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Magnetic Z_N symmetry in hot QCD and the spatial Wilson loop.

C. Korthals-Altes[1] and A. Kovner[2]

[1]Centre Physique Theorique au CNRS, Case 907, Luminy
F13288, Marseille Cedex, France
[2]Theoretical Physics, University of Oxford, 1 Keble Road,
Oxford, OX1 3NP, UK

ABSTRACT

We discuss the relation between the deconfining phase transition in gauge theories and the realization of the magnetic Z_N symmetry. At low temperature the Z_N symmetry is spontaneously broken while above the phase transition it is restored. This is intimately related to the change of behaviour of the spatial 't Hooft loop discussed in [1]. We also point out that the realization of the magnetic symmetry has bearing on the behaviour of the spatial Wilson loop. We give a physical argument to the effect that the spatial Wilson loop must undergo a change of behaviour at the deconfining phase transition and must have a perimeter law behaviour in the hot phase.

1 Introduction

This paper is devoted to further study of theoretical aspects of the deconfining temperature phase transition in nonabelian gauge theories. It is an immediate continuation of our earlier work [1]. In [1] we showed that the deconfining phase transition in the pure Yang Mills theory is characterised by the change of behaviour of the 't Hooft loop operator $V(C)$. In the "cold" phase the 't Hooft loop has a perimiter law behaviour $< V(C) > \propto \exp\{-aP(C)\}$, while in the "hot" phase it has an area law behaviour $< V(C) > \propto \exp\{-\alpha S(C)\}$.

In the present paper we want to sharpen somewhat this observation and further discuss related questions. We wish to point out that V is in fact an order parameter which probes the breaking of a physical symmetry of the Yang Mills theory. The symmetry in question is the magnetic Z_N symmetry discussed by 't Hooft [2]. The deconfining phase transition is therefore characterized by the change in the mode of realization of a global Z_N symmetry: the symmetry is broken spontaneously in the "cold" phase while it is restored in the "hot“ phase.

The previous two paragraphs may sound at first like a red herring. After all an order parameter for the deconfining phase transition as well as a related Z_N symmetry have been discussed for many years. The order parameter in question is the free energy of an external static colour source in the fundamental representation: the Polyakov line $P = \mathrm{Tr} P \exp\{ig \int_0^\beta dt A_0\}$. The Z_N symmetry is the transformation $P \to \exp\{i\frac{2\pi}{N}\}P$. We will refer to this transformation as the electric Z_N. There is however a great difference between the physical nature of P and V and the associated Z_N symmetries. The operator V is a canonical operator in the physical Hilbert space of the Yang Mills theory. The magnetic Z_N symmetry similarly is a transformation that acts on quantum states in the physical Hilbert space. On the other hand P has a very different status. It is not an operator in the Hilbert space and as such not a canonical order parameter. It appears as an auxilliary object when projecting onto gauge invariant physical subspace of the Hilbert space. The "electric" Z_N - the operation that transforms P by multiplying it by a phase - similarly is not a canonical symmetry. There is no transformation of states in the physical Hilbert space that is related to this "symmetry", although it is indeed a symmetry of the Euclidean path integral representing the statistical sum.

This is not to say of course that P and electric Z_N are useless concepts. The standard effective action, defined by the constrained path integral

$$\exp -S_{eff}(P) = \int DA_0 \delta(P - P(A_0)) \exp -S(A) \tag{1}$$

is gauge invariant. It is instrumental in computing the vortex expectation value. The way the electric $Z(N)$ symmetry is realized in S_{eff} is also related to the behaviour of the order parameter of the magnetic $Z(N)$. We will discuss this in detail in section 3 [1].

However if one wants to describe the deconfinement phase transition in terms of a canonical order parameter in the same way as the Ising transition is described in terms of magnetisation, one should zero in on V rather than on P and should study the magnetic Z_N symmetry rather than electric Z_N. This is what we intend to do in this paper.

The action of the magnetic Z_N symmetry is very different in 2+1 and 3+1 dimensional cases. In 2+1 dimensions it acts very much like usual global symmetry in a scalar theory with the order parameter being a scalar vortex field. In 3+1 dimensions the symmetry acts not like a standard global symetry - its "charge" is an integral over a two dimensional spacelike surface rather than over the whole of the three dimensional space[1]. As a consequence its order parameter is not a local field but rather a magnetic vortex stretching over macroscopic distances.

It is therefore convenient to start the discussion with the three dimensional gauge theories and to present all the arguments in this case. The generalization of appropriate aspects of this discussion to 3+1 dimensions will be given in the last part of every section.

