MAGNETOSTATICS

Creation of magnetic field B.

Effect of B on a moving charge.

Take the second case:

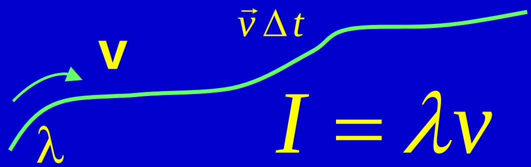
$$\vec{F}_{mag} = Q\left(\vec{v} \times \vec{B}\right)$$ → On moving charges only

$$\vec{F} = Q\left[\vec{E} + \left(\vec{v} \times \vec{B}\right)\right]$$ →Stationary and moving charges

Analysis on $\mathbf{F_{mag}}$: $dW_{mag} = \vec{F}_{mag} \cdot d\vec{l} = Q\left(\vec{v} \times \vec{B}\right).\vec{v}dt = 0$

Magnetic forces do no work!

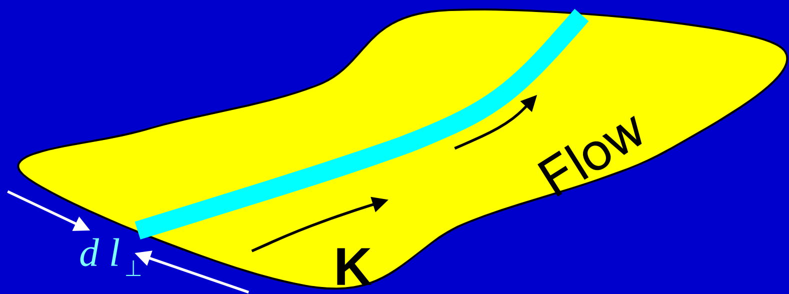
Current-carrying wire



$$I = \lambda v$$

$$\vec{F}_{mag} = \int \left(\vec{v} \times \vec{B}\right) dq$$
$$= \int \left(\vec{v} \times \vec{B}\right) \lambda dl$$
$$= \int I \left(d\vec{l} \times \vec{B}\right)$$

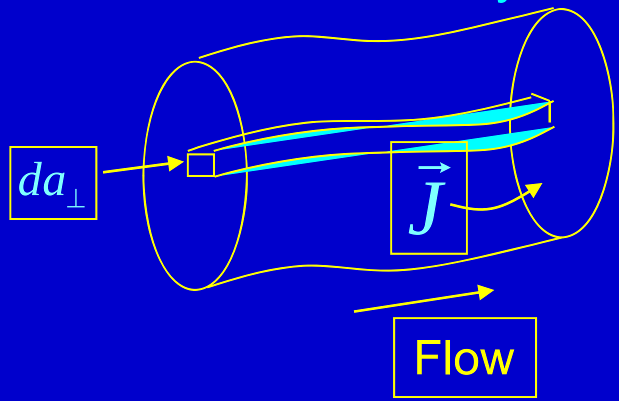
Charge-flow over a surface with a surface current density $\vec{K}$



$$\vec{K} = \frac{dI}{dl_\perp}\hat{K} \qquad \vec{K} = \sigma \vec{v}$$

$$\vec{F}_{mag} = \int \left(\vec{v} \times \vec{B}\right) \sigma da$$
$$= \int \left(\vec{K} \times \vec{B}\right) da$$

Volume current density $\vec{J}$



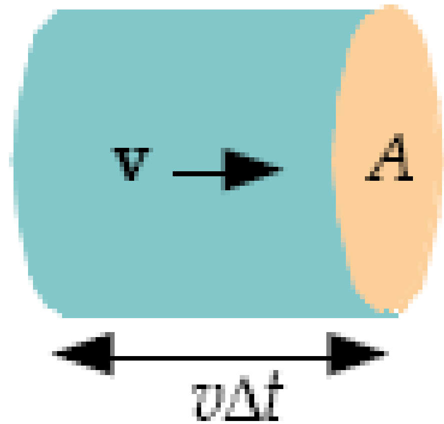
$$\vec{J} = \frac{dI}{da_\perp}\hat{v}$$
$$\vec{J} = \rho \vec{v}$$

$$\vec{F}_{mag} = \int \left(\vec{v} \times \vec{B}\right) \rho d\tau$$
$$= \int \left(\vec{J} \times \vec{B}\right) d\tau$$

The amount of charge passing through area A in time Δt is $nq(Av\Delta t)$.

Amount flowing per unit area per unit time is nqv, giving you the current density.

$$|\mathbf{j}| = nqv \ .$$



$$I = \oint_s \mathbf{j} \cdot d\mathbf{s} = \frac{dq}{dt} \text{ (rate of flow of charge)}$$

Consider a closed surface enclosing volume V. If ρ be the charge density for an infinitesimal volume dV, then $\int \rho dV$ represents the total charge inside the volume V.

According to the law of conservation of charge, the rate of flow of charge through the enclosed surface is equal to the rate of decrease of charge in it.

$$\oint_s \mathbf{j} \cdot d\mathbf{s} = -\frac{\partial}{\partial t}\int_V \rho \, dV = -\int_V \left(\frac{\partial \rho}{\partial t}\right) dV$$

According to divergence theorem

$$\oint_s \mathbf{j}\cdot d\mathbf{s} = \int_V (\nabla\cdot\mathbf{j})\, dV \qquad \text{-->} \qquad \int_V (\nabla\cdot\mathbf{j})\ \mathrm{dV} = \int_V \left(-\frac{\partial\rho}{\partial t}\right) \mathrm{dV}$$

$$\nabla\cdot\mathbf{j} = -\frac{\partial\rho}{\partial t}$$

This is called the equation of continuity and represents the physical facts of conservation of charges.

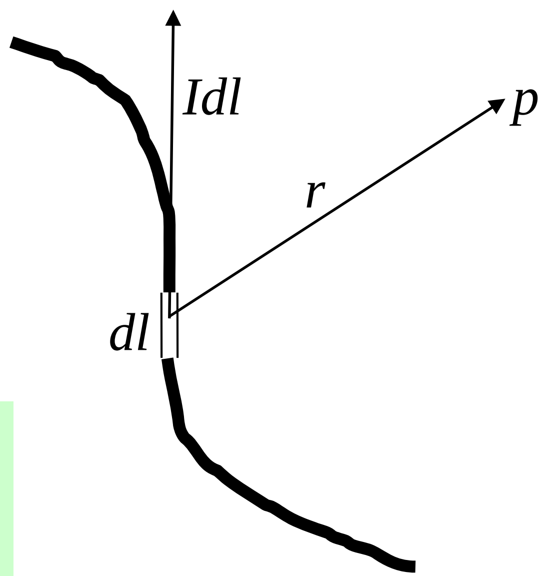
BIOT – SAVART LAW

Creation of magnetic field with the movement of charges.

Under steady state movement of charges (steady current), the magnetic field produced is given by the **Biot-Savart Law**:

(i) $d\mathbf{B} \propto I$ (ii) $d\mathbf{B} \propto dl$

(iii) $d\mathbf{B} \propto \frac{1}{r^2}$ (iv) $d\mathbf{B} \propto \sin\theta$



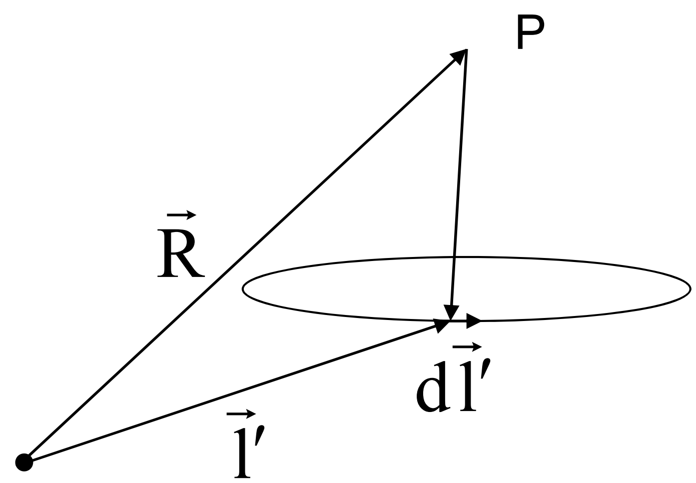
with *k* as constant of proportionality, in SI units,

$$k = \frac{\mu_0}{4\pi} = 10^{-7}\, NA^{-2}$$

$$dB = \frac{\mu_0}{4\pi}\frac{Idl\, sin\theta}{r^2}$$

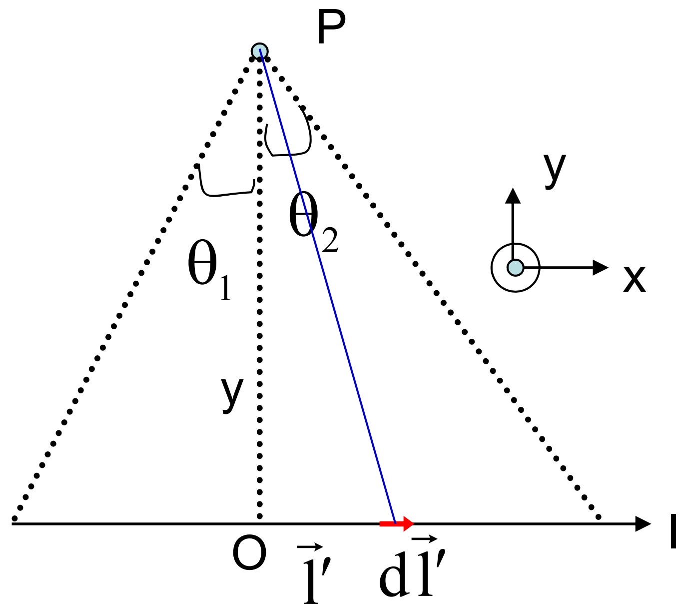
$$d\vec{B} = \frac{\mu_0}{4\pi} \frac{I d\vec{l}' \times (\vec{R} - \vec{l}')}{\left|(\vec{R} - \vec{l}')\right|^3}$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{1}{\left|(\vec{R} - \vec{l}')\right|^3} \left(d\vec{l}' \times (\vec{R} - \vec{l}') \right)$$



