section.

(*b*) Determine the yielding factor of safety.

Example 7–1 (continued)

(*a*) $D/d = 1.65/1.100 = 1.50$, $r/d = 0.11/1.100 = 0.10$, $K_t = 1.68$ (Fig. A–15–9), $K_{ts} = 1.42$ (Fig. A–15–8), $q = 0.85$ (Fig. 6–20), $q_{\text{shear}} = 0.88$ (Fig. 6–21).

From Eq. (6–32),

$$K_f = 1 + 0.85(1.68 - 1) = 1.58$$

$$K_{fs} = 1 + 0.88(1.42 - 1) = 1.37$$

Eq. (6–8): $$S'_e = 0.5(105) = 52.5\,\text{kpsi}$$

Eq. (6–19): $$k_a = 2.70(105)^{-0.265} = 0.787$$

Eq. (6–20): $$k_b = \left(\frac{1.100}{0.30}\right)^{-0.107} = 0.870$$

$$k_c = k_d = k_f = 1$$

Table 6–6: $$k_e = 0.814$$

$$S_e = 0.787(0.870)0.814(52.5) = 29.3\,\text{kpsi}$$

Example 7–1 (continued)

For a rotating shaft, the constant bending moment will create a completely reversed bending stress.

$$M_a = 1260 \text{ lbf} \cdot \text{in} \qquad T_m = 1100 \text{ lbf} \cdot \text{in} \qquad M_m = T_a = 0$$

Applying Eq. (7–7) for the DE-Goodman criteria gives

$$\frac{1}{n} = \frac{16}{\pi(1.1)^3}\left\{\frac{\left[4\,(1.58 \cdot 1260)^2\right]^{1/2}}{29\,300} + \frac{\left[3\,(1.37 \cdot 1100)^2\right]^{1/2}}{105\,000}\right\} = 0.615$$

$n = 1.63$ DE-Goodman Answer

Similarly, applying Eqs. (7–9), (7–11), and (7–13) for the other failure criteria,

$n = 1.87$ DE-Gerber Answer

$n = 1.88$ DE-ASME Elliptic Answer

$n = 1.56$ DE-Soderberg Answer

Example 7–1 (continued)

For comparison, consider an equivalent approach of calculating the stresses and applying the fatigue failure criteria directly. From Eqs. (7–5) and (7–6),

$$\sigma_a' = \left[\left(\frac{32 \cdot 1.58 \cdot 1260}{\pi\,(1.1)^3}\right)^2\right]^{1/2} = 15\,235 \text{ psi}$$

$$\sigma_m' = \left[3\left(\frac{16 \cdot 1.37 \cdot 1100}{\pi\,(1.1)^3}\right)^2\right]^{1/2} = 9988 \text{ psi}$$

Taking, for example, the Goodman failure critera, application of Eq. (6–46) gives

$$\frac{1}{n} = \frac{\sigma_a'}{S_e} + \frac{\sigma_m'}{S_{ut}} = \frac{15\,235}{29\,300} + \frac{9988}{105\,000} = 0.615$$

$$n = 1.63$$ Answer

which is identical with the previous result. The same process could be used for the other failure criteria.

Example 7–1 (continued)

(*b*) For the yielding factor of safety, determine an equivalent von Mises maximum stress using Eq. (7–15).

$$\sigma'_{\max} = \left[\left(\frac{32(1.58)\,(1260)}{\pi\,(1.1)^3}\right)^2 + 3\left(\frac{16(1.37)\,(1100)}{\pi\,(1.1)^3}\right)^2\right]^{1/2} = 18\ 220\,\text{psi}$$

$$n_y = \frac{S_y}{\sigma'_{\max}} = \frac{82\,000}{18\,220} = 4.50$$

For comparison, a quick and very conservative check on yielding can be obtained by replacing $\sigma'_{\max}$ with $\sigma'_a + \sigma'_m$. This just saves the extra time of calculating $\sigma'_{\max}$ if σ'_a and σ'_m have already been determined. For this example,

$$n_y = \frac{S_y}{\sigma'_a + \sigma'_m} = \frac{82\,000}{15\,235 + 9988} = 3.25$$

which is quite conservative compared with $n_y = 4.50$.