The plan of this paper is the following. In Section 2 we recap the definition of the 't Hooft loop and its 2+1 dimensional analog - the magnetic vortex operator. We formulate the arguments for the existence of the magnetic Z_N symmetry in theories without fundamental matter fields. We also show by explicit construction that the generator of this symmetry in the pure gluodynamics is none other than the spatial Wilson loop.

In Section 3 we discuss the relation between the behaviour of the 't Hooft loop and the realization of the magnetic Z_N in the ground state of the theory. We demonstrate that the mode of the realization of the symmetry changes at the deconfining phase transition, while spontaneously broken at low temperature the symmetry is restored above the phase transition.

In Section 4 we present a simple physical picture explaining why the Wilson loop must have an area law in the low temperature phase where the Z_N symmetry is broken and perimeter law in the high temperature unbroken phase. We also point out that there is a difference between the Abelian and nonabelian theories in this respect.

Finally in Section 5 we conclude with a discussion. We are well aware that our conclusion about the perimeter behaviour of the Wilson loop at high temperature is controversial since it contradicts the existing lore. If the simple reduced theory calculations are to be believed at asymptotically high temperatures, the spatial Wilson loop must have area law behaviour. We point out that the argument based on the reduced theory is not water tight and suggest a simple lattice calculation which could verify/falsify our conclusion.

[1] These type of symmetries nowadays are frequently discussed in the context of "M - theory"[3].

2 The magnetic Z_N symmetry and the 't Hooft loop operator.

In this section we discuss the notion of the magnetic Z_N symmetry and its order parameter - 't Hooft loop, or magnetic vortex operator. Most of the material contained here is not new and, perhaps with the exception of explicit identification of the Z_N generator with the spatial Wilson loop, is contained in [2, 4, 5]. At the risk of being repetitive we have decided nevertheless to include this extended introductory part, since we feel that the concept of magnetic Z_N symmetry is not widely appreciated in the community. The Z_N symmetry structure is the basis for our discussion of the deconfining phase transition in the following sections.

Let us start by recalling the argument due to 't Hooft that a nonabelian $SU(N)$ gauge theory with charged fields in adjoint representation posesses a global Z_N symmetry [2].

We discuss the 2+1 dimensional case first. Consider a theory with several adjoint Higgs fields so that varying parameters in the Higgs sector the $SU(N)$ gauge symmetry can be broken completely. In this phase the perturbative spectrum will contain the usual massive "gluons" and Higgs particles. However in addition to that there will be heavy stable magnetic vortices. Those are the analogs of Abrikosov-Nielsen-Olesen vortices in the superconductours and they must be stable by virtue of the following topological argument. The vortex configuration away from the vortex core has all the fields in the pure gauge configuration

$$H^\alpha(x) = U(x)h^\alpha, \qquad A^\mu = iU\partial^\mu U^\dagger \tag{2}$$

Here the index α labels the scalar fields in the theory, h^α are the constant vacuum expectation values of these fields, and $U(x)$ is a unitary matrix. As one goes around the location of the vortex in space, the matrix U winds nontrivially in the gauge group. This is possible, since the gauge group in the theory without fundamental fields is $SU(N)/Z_N$ and it has a nonvanishing first homotopy group $\Pi_1(SU(N)/Z_N) = Z_N$. Practically it means that when going around the vortex location full circle, U does not return to the same $SU(N)$ group element U_0, but rather ends up at $\exp\{i\frac{2\pi}{N}\}U_0$. Adjoint fields do not feel this type of discontinuity in U and therefore the energy of such a configuration is finite. Since such a configuration can not be smoothly deformed into a trivial one, a single vortex is stable. Processes involving annihilation of N such vortices into vacuum are allowed since N-vortex configurations are topologically trivial. One can of course find explicit vortex solutions once the Higgs potential is specified. As any other semiclassical solution in the weak coupling limit the energy of such a vortex is inversely proportional to the gauge coupling constant and therefore very large. One is therefore in a situation where the spectrum of the theory contains a stable particle even though its mass is much higher than masses of many other particles (gauge and Higgs bosons) and the phase space for its decay into these particles is enormous. The only possible reason for the existence of such a heavy stable particle is that it must carry a conserved quantum number. The theory therefore must possess a global symmetry which is unbroken in the completely higgsed phase. The symmetry group must be Z_N since the number of vortices is only conserved modulo N.