EXAMPLE - 1

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{l}' \times (y\hat{e}_z - l'\hat{e}_x)}{\left(y^2 + l'^2\right)^{3/2}}$$



Using the tan $\theta = l'/y$, we get the final result as

$$\boxed{\vec{B} = \frac{\mu_0 I}{4\pi y}\left(\sin\theta_2 - \sin\theta_1\right)\hat{e}_z}$$

If $-\theta_1 = \theta_2$:

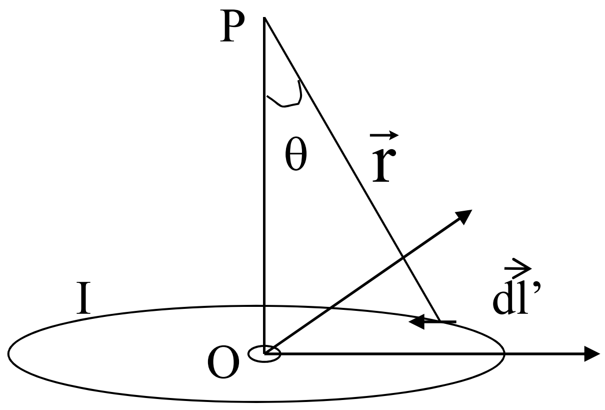
$$\vec{B} = \frac{\mu_0 I}{4\pi y}\left(2\sin\theta_2\right)\hat{e}_z$$

If $\theta_2 = \pi/2$ → Infinite line:

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi y}\hat{e}_z}$$

EXAMPLE - 2

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{\left(ad\varphi\hat{e}_\varphi \times \left(z\hat{e}_z - a\hat{e}_\rho\right)\right)}{\left(z^2 + a^2\right)^{3/2}}$$



$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint \frac{\left(azd\varphi\hat{e}_\rho + a^2 d\varphi\hat{e}_z\right)}{\left(z^2 + a^2\right)^{3/2}}$$

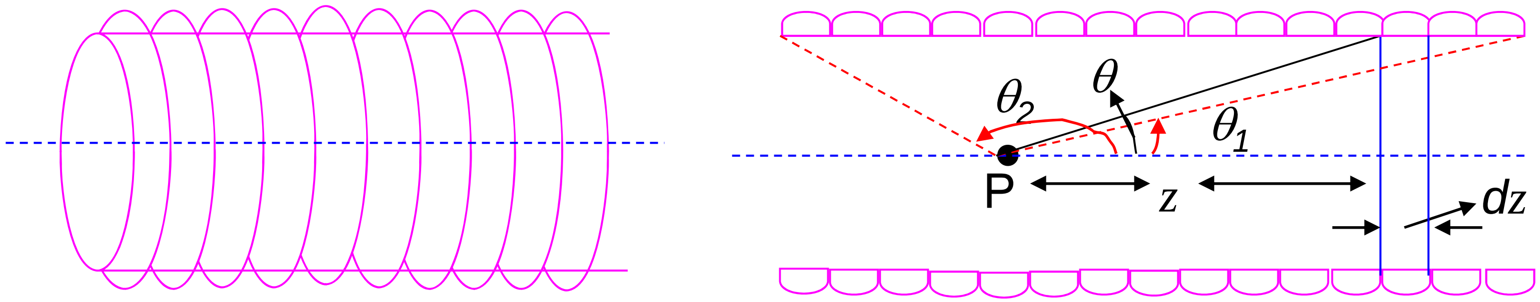
$$\oint d\varphi\hat{e}_\rho = 0$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \frac{\left(2\pi a^2\right)}{\left(z^2 + a^2\right)^{3/2}} \hat{e}_z = \frac{\mu_0 I}{2} \frac{a^2}{\left(z^2 + a^2\right)^{3/2}} \hat{e}_z$$

If there are 'n' turns, then $\vec{B} = \frac{\mu_0 n I}{2} \frac{a^2}{\left(z^2 + a^2\right)^{3/2}} \hat{e}_z$

At the center of the loop: $\vec{B} = \frac{\mu_0 I}{2a} \hat{e}_z$

EXAMPLE - 3



Use the evaluation of the magnetic field for a loop and follow superposition principle to evaluate for a solenoid.

The variable is now 'z'.

What about current? If there are 'n' no. of turns per unit length and each turn having current I, then

$$d\vec{B} = \frac{\mu_0 I}{2} \frac{a^2}{\left(z^2 + a^2\right)^{3/2}} (n\,dz)\hat{e}_z = \frac{\mu_0 n I}{2} \frac{a^2 dz}{\left(z^2 + a^2\right)^{3/2}} \hat{e}_z$$

$$z = a\ \cot\theta$$

$$dz = -a\ \mathrm{cosec}^2\theta\ d\theta$$

$$z^2 + a^2 = a^2\ \cot^2\theta + a^2 = a^2\mathrm{cosec}^2\theta$$

$$\vec{B} = \int_{\theta_1}^{\theta_2} \frac{\mu_0 n I}{2} \sin\theta\, d\theta\, \hat{e}_z = \frac{\mu_0 n I}{2}\left(\cos\theta_1 - \cos\theta_2\right)\hat{e}_z$$

Case 1 **If we take a long solenoid (radius of the solenoid very small compared to its length) and observation point p is well with in the solenoid, then $\theta_1 = 0$ and $\theta_2 = \pi$**

$$\vec{B} = \mu_0 n I \hat{e}_z$$

Case 2 When the observation point P is taken one end of the solenoid

$$\theta_1 = 0 \text{ and } \theta_2 = \pi/2$$

$$\left|\vec{B}\right| = \frac{\mu_0 n I}{2}$$

Hence in case of semi-infinitely long solenoid, the magnetic field at a point at the end of the solenoid is half the magnetic field at a point well inside the solenoid.

For surface and volume currents, Biot-Savart law becomes

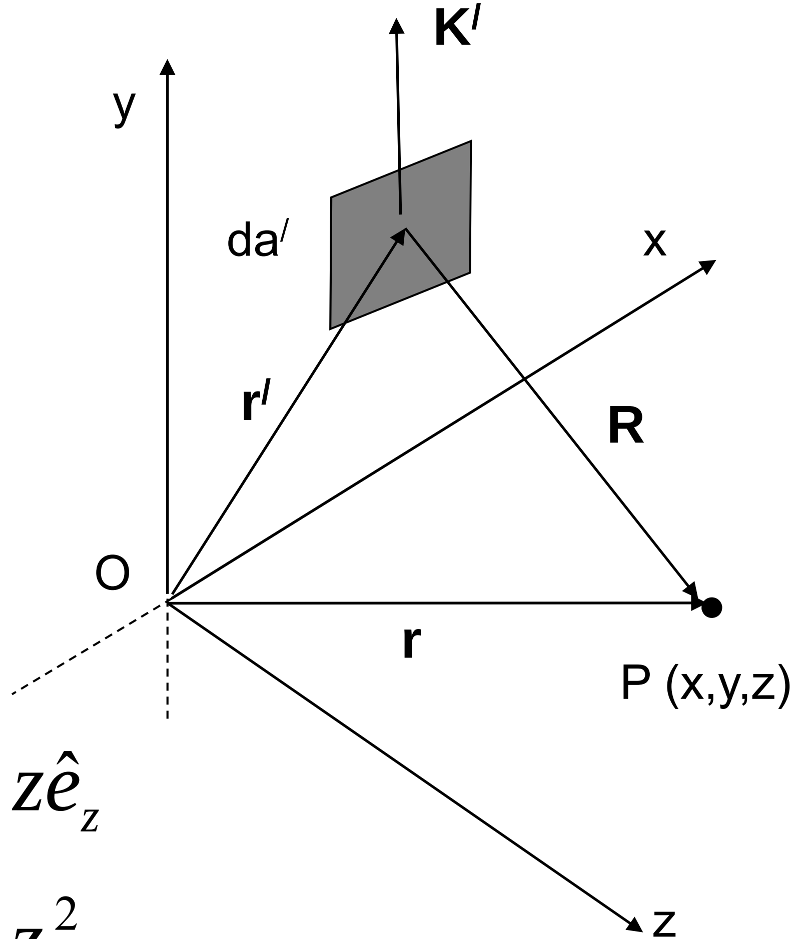
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}(\vec{r}\,')\times\hat{R}}{R^2}da'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')\times\hat{R}}{R^2}d\tau'$$