Estimating Stress Concentrations

- Stress analysis for shafts is highly dependent on stress concentrations.
- Stress concentrations depend on size specifications, which are not known the first time through a design process.
- Standard shaft elements such as shoulders and keys have standard proportions, making it possible to estimate stress concentrations factors before determining actual sizes.

Estimating Stress Concentrations

Table 7–1

First Iteration Estimates for Stress-Concentration Factors K_t and K_{ts}.

Warning: These factors are only estimates for use when actual dimensions are not yet determined. Do *not* use these once actual dimensions are available.

	Bending	Torsional	Axial
Shoulder fillet—sharp ($r/d = 0.02$)	2.7	2.2	3.0
Shoulder fillet—well rounded ($r/d = 0.1$)	1.7	1.5	1.9
End-mill keyseat ($r/d = 0.02$)	2.14	3.0	—
Sled runner keyseat	1.7	—	—
Retaining ring groove	5.0	3.0	5.0

Missing values in the table are not readily available.

Example 7–2

This example problem is part of a larger case study. See Chap. 18 for the full context.

A double reduction gearbox design has developed to the point that the general layout and axial dimensions of the countershaft carrying two spur gears has been proposed, as shown in Fig. 7-10. The gears and bearings are located and supported by shoulders, and held in place by retaining rings. The gears transmit torque through keys. Gears have been specified as shown, allowing the tangential and radial forces transmitted through the gears to the shaft to be determined as follows.

$$W_{23}^t = 540 \text{ lbf} \qquad W_{54}^t = 2431 \text{ lbf}$$

$$W_{23}^r = 197 \text{ lbf} \qquad W_{54}^r = 885 \text{ lbf}$$

where the superscripts t and r represent tangential and radial directions, respectively; and, the subscripts 23 and 54 represent the forces exerted *by* gears 2 and 5 (not shown) *on* gears 3 and 4, respectively.

Proceed with the next phase of the design, in which a suitable material is selected, and appropriate diameters for each section of the shaft are estimated, based on providing sufficient fatigue and static stress capacity for infinite life of the shaft, with minimum safety factors of 1.5.

Example 7–2 (continued)