Now imagine changing smoothly the parameters in the Higgs sector so that the expectation values of the Higgs fields become smaller and smaller, and finally the theory undergoes a phase transition into the confining phase. One can further change the parameters so that the adjoint scalars become heavy and eventually decouple completely from the glue. This limiting process does not change the topology of the gauge group and therefore does not change the symmetry content of the theory. We conclude that the pure Yang-Mills theory also posesses a Z_N symmetry. Of course since the confining phase is separated from the completely Higgsed phase by a phase transition one may expect that the Z_N symmetry in the confining phase is represented differently. In fact the original paper of 't Hooft as well as subsequent work[4] convincingly argued that in the confining phase the Z_N symmetry is spontaneously broken and this breaking is related to the confinement phenomenon.

The physical considerations given above can be put on firmer formal basis. In particular one can construct explicitly the generator of the Z_N as well as the order parameter associated with it - the operator that creates the magnetic vortex [5]. We will now describe this construction.

2.1 The Abelian case

Consider first an Abelian gauge theory. In this case the homotopy group is Z and therefore we expect the $U(1)$ rather than Z_N magnetic symmetry. It is in fact absolutely straightforward to identify the relevant charge. It is none other than the magnetic flux through the equal time plane, with the associated conserved current being the dual of the electromagnetic field strength

$$\Phi = \int d^2x B(x), \qquad \partial^\mu \tilde{F}_\mu = 0 \tag{3}$$

The current conservation is insured by the Bianchi identity. A group element of the $U(1)$ magnetic symmetry group is $\exp\{i\alpha\Phi\}$ for any value of α. A local order parameter - a local field $V(x)$ which carries the magnetic charge - is also readily constructed. It has a form of the singular gauge transformation operator with the singularity at the point x

$$V(x) = \exp\frac{i}{g}\int d^2y \; \left[\epsilon_{ij}\frac{(x-y)_j}{(x-y)^2}E_i(y) + \Theta(x-y)J_0(y)\right] \tag{4}$$

where $\Theta(x-y)$ is the polar angle function and J_0 is the electric charge density of whatever matter fields are present in the theory. The cut discontinuity in the function Θ is not physical and can be chosen parallel to the horizontal axis. Using the Gauss' law constraint this can be cast in a different form, which we will find more convenient for our discussion

$$V(x) = \exp \frac{2\pi i}{g} \int_C dy^i \ \epsilon_{ij} E_i(y) \tag{5}$$

where the integration goes along the cut of the function Θ which starts at the point x and goes to spatial infinity. The operator does not depend on where precisely one chooses the cut to lie. To see this, note that changing the position of the cut C to C' adds to the phase $\frac{2\pi}{g}\int_S d^2x\partial_i E^i$ where S is the area bounded by $C - C'$. In the theory we consider only charged particles with charges multiples of g are present. Therefore the charge within any closed area is a multiple integer of the gauge coupling $\int_S d^2x\partial_i E^i = gn$ and the extra phase factor is always unity.

The meaning of the operator V is very simple. From the commutation relation

$$V(x)B(y)V^\dagger(x) = B(y) + \frac{2\pi}{g}\delta^2(x-y) \tag{6}$$

it is obvious that V creates a pointlike magnetic vortex of flux $2\pi/g$. Despite its nonlocal appearance the operator V can be proven to be a local Lorentz scalar field[8]. The locality is the consequence of the fact that $V(x)$ commutes with any local gauge invariant operator in the theory $O(y)$ except when $x = y$. This is due to the coefficient $2\pi/g$ in the exponential which ensures that the Aharonov-Bohm phase of the vortex created by V and any dynamical charged particle present in the theory vanishes. Eqs.(3,5) formalize the physical arguments of 't Hooft in the abelian case.

2.2 The non-Abelian case at weak coupling.

Let us now move onto the analogous construction for nonabelian theories. Ultimately we are interested in the pure Yang - Mills theory. It is however illuminating to start with the theory with an adjoint Higgs field and take the decoupling limit explicitly later. For simplicity we discuss the $SU(2)$ gauge theory. Consider the Georgi-Glashow model - $SU(2)$ gauge theory with an adjoint Higgs field.