Infinite plane of uniform current sheet

Constant current density $\vec{\mathbf{K}}'$

Current sheet lies in the xy plane and



$$\vec{K}' = K'\hat{e}_y$$

$$\vec{r} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z$$

$$\vec{r}\,' = x'\hat{e}_x + y'\hat{e}_y$$

$$\vec{R} = \left(x - x'\right)\hat{e}_x + \left(y - y'\right)\hat{e}_y + z\hat{e}_z$$

$$R^2 = \left(x - x'\right)^2 + \left(y - y'\right)^2 + z^2$$

$$da' = dx'dy' \qquad \vec{K}' \times \vec{R} = K'\left[z\hat{e}_x + \left(x' - x\right)\hat{e}_z\right]$$

Use Biot-Savart law for a surface current density and integrate

$$\vec{B} = \frac{\mu_0 K'}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\left[z\hat{e}_x + \left(x' - x \right) \hat{e}_z \right] dx' dy'}{\left(\left(x - x' \right)^2 + \left(y - y' \right)^2 + z^2 \right)^{3/2}}$$

$$= \frac{\mu_0 K'}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\left[z\hat{e}_x + X'\hat{e}_z \right] dX' dY'}{\left(X'^2 + Y'^2 + z^2 \right)^{3/2}}$$

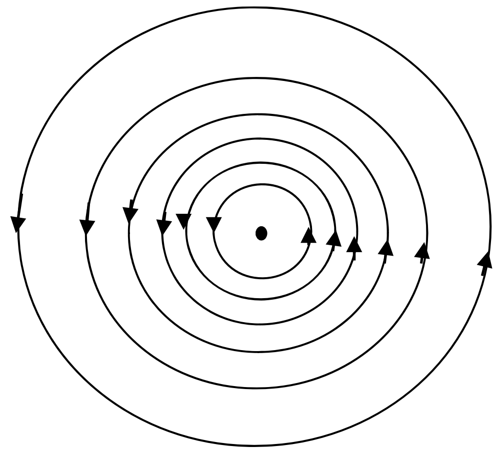
$$X' = x' - x \quad Y' = y' - y$$

z-component will vanish because the integrand is an odd function of X'

$$\vec{B} = \pm \frac{1}{2} \mu_0 K' \hat{e}_x = \frac{1}{2} \mu_0 K' \left(\frac{z}{|z|} \right) \hat{e}_x$$

The Divergence and curl of B

Nonzero curl??



$$\oint \vec{B}\cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi s} dl = \frac{\mu_0 I}{2\pi s}\oint dl$$

$$= \mu_0 I$$

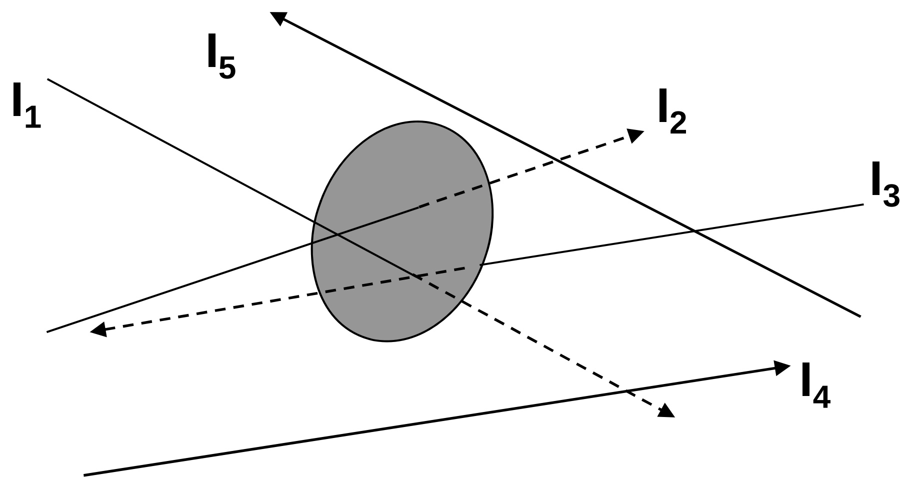
If we use cylindrical coordinates (s, ϕ, z) with current along z axis

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{e}_\phi$$

$$d\vec{l} = ds\hat{e}_s + sd\phi\hat{e}_\phi + dz\hat{e}_z$$

$$\vec{B} = \frac{\mu_0 I}{2\pi}\oint \frac{1}{s} sd\phi = \frac{\mu_0 I}{2\pi}\oint d\phi = \mu_0 I$$

Bundle of straight wires



Each wire that passes through the loop contributes $\mu_0 I$

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

If the flow of charge is represented by a volume charge density $\vec{\mathbf{J}}$

$$I_{enc} = \int \vec{J} \cdot d\vec{a}$$ Integral taken over the surface bounded by the loop

Applying stokes' theorem

$$\int \left(\vec{\nabla} \times \vec{B} \right) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

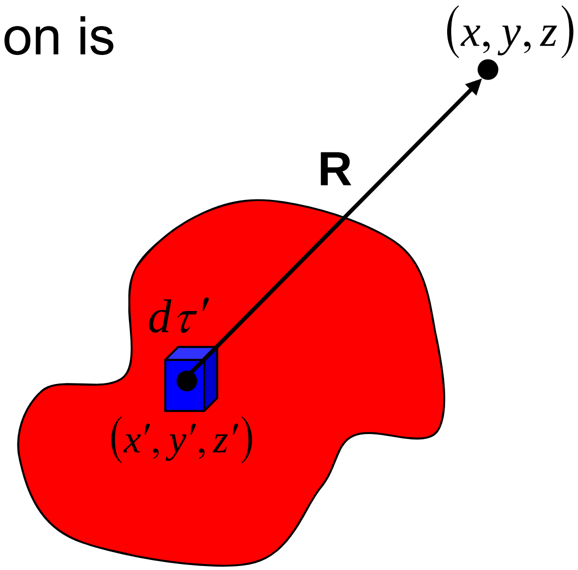
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

Above derivation is restricted by the condition that we need infinitely straight line currents

Divergence and curl of B

Biot-Savart law for a volume current distribution is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}')\times\hat{R}}{R^2}dV'$$



$\vec{\mathbf{B}}$ is a function of (x, y, z)

$\vec{\mathbf{J}}$ is a function of (x', y', z')

$$\vec{R} = (x - x')\hat{e}_x + (y - y')\hat{e}_y + (z - z')\hat{e}_z$$

$$dV' = dx'dy'dz'$$

Integration is done over the primed coordinates
Divergence and curl is done over the unprimed coordinates

Applying divergence to the magnetic field B due to a volume charge distribution

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{R}}{R^2}\right)d\tau'$$

$$\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right) = \vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right) - \vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$$

$$\vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{R}}{R^2}\right) = \frac{\hat{R}}{R^2}\cdot\left(\vec{\nabla}\times\vec{J}\right) - \vec{J}\cdot\left(\vec{\nabla}\times\frac{\hat{R}}{R^2}\right)$$

$\vec{\nabla}\times\vec{J} = 0$ because $\vec{J}$ **does not depend on unprimed coordinates**

and $\vec{\nabla}\times\frac{\hat{R}}{R^2} = 0$

$$\boxed{\vec{\nabla}\cdot\vec{B} = 0}$$

Applying curl to the magnetic field B due to a volume charge distribution

$$\vec{\nabla} \times \vec{B} = \frac{\mu_0}{4\pi} \int \vec{\nabla} \times \left(\vec{J} \times \frac{\hat{R}}{R^2} \right) d\tau'$$

$$\vec{\nabla} \times \left(\vec{A} \times \vec{B} \right) = \left(\vec{B} \cdot \vec{\nabla} \right) \vec{A} - \left(\vec{A} \cdot \vec{\nabla} \right) \vec{B} + \vec{A} (\vec{\nabla} \cdot \vec{B}) - \vec{B} (\vec{\nabla} \cdot \vec{A})$$

$$\vec{\nabla} \times \left(\vec{J} \times \frac{\hat{R}}{R^2} \right) = \vec{J} \left(\vec{\nabla} \cdot \frac{\hat{R}}{R^2} \right) - \left(\vec{J} \cdot \vec{\nabla} \right) \frac{\hat{R}}{R^2} \longrightarrow \textbf{(a)}$$

The terms involving derivatives of $\vec{\mathbf{J}}$ is dropped since $\vec{\mathbf{J}}$ does not depend on (x,y,z)

The second term in (a) integrates to zero

$$\vec{\nabla} \cdot \frac{\hat{R}}{R^2} = 4\pi \delta^3 (\vec{R})$$

$$\vec{\nabla} \times \vec{\mathrm{B}} = \frac{\mu_0}{4\pi} \int \vec{\mathrm{J}}(\vec{\mathrm{r}}') 4\pi \delta^3 (\vec{\mathrm{r}} - \vec{\mathrm{r}}') d\tau' = \mu_0 \vec{\mathrm{J}}(\vec{\mathrm{r}})$$

How does the other term vanish?