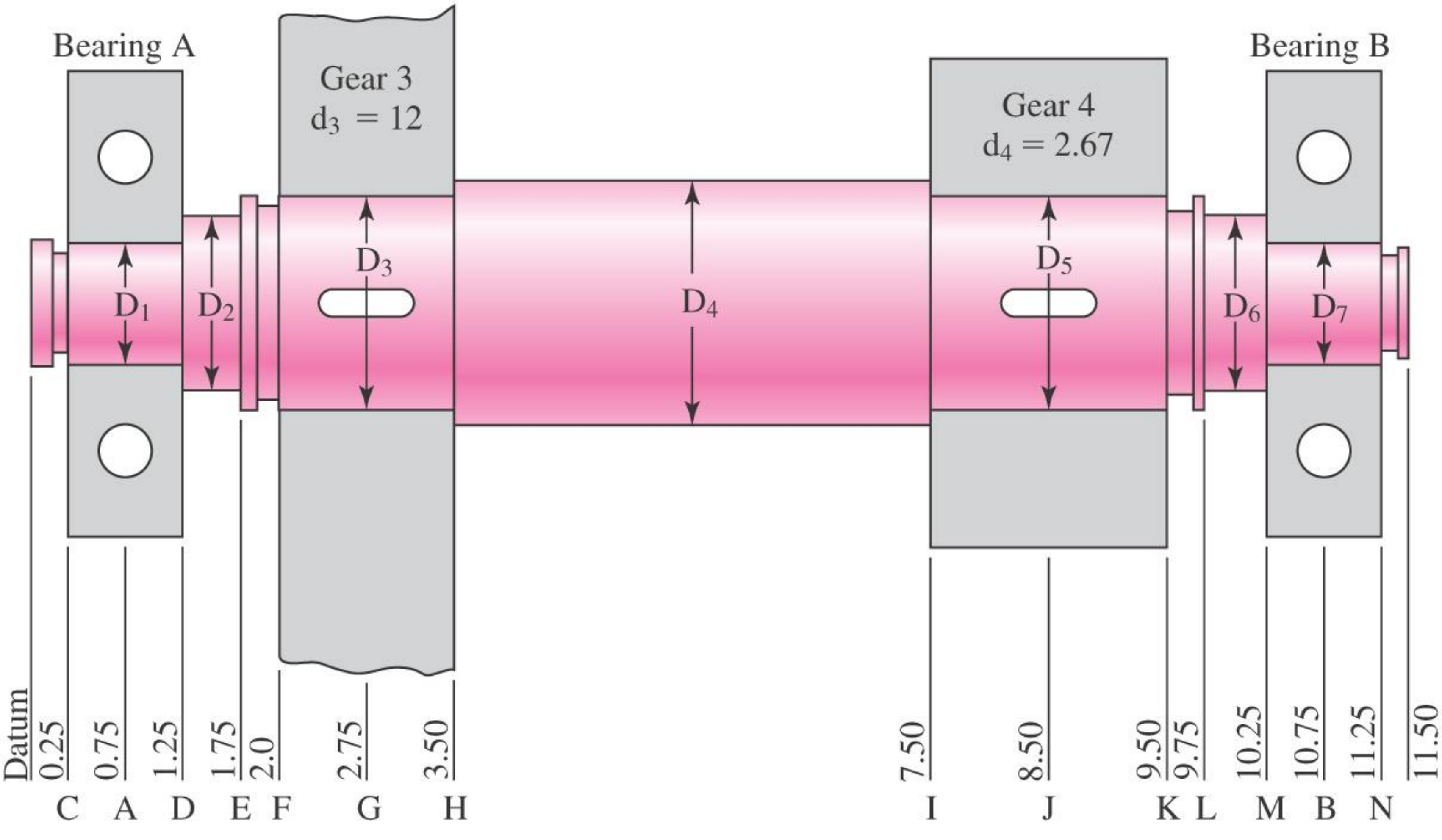


Fig. 7–10

Example 7–2 (continued)

Solution