Because the derivative acts only on R term, we can switch to

$$-\left(\vec{J}\cdot\vec{\nabla}\right)\frac{\hat{R}}{R^2}=\left(\vec{J}\cdot\vec{\nabla}'\right)\frac{\hat{R}}{R^2}$$

Consider the x-component

$$\left(\vec{J}\cdot\vec{\nabla}'\right)\left(\frac{x-x'}{R^3}\right)=\vec{\nabla}'\cdot\left[\frac{(x-x')}{R^3}\vec{J}\right]-\left(\frac{(x-x')}{R^3}\right)\left(\vec{\nabla}'\cdot\vec{J}\right)$$

We are dealing with steady currents, hence second terms in zero

$$-\left(\vec{J}\cdot\vec{\nabla}\right)\frac{\hat{R}}{R^2}=\vec{\nabla}'\cdot\left[\frac{(x-x')}{R^3}\vec{J}\right]$$

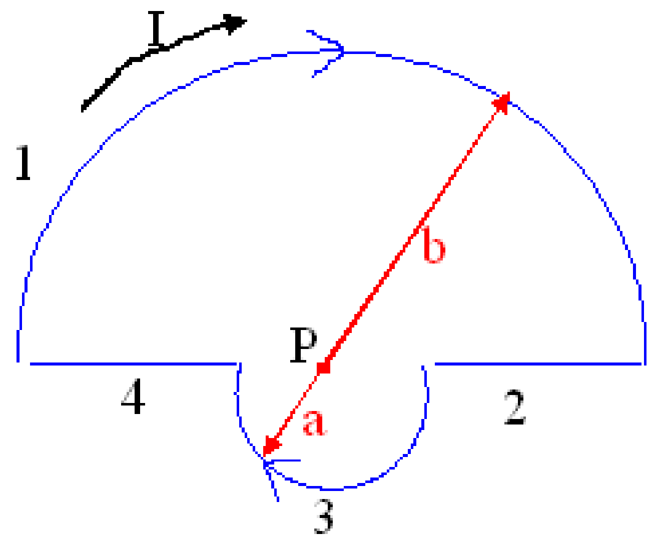
Contribution to the integral from this term is

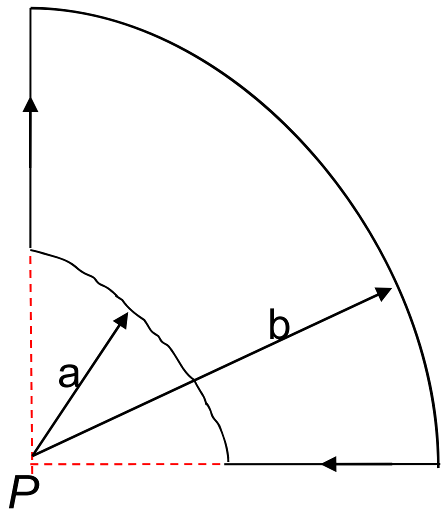
$$\int_V \vec{\nabla}'\cdot\left[\frac{(x-x')}{R^3}\vec{J}\right]d\tau'=\oint_S \frac{(x-x')}{R^3}\vec{J}\cdot d\vec{a}$$

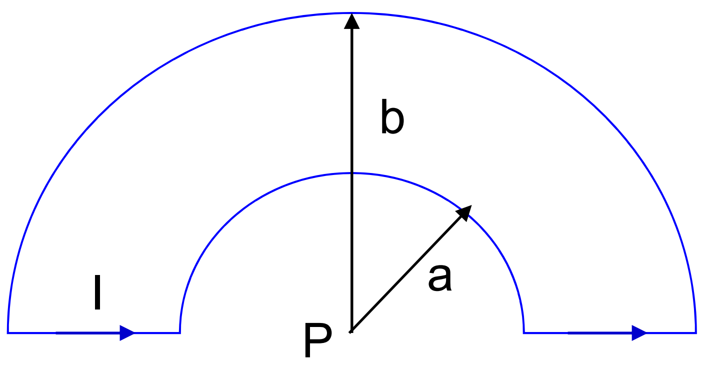
We are integrating over the source region that include all the current.

On the boundary the current is zero and hence the surface integral vanishes

FURTHER EXAMPLES FOR BIOT – SAVART LAW







AMPERE'S LAW

$$\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

in differential form

Using Stokes' theorem

$$\int \left(\vec{\nabla} \times \vec{B}\right) \cdot d\vec{a} = \oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a}$$

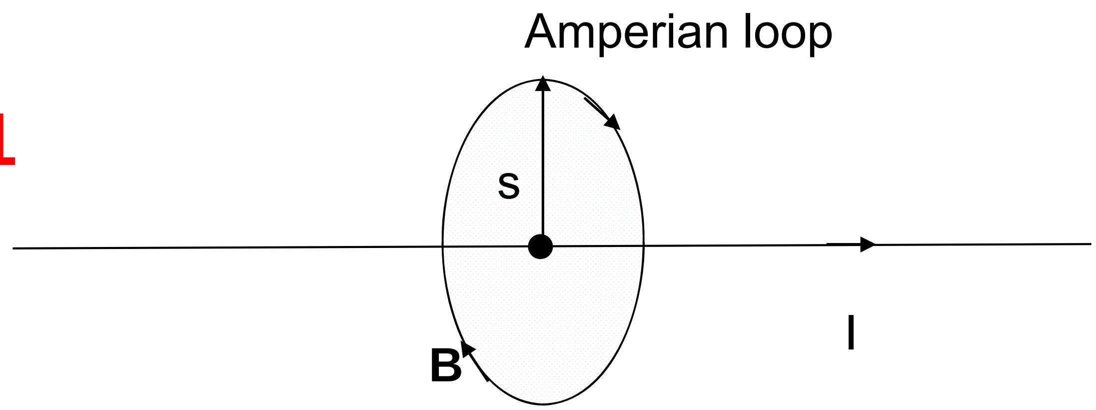
$\int \vec{J} \cdot d\vec{a}$ is the total current passing through the surface-- I_{enc}

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}}$$

in integral form

EXAMPLES

Example 1



By symmetry, the magnitude of $\vec{\mathbf{B}}$ is constant around an amperian loop of radius s

$$\oint \vec{B} \cdot d\vec{l} = B\oint dl = B2\pi s = \mu_0 I_{enc} = \mu_0 I$$

$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{e}_\varphi$$

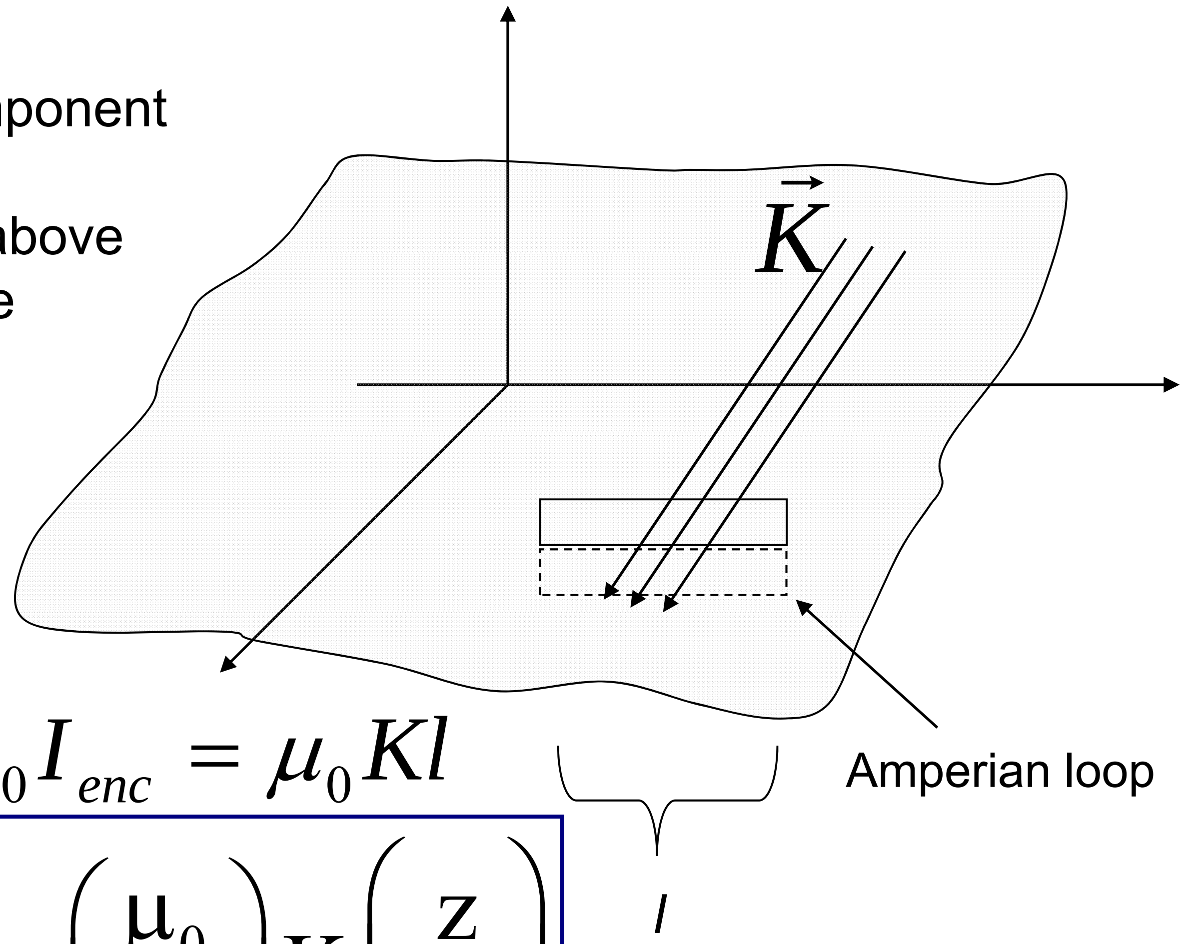
Example 2

Magnetic field of an infinite uniform surface current flowing over the xy plane

$$\boxed{\vec{K} = K\hat{e}_x}$$

$\vec{\mathbf{B}}$ can only have a y-component

It points towards the left above the plane and towards the right in the plane below



$$\oint \vec{B}\cdot d\vec{l} = 2Bl = \mu_0 I_{enc} = \mu_0 K l$$

$$\boxed{B = \left(\frac{\mu_0}{2}\right)K}$$

$$\boxed{\vec{\mathrm{B}} = \left(\frac{\mu_0}{2}\right)\mathrm{K}\left(\frac{z}{|z|}\right)}$$

Example 3

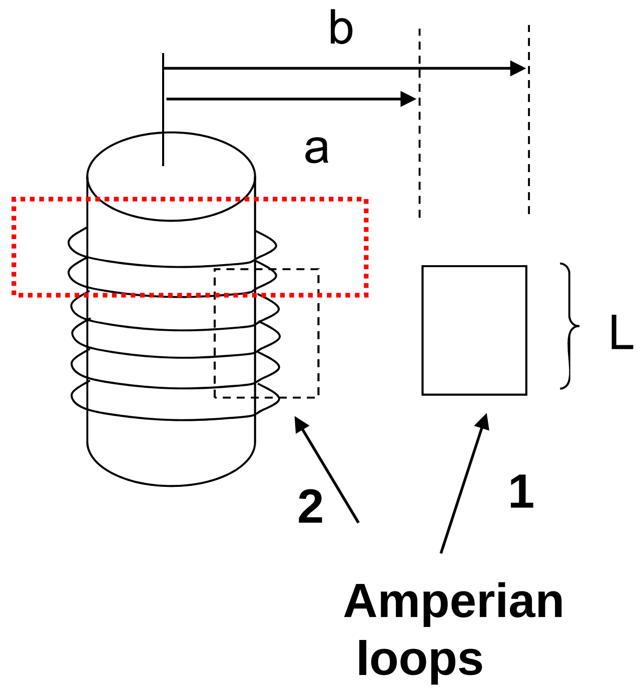
Magnetic field of a very long solenoid consisting of n closely wound turns per unit length on a cylinder of radius R and carrying a steady current I

Loop1

$$\oint \vec{B}\cdot d\vec{l} = \left[B(a) - B(b)\right]L = \mu_0 I_{enc} = 0$$

$$\mathrm{B(a) = B(b) = 0}$$

$$K = nI$$



Loop 2

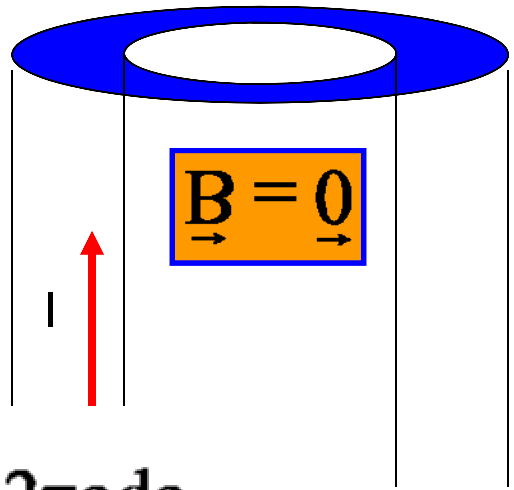
$$\oint \vec{B}\cdot d\vec{l} = BL = \mu_0 I_{enc} = \mu_0 nIL$$

$$\vec{\mathrm{B}} = \begin{cases} \mu_0 n I \hat{e}_z = \mu_0 K \hat{e}_z & \text{Inside the solenoid} \\ 0 & \text{Outside the solenoid} \end{cases}$$

EXAMPLE 4

A long copper pipe with thick walls has an inner radius R and an outer radius 2R. What is the current density J for all r? Use Ampere's Law to find the magnetic fields as a function of radial distance from the centre of the pipe.

$$\vec{J}(\vec{r}) = \frac{I}{3\pi R^2}\hat{e}_z$$



$0 \le \rho \le R$:

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 \iint \vec{J} \bullet d\vec{a} = 0$$

$R \le \rho \le 2R$:

$$\oint \vec{B} \bullet d\vec{l} = \mu_0 \iint \vec{J} \bullet d\vec{a} = \mu_0 \int_R^\rho \frac{I}{3\pi R^2} 2\pi\rho d\rho$$

$\rho > 2R$:

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi\rho}\hat{e}_\varphi}$$

$$\boxed{\vec{B} = \frac{\mu_0 J}{2\rho}\left(\rho^2 - R^2\right)\hat{e}_\varphi}$$

$$B\,2\pi\rho = \frac{\mu_0 I}{3\pi R^2}\frac{2\pi}{2}\left(\rho^2 - R^2\right)$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{6\pi R^2}\frac{1}{\rho}\left(\rho^2 - R^2\right)\hat{e}_\varphi}$$

EXAMPLE 5

A long copper wire of cross-sectional radius R carries a current density $\vec{J}(\vec{\rho}) = \alpha e^{-k\rho}\hat{e}_z$. Use Ampere's Law to determine B as a function of the distance from the centre of the wire.

B inside the wire:

$$B\rho = \mu_0\alpha\left[-\rho\frac{e^{-k\rho}}{k} - \frac{e^{-k\rho}}{k^2}\right]_0^{\rho}$$

$$\vec{B} = \frac{\mu_0\alpha}{\rho k^2}\left(1 - \rho k e^{-k\rho} - e^{-k\rho}\right)\hat{e}_\varphi$$

B outside the wire:

$$B\rho = \mu_0\alpha\left[-\rho\frac{e^{-k\rho}}{k} - \frac{e^{-k\rho}}{k^2}\right]_0^{R}$$

$$\vec{B} = \frac{\mu_0\alpha}{\rho k^2}\left(1 - R k e^{-kR} - e^{-kR}\right)\hat{e}_\varphi$$

POISSON'S EQN. IN MAGNETOSTATICS

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \rightarrow \quad \oint_S \vec{B} \cdot d\vec{a} = 0$$

Flux through any closed surface is always zero.
No monopole for the 'magnetic charge'

$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r})$ Divergence of a curl is always zero

$\vec{A}(\vec{r})$ is the vector potential

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

We can add to magnetic potential any function whose curl vanishes, with no effect on B

We use this freedom to eliminate the divergence of $\vec{A}$

$$\vec{\nabla} \bullet \vec{A} = 0$$

If the original potential A_0 is not divergenceless, then add to it gradient of function such that, A becomes divergenceless. $\vec{A} = \vec{A}_0 + \vec{\nabla}\lambda$

Gauge Transformation:
Coulomb gauge

$$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}_0 + \nabla^2 \lambda \rightarrow \nabla^2 \lambda = -\nabla \cdot A_0$$

$$\rightarrow \lambda = \frac{1}{4\pi} \int \frac{\vec{\nabla} \cdot \vec{A}_0}{R} d\tau'$$

$$\rightarrow \boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

Poisson's eq

In the current free region,

$$\boxed{\nabla^2 \vec{A} = \vec{0}}$$

Laplace eqn.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}\,' \times \hat{R}}{R^2}$$

$$\frac{d\vec{l}\,' \times \hat{R}}{R^2} = -d\vec{l}\,' \times \nabla\left(\frac{1}{R}\right) = \vec{\nabla} \times \left(\frac{d\vec{l}'}{R}\right) - \frac{\left(\nabla \times d\vec{l}'\right)}{R} = \nabla \times \left(\frac{d\vec{l}'}{R}\right)$$

$$\vec{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \nabla' \times \left(\frac{d\vec{l}'}{R}\right) = \nabla \times \left(\frac{\mu_0 I}{4\pi} \oint_{C'} \frac{d\vec{l}'}{R}\right) = \nabla \times \vec{A}$$

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \oint_{C'} \frac{I d\vec{l}\,'}{R}} \quad \boxed{= \frac{\mu_0}{4\pi} \oint_{S'} \frac{\vec{K}(\vec{r}')da'}{R}} \quad \boxed{= \frac{\mu_0}{4\pi} \oint_{V'} \frac{\vec{J}(\vec{r}')d\tau'}{R}}$$