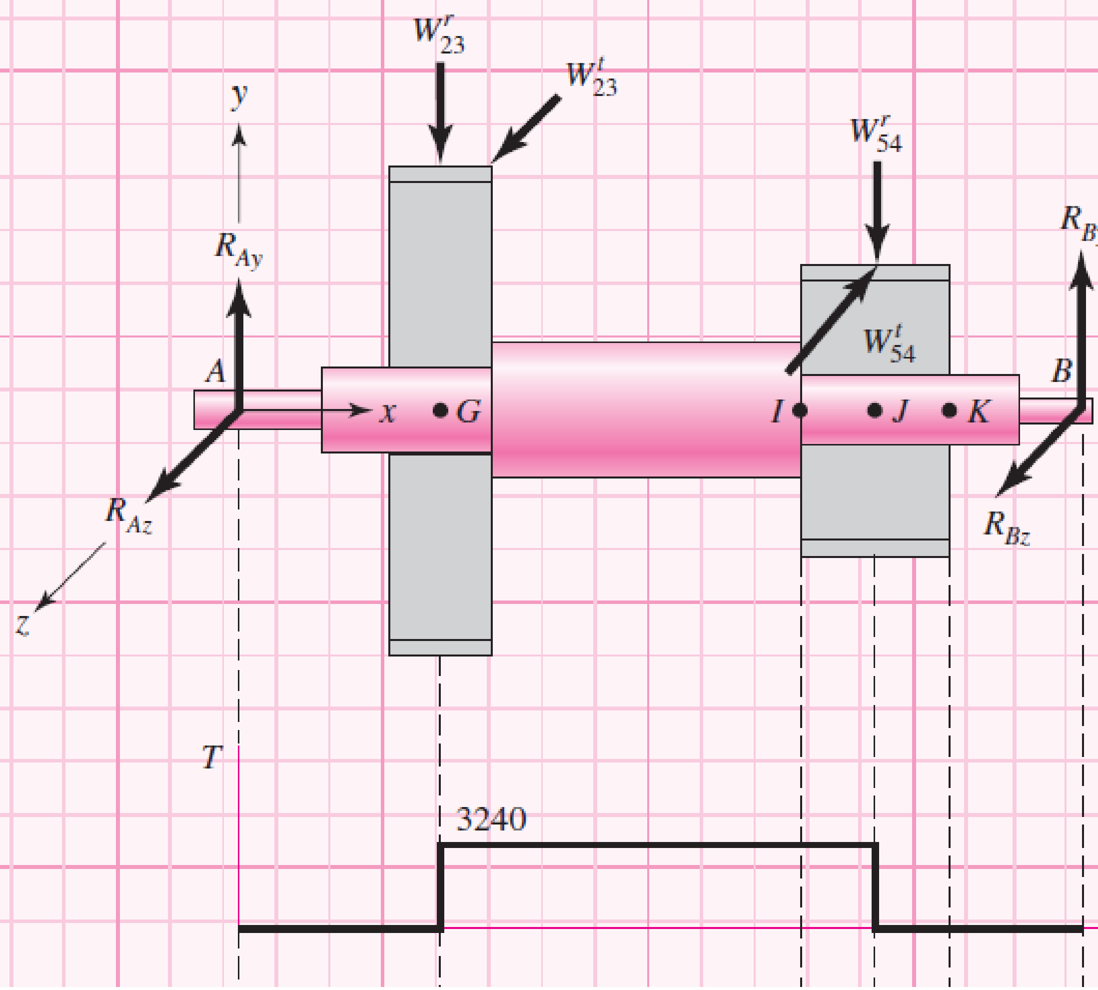
Perform free body diagram analysis to get reaction forces at the bearings.