DIVERGENCE OF $\vec{A}$

$$\vec{\nabla}\cdot\vec{A}(\vec{r}) = \vec{\nabla}\cdot\left(\frac{\mu_0}{4\pi}\oint_{C'}\frac{Id\vec{l}\,'}{R}\right) = \frac{\mu_0 I}{4\pi}\oint_{C'}\overline{\nabla}\cdot\left(\frac{d\vec{l}\,'}{R}\right)$$

$$\overline{\nabla}\cdot\left(\frac{d\vec{l}\,'}{R}\right) = d\vec{l}\,'\cdot\vec{\nabla}\left(\frac{1}{R}\right) + \frac{1}{R}\left(\vec{\nabla}\cdot d\vec{l}\,'\right) = -\vec{\nabla}'\left(\frac{1}{R}\right)\cdot d\vec{l}\,'$$

$$\vec{\nabla}\cdot\vec{A}(\vec{r}) = -\frac{\mu_0 I}{4\pi}\oint_{C'}\vec{\nabla}'\left(\frac{1}{R}\right)\cdot d\vec{l}\,'$$

$$= -\frac{\mu_o I}{4\pi}\oint_{S'}\left[\vec{\nabla}'\times\vec{\nabla}'\left(\frac{1}{R}\right)\right]\cdot d\vec{a}' = 0$$

MAGNETIC SCALAR POTENTIAL

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

In the current free region, $\nabla \times \vec{B} = \vec{0}$

Therefore, $\vec{B}$ can be expressed as: $\overline{B} = -\overline{\nabla}\Phi_m$

We call Φ_m as magnetic scalar potential.

$$\overline{\nabla} \bullet \overline{B} = -\overline{\nabla} \bullet \overline{\nabla}\Phi_m = -\overline{\nabla}^2\Phi_m = 0$$

We see that Φ_m satisfies the Laplace's equation.

ELECTROSTATICS AND MAGNETOSTATICS

$$\nabla \bullet \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \times \vec{E} = \vec{0}$$

$$\oint \vec{E} \bullet d\vec{s} = \frac{q_{enc}}{\varepsilon_0}$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$\nabla^2 V = 0$$

$$\nabla \bullet \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

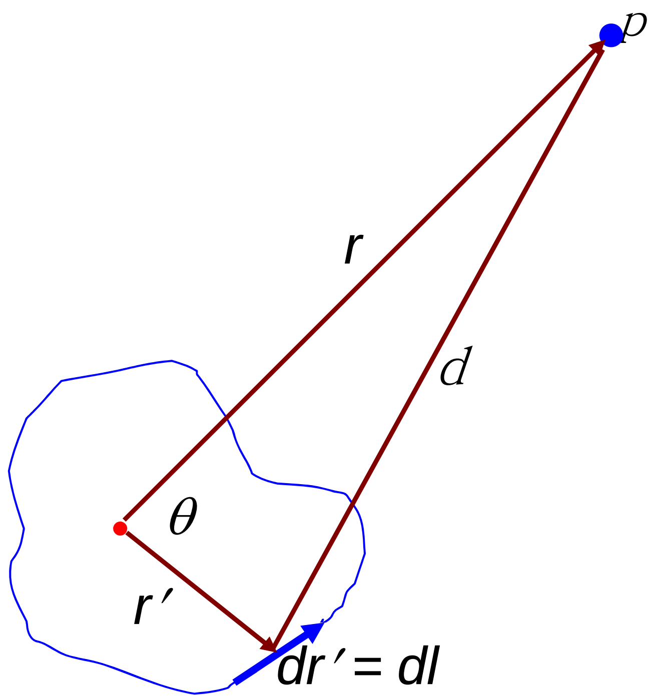
$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\nabla^2 \vec{A} = \vec{0}$$

MULTIPOLE EXPANSION FOR MAGNETIC VECTOR POTENTIAL

Similar to the electric scalar potential, one can use multipole expansion to find out the magnetic vector potential at a far away place due to a current distribution.



$$d^2 = r^2 + r'^2 - 2rr'\cos\theta = r^2\left(1+\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta\right)$$

This expression can be rewritten as

$$\frac{1}{d} = \frac{1}{r}\frac{1}{\sqrt{\left(1+\left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta\right)}}$$

Using bionomial expansion

$$\frac{1}{\sqrt{(1+x)}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots\cdots\cdots$$

$$\Rightarrow \frac{1}{d} = \frac{1}{r}\left\{1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\left(\frac{r'}{r}\right) - 2\cos\theta\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\left(\frac{r'}{r}\right) - 2\cos\theta\right)^2 - \cdots\cdots\right\}$$

Vector potential due to the current loop is $\vec{A} = \dfrac{\mu_0 I}{4\pi}\oint\dfrac{dl}{d}$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\left\{\frac{1}{r}\oint d\vec{l} + \frac{1}{r^2}\oint r'\cos\theta\, d\vec{l} + \frac{1}{r^3}\oint (r')^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) d\vec{l} + \cdots\right\}$$

As in multipole expression of *V*,

→The first term, goes like *1/r,* is monopole term.

→The second term, which goes like $1/r^2$, is dipole term.

→The third term, goes like $1/r^3$, is quadrupole term.

The magnetic monopole term is always zero as the total vector displacement around close loop is

$$\oint d\vec{l} = 0$$

Hence no magnetic monopole exists in nature. In the absence of the monopole term, the dominant term is the dipole-except in the rare case where it, too vanishes.

$$A_{dipole}(\bar{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r' \cos\theta \, dl = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r} \bullet r') dl$$

The integral in the above expression after some manipulations can be written as

$$\oint (\hat{r} \bullet r') d\bar{l} = -\frac{1}{2}\hat{r} \times \oint (r' \times d\bar{l})$$

$$\vec{A}_{dipole}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\left[-\frac{1}{2}\hat{r} \times \oint (\vec{r}' \times d\vec{l})\right]$$

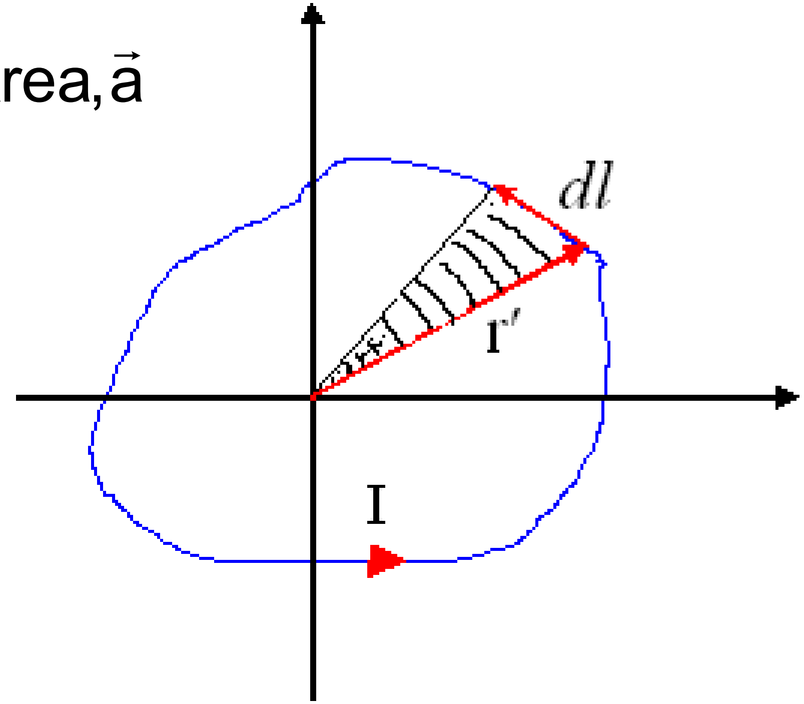
$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

Where *m* is the magnetic dipole moment of the loop, defined as $\vec{m} = \frac{1}{2} I \oint \left(\vec{r}' \times d\vec{l}\right)$

If the current loop is a plane loop (current located on the surface of a plane, then

$\dfrac{\left(\vec{r}' \times d\vec{l}\right)}{2}$ Is the area of the shaded triangle as shown in figure. So the integral is the area of whole loop

$$\oint_{line} \frac{1}{2}\left(\vec{r}' \times d\vec{l}\right) = \text{Area}, \vec{a}$$



In this case, the dipole moment of the current loop is equal to $\vec{m} = I\,\vec{a}$

Where the direction of *a* must be consistent with the direction of the current loop (right hand rule)

→ Since the magnetic monopole term is always zero, the magnetic dipole moment is always independent of origin.

Assuming that the magnetic dipole is located at the origin of our coordinate system and that ***m*** is pointing along the positive z axis,

$$\vec{A}_{dipole} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{e}_\varphi$$

Corresponding magnetic field is equal to

$$\overline{B}_{dipole} = \overline{\nabla}\times\overline{A}_{dipole} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\right)\hat{\theta}$$

$$\vec{B}_{dipole} = \frac{\mu_0}{4\pi}\frac{m}{r^3}\left\{2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta\right\}$$

$$\Rightarrow \quad \boxed{\vec{B} = \frac{\mu_0}{4\pi}\frac{1}{r^3}\left[3\left(\vec{m}\bullet\hat{e}_r\right)\hat{e}_r - \vec{m}\right]}$$

Problem 5.38: A phonograph record of radius R, carrying a uniform surface charge σ, is rotating with a constant angular velocity ω. Find the magnetic dipole moment.

Problem 5.39: Find the magnetic dipole moment of a spinning spherical shell of radius R, carries a uniform surface charge σ. Show that for $r>R$ the potential is that of prefect dipole.