$R_{Az} = 115.0\,\text{lbf}$

$R_{Ay} = 356.7\,\text{lbf}$

$R_{Bz} = 1776.0\,\text{lbf}$

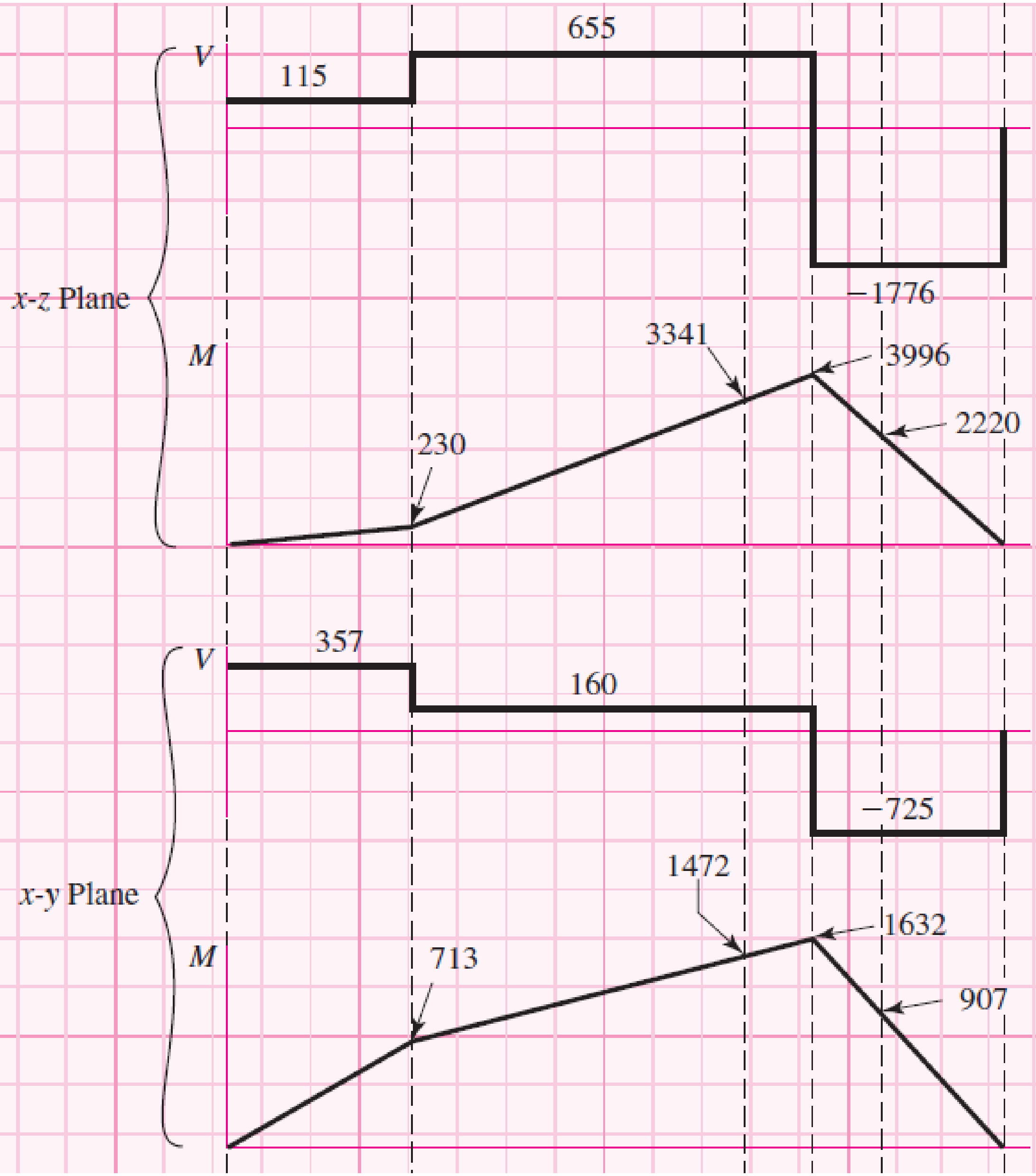
$R_{By} = 725.3\,\text{lbf}$



From ΣM_x, find the torque in the shaft between the gears, $T = W^t_{23}(d_3/2) = 540\,(12/2) = 3240\,\text{lbf}\cdot\text{in.}$

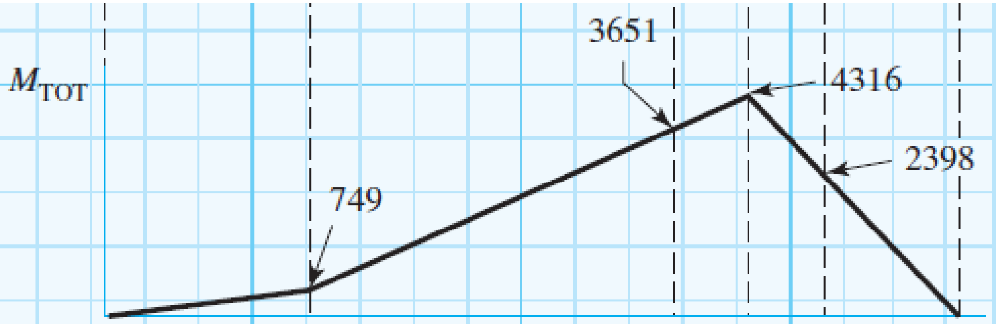
Example 7–2 (continued)

Generate shear-moment diagrams for two planes.



Example 7–2 (continued)

Combine orthogonal planes as vectors to get total moments, e.g., at J, $\sqrt{3996^2 + 1632^2} = 4316$ lbf · in.



Start with point I, where the bending moment is high, there is a stress concentration at the shoulder, and the torque is present.

$$\text{At } I,\ M_a = 3651 \text{ lbf} \cdot \text{in},\ T_m = 3240 \text{ lbf} \cdot \text{in},\ M_m = T_a = 0$$

Assume generous fillet radius for gear at I.

From Table 7–1, estimate $K_t = 1.7$, $K_{ts} = 1.5$. For quick, conservative first pass, assume $K_f = K_t$, $K_{fs} = K_{ts}$.

Example 7–2 (continued)

Choose inexpensive steel, 1020 CD, with $S_{ut} = 68$ kpsi. For S_e,

Eq. (6–19) $$k_a = aS_{ut}^b = 2.7(68)^{-0.265} = 0.883$$

Guess $k_b = 0.9$. Check later when d is known.

$$k_c = k_d = k_e = 1$$

Eq. (6–18) $$S_e = (0.883)(0.9)(0.5)(68) = 27.0 \text{ kpsi}$$

For first estimate of the small diameter at the shoulder at point I, use the DE-Goodman criterion of Eq. (7-8). This criterion is good for the initial design, since it is simple and conservative. With $M_m = T_a = 0$, Eq. (7-8) reduces to

$$d = \left\{ \frac{16n}{\pi} \left(\frac{2\left(K_f M_a\right)}{S_e} + \frac{\left[3\left(K_{fs} T_m\right)^2\right]^{1/2}}{S_{ut}} \right) \right\}^{1/3}$$

$$d = \left\{ \frac{16(1.5)}{\pi} \left(\frac{2\,(1.7)\,(3651)}{27\,000} + \frac{\{3\,[(1.5)\,(3240)]^2\}^{1/2}}{68\,000} \right) \right\}^{1/3}$$

$$d = 1.65 \text{ in}$$

All estimates have probably been conservative, so select the next standard size below 1.65 in. and check, $d = 1.625$ in.