Problem 5.41: Show that magnetic dipole moment of an arbitrary localized current loop is independent of the location of reference point.

Magnetic dipole and magnetic dipole moment:

A magnetic dipole consists of pair of magnetic dipole of equal and opposite strength separated by small distance.

Examples of magnet dipoles are

- Magnetic needle,
- Bar magnet,
- Current carrying solenoid,
- A current loop etc.

Atom is also considered to behave like a dipole - so the fundamental magnetic dipole in nature is associated with the electrons.

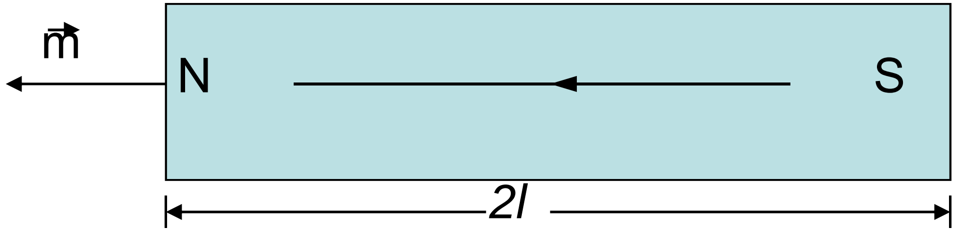
The product of pole strength of either poles and distance between them is called as magnetic dipole moment.

The distance between two poles is called magnetic length.

$$m = M \times 2l$$

or

$$\vec{m} = M \times (2\vec{l}\,)$$



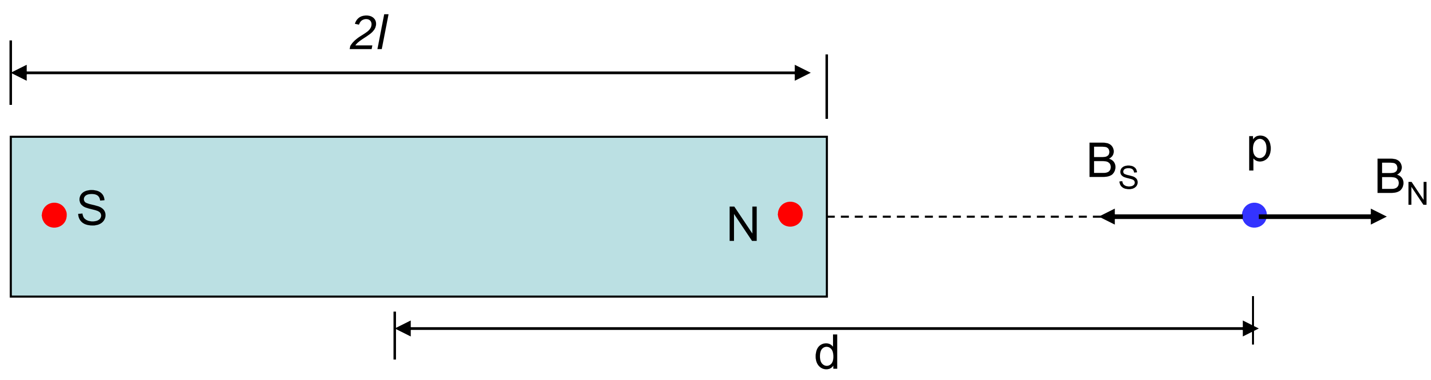
The vector *2l* is directed from south to north. Thus the direction of magnetic dipole moment is from south to north

S.I. units of *M* is *ampere-meter* (*Am*) and that of

m is *ampere-meter*^{2}(Am^2) or *joule tesla*$^{-1}$

Field at a point due to magnetic dipole in the end on-position (on the axis)

Let NS be a magnetic dipole of pole strength *M* and length *2l.* Let P be the point on its axis at a distance *d* from the center of the dipole.



The magnetic field at point P due to north pole is

$$\frac{\mu_0}{4\pi}\frac{M}{(d-l)^2}$$

And will be directed away from the magnet.

The magnetic field at point P due to South pole pole is

$$\frac{\mu_0}{4\pi}\frac{M}{(d+l)^2}$$

And will be directed towards from the magnet.

Therefore resultant field is

$B = B_N - B_S$

$$B = \frac{\mu_0}{4\pi}\frac{4Mld}{\left(d^2 - l^2\right)^2}$$

$$= \frac{\mu_0}{4\pi}\frac{2md}{\left(d^2 - l^2\right)^2}$$

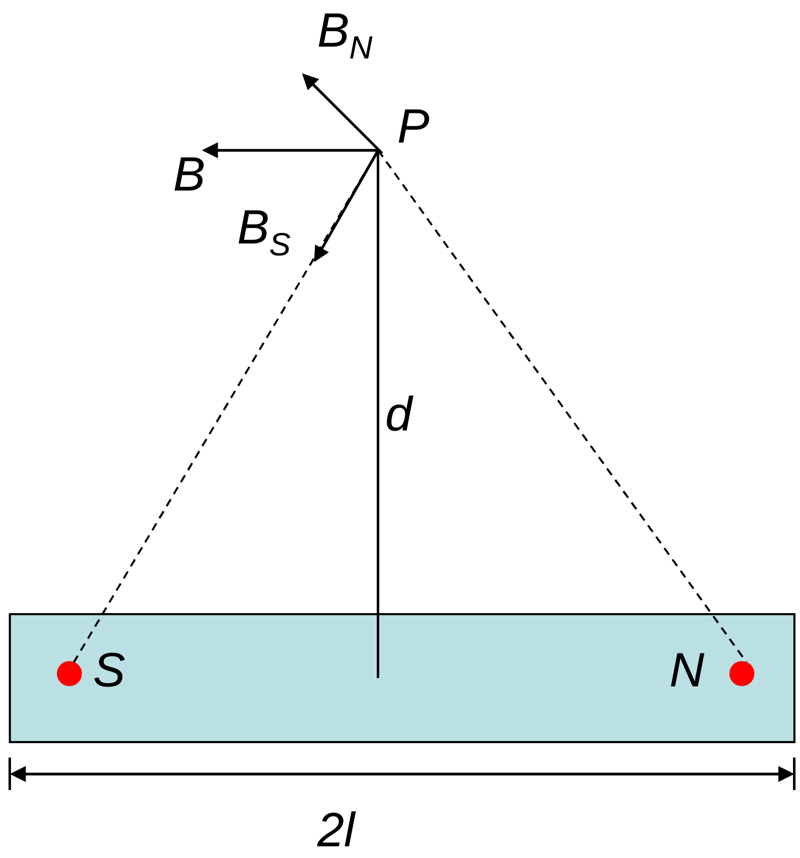
Here $m = 2Ml$ is dipole moment of the magnet

If $d >> l$

$$B = \frac{\mu_0}{4\pi}\frac{2m}{d^3}$$

Field due to a magnetic dipole in the broad side on position

Let NS be a magnetic dipole of pole strength *M* and length *2l. Let P* be a point on the broad side on position of the dipole at the distance *d* from its center.



The magnetic field at point *P* due to north pole is

$$\vec{B}_N = \frac{\mu_0}{4\pi}\frac{M\ \overline{PN}}{PN^3}$$

The magnetic field at point *P* due to north pole is

$$\vec{B}_S = \frac{\mu_0}{4\pi}\frac{M\ \overline{PS}}{PS^3}$$

Now $NP = PS = (d^2+l^2)^{1/2}$

Therefore resultant field at P

$$= \frac{\mu_0}{4\pi}\frac{M}{\left(d^2+l^2\right)^{3/2}}(\overrightarrow{NP}+\overrightarrow{PS})$$

$$= \frac{\mu_0}{4\pi}\frac{M}{\left(d^2+l^2\right)^{3/2}}\overrightarrow{NS}$$

$$= \frac{\mu_0}{4\pi}\frac{M}{\left(d^2+l^2\right)^{3/2}}2l$$

$$= \frac{\mu_0}{4\pi}\frac{m}{\left(d^2+l^2\right)^{3/2}}$$

If $d >> l$

$$B = \frac{\mu_0}{4\pi}\frac{m}{d^3}$$

Thus

$$B_{axial} = 2B_{equatorial}$$

The same conclusion we had drawn for electric field

When a permanent magnet is placed in a field, North pole will experience a force in the direction of field and south pole has a force opposite to the pole.

If the field is uniform the net force is zero, but there is a torque.

For electric dipole, the torque is given by relation

$$\vec{N} = \vec{p} \times \vec{E}$$

$\vec{p}$: electric dipole moment and

$\vec{E}$: uniform electric field

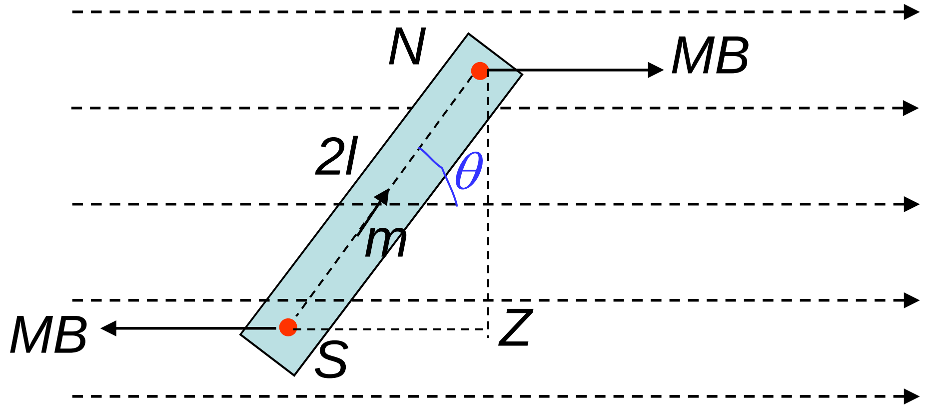
The corresponding expression for torque of magnetic dipole in magnetic field is

$$\vec{N} = \vec{m} \times \vec{B}$$

In addition to the permanent magnets being dipole, we see that current loops are also magnetic dipole.

Torque on a dipole (bar magnet) in a magnetic field:

If a magnetic dipole is placed in a uniform magnetic field *B* as shown in figure, the North and South poles of the magnet will experience equal and opposite forces.



Let *M* be the pole strength of each pole and θ be the angle between magnetic dipole moment *m* and magnetic field *B*, then

Force on North pole = *MB* along *B*

Force on South pole = *MB* opposite to *B*

These forces will constitute a couple which tends to rotate the magnet in the direction of *B*. Thus a magnet experience a torque

N = Force $\times\perp$ distance between the force

= $MB \times ZN$

= $MB \times (SN \sin\theta)$ = $MB \times (2l \sin\theta)$

because In triangle SZN, $\sin\theta = ZN/SN$ or $ZN = SN \sin\theta$

or

$N = (M \times 2l)B \sin\theta$

$= mB\sin\theta$

In vector form

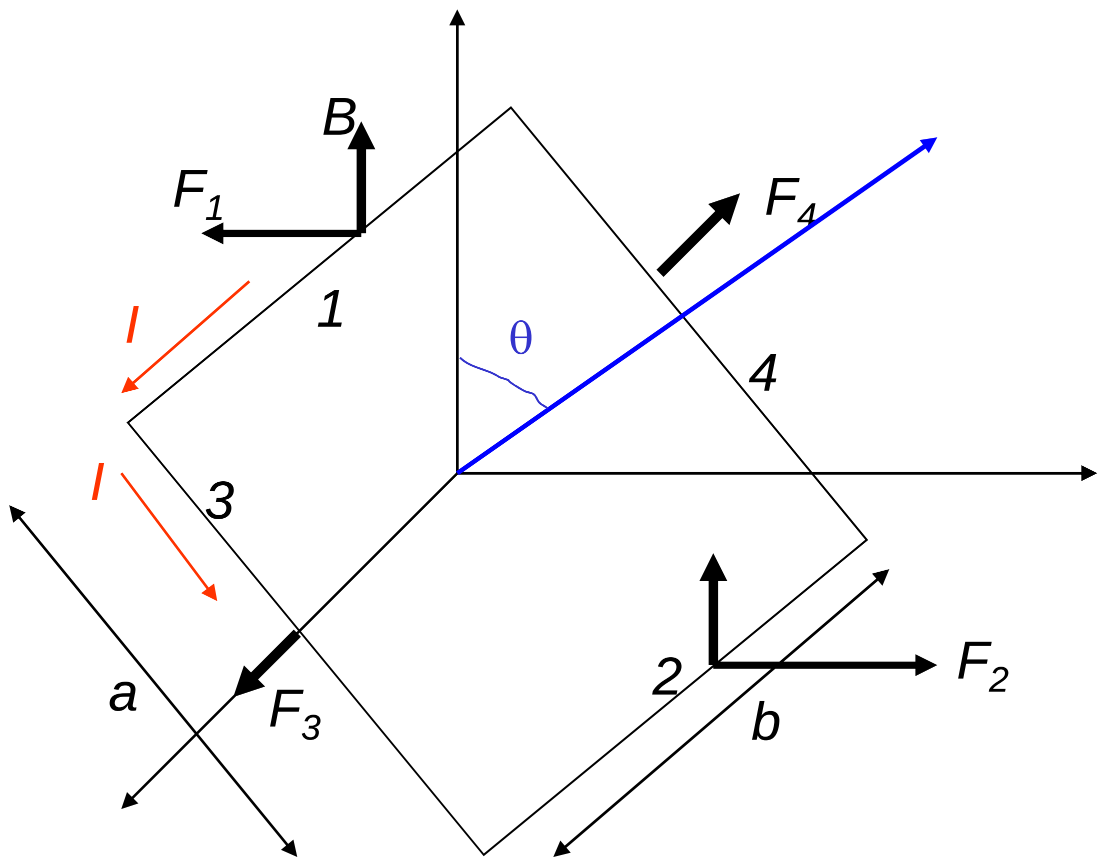
$$\vec{N} = \vec{m} \times \vec{B}$$

When $\vec{B}$ = 1 unit and $\theta = 90°$, then $\vec{N} = \vec{m}$

Thus the magnetic dipole moment can be defined as the torque acting on a magnetic dipole placed normal to the uniform magnetic field of unit strength

Let us calculate the torque on a rectangular current loop in a uniform field $\overline{B}$.

Consider a rectangular loop of sides a and b as shown in figure placed in a uniform magnetic field and let the direction of the field is along z-axis. The magnetic dipole moment is perpendicular to the current loop and makes an angle θ with magnetic field.



Since the currents are opposite on opposite sides of the loop, the forces are also opposite, so there is no net force on the loop (when the field is uniform). The forces on the loop sides 3 and 4 tend to stretch the loop, but do not rotate the loop. Because of the forces on the two sides marked 1 and 2 , tend to rotate the loop about y-axis and generates the torque.

The magnitude of forces F_1 and F_2 is

$$F_1 = F_2 = IBb$$

And their moment or lever arm is

$$a \sin\theta$$

So the torque N is equal to

$$N = IabB\sin\theta$$
$$= mB\sin\theta$$

Where $m = Iab$ is the magnetic dipole moment of the loop

Or

$$\overline{N} = \vec{m} \times \overline{B}$$

The torque given by above equation is a special case, the result is right for small current loop of any shape in the uniform magnetic field.

Work Done on a Magnetic Dipole.

Since a magnetic dipole placed in an external magnetic field experiences a torque, work (positive or negative) must be done by an external agent in order to change the orientation of the dipole.

Let us calculate how much work is done by the field when rotating the dipole from angle θ_A to θ_B.

$$W = \int_A^B N d\theta$$

$$W = mB\int_{\theta_A}^{\theta_B} \sin\theta d\theta$$

$$= -mB\left[\cos\theta_B - \cos\theta_A\right]$$

If the dipole is initially at right angle to the field i.e. $\theta_A = 90°$ and finally makes an angle θ with field i.e. $\theta_B = \theta$, then

$$W = -mB\left[\cos\theta - \cos 90\right]$$

$$= -mB\cos\theta$$

This work done is equal to the potential of the dipole

$$U_p = -mB\cos\theta$$

$$= -\vec{m} \bullet \vec{B}$$

In case of non-uniform field the above discussion is exact only for a perfect dipole of infinitesimal size. Now we will calculate force of a infinitesimal loop of dipole moment *m* in the field *B.*

We have seen that the potential energy of a magnetic dipole *m* in a magnetic field is

$$U_p = -mB\cos\theta$$

$$= -\vec{m} \bullet \vec{B}$$

We know force is related to potential energy by the relation

$$\vec{F} = -\nabla U$$

Therefore

$$\vec{F} = \vec{\nabla}\left(\vec{m} \bullet \vec{B}\right)$$

Using product rule (rule 4)

$$\overline{\nabla}\left(\overline{A}\bullet\overline{B}\right)=\overline{A}\times\left(\overline{\nabla}\times\overline{B}\right)+\overline{B}\times\left(\overline{\nabla}\times\overline{A}\right)+\left(\overline{A}\bullet\overline{\nabla}\right)\overline{B}+\left(\overline{B}\bullet\overline{\nabla}\right)\overline{A}$$

Therefore

$$\vec{F}=\overline{\nabla}\left(\overline{m}\bullet\overline{B}\right)=\overline{m}\times\left(\overline{\nabla}\times\overline{B}\right)+\overline{B}\times\left(\overline{\nabla}\times\overline{m}\right)+\left(\overline{m}\bullet\overline{\nabla}\right)\overline{B}+\left(\overline{B}\bullet\overline{\nabla}\right)\overline{m}$$

Since $\overline{m}$ is not function of space co-ordinate

Therefore

$$\left(\overline{B}\bullet\overline{\nabla}\right)\overline{m}=0 \qquad \text{and} \qquad \overline{\nabla}\times\overline{m}=0$$

And

$$\overline{\nabla}\times\overline{B}=0$$

Therefore

$$\overline{F} = (\overline{m} \bullet \overline{\nabla})\overline{B}$$

Provided there is no external current at the actual location of the dipole

We must be very careful about the analogies between electric and magnetic dipoles. For example force on a magnetic dipole in non uniform field is

$$\vec{F} = \vec{\nabla}(\vec{m} \bullet \vec{B})$$

Where as for electric field

$$\overline{F} = (\overline{p} \bullet \nabla)\overline{E}$$

So one should be very alert when solving the problems.