$$\mathcal{L} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + \frac{1}{2}(\mathcal{D}^{ab}_\mu H^b)^2 + \tilde{\mu}^2 H^2 - \tilde{\lambda}(H^2)^2 \tag{7}$$

where

$$\mathcal{D}^{ab}_\mu H^b = \partial_\mu H^a - gf^{abc}A^b_\mu H^c \tag{8}$$

At large and positive $\tilde{\mu}^2$ the model is weakly coupled. The $SU(2)$ gauge symmetry is broken down to $U(1)$ and the Higgs mechanism takes place. Two gauge bosons, $W^{\pm}$, acquire a mass, while the third one, the "photon", remains massless to all orders in perturbation theory. The theory in this region of parameter space resembles very much electrodynamics with vector charged fields. The Abelian construction can therefore be repeated. The $SU(2)$ gauge invariant analog of the conserved dual field strength is

$$\tilde{F}^{\mu}=\frac{1}{2}[\epsilon_{\mu\nu\lambda}F^{a}_{\nu\lambda}n^{a}-\frac{1}{g}\epsilon^{\mu\nu\lambda}\epsilon^{abc}n_{a}(\mathcal{D}_{\nu}n)^{b}(\mathcal{D}_{\lambda}n)^{c}] \tag{9}$$

where $n^{a}\equiv\frac{H^{a}}{|H|}$ is the unit vector in the direction of the Higgs field. Classically this current satisfies the conservation equation

$$\partial^{\mu}\tilde{F}_{\mu}=0 \tag{10}$$

The easiest way to see this is to choose a unitary gauge of the form $H^{a}(x)=H(x)\delta^{a3}$. In this gauge $\tilde{F}$ is equal to the abelian part of the dual field strength in the third direction in colour space. Its conservation then follows by the Bianchi identity. Thus classically the theory has a conserved $U(1)$ magnetic charge $\Phi=\int d^{2}x\tilde{F}_{0}$ just like QED. However the unitary gauge can not be imposed at the points where H vanishes, which necessarilly happens in the core of an 't Hooft-Polyakov monopole. It is well known of course [7] that the monopoles are the most important nonperturbative configurations in this model. Their presence leads to a nonvanishing small mass for the photon as well as to confinement of the charged gauge bosons with a tiny nonperturbative string tension. As far as the monopole effects on the magnetic flux, their presence leads to a quantum anomaly in the conservation equation (10). As a result only the discrete Z_2 subgroup of the transformation group generated by Φ remains unbroken in the quantum theory. The detailed discussion of this anomaly, the residual Z_2 symmetry and their relation to monopoles is given in [5].

The order parameter for the magnetic Z_2 symmetry is constructed analogously to QED as a singular gauge transformation generated by the gauge invariant electric charge operator

$$J^{\mu}=\epsilon^{\mu\nu\lambda}\partial_{\nu}(\tilde{F}^{a}_{\lambda}n^{a}),\qquad Q=\int d^{2}xJ_{0}(x) \tag{11}$$

Explicitly

$$\begin{aligned} V(x) &= \exp\frac{i}{g}\int d^{2}y\ \left[\epsilon_{ij}\frac{(x-y)_{j}}{(x-y)^{2}}n^{a}(y)E^{a}_{i}(y)+\Theta(x-y)J_{0}(y)\right] \\ &= \exp\frac{2\pi i}{g}\int_{C}dy^{i}\epsilon_{ij}n^{a}E^{a}_{i}(y) \end{aligned} \tag{12}$$

One can think of it as a singular $SU(2)$ gauge transformation with the field dependent gauge function

$$\lambda^{a}(y)=\frac{1}{g}\Theta(x-y)n^{a}(\vec{y}) \tag{13}$$

This field dependence of the gauge function ensures the gauge invariance of the operator V. Just like in QED it can be shown[5], [8] that the operator V is a local scalar field. Again like in QED, the vortex operator V is a local eigenoperator of the abelian magnetic field $B(x) = \tilde{F}_0$.

$$[V(x), B(y)] = -\frac{2\pi}{g} V(x)\delta^2(x-y) \tag{14}$$

That is to say, when acting on a state it creates a pointlike magnetic vortex which carries a quantized unit of magnetic flux. The Z_2 magnetic symmetry transformation is generated by the operator

$$U = \exp\{i\frac{g}{2}\Phi\} \tag{15}$$

and acts on the vortex field V as a phase rotation by π

$$e^{i\frac{g}{2}\Phi} V(x) e^{-i\frac{g}{2}\Phi} = -V(x) \tag{16}$$

An operator closely related to U and which will be of interest to us in the following, is the generator of the magnetic Z_2 transformation only inside some closed contour C

$$U(C) = \exp\{i\frac{g}{2}\int_S d^2x B(x)\} \tag{17}$$

where the integration is over the area S bounded by C. The analog of the commutator eq.(16) for this operator is

$$U_C V(x) U_C^\dagger = \begin{matrix} - & V(x) & , & x \in S \\ & V(x) & , & x \notin S \end{matrix} \tag{18}$$

Taking the contour C to run at infinity U_C becomes the generator of Z_2.

We now have the explicit realization of the magnetic Z_2 symmetry in the Georgi-Glashow model.

2.3 The pure gauge theory.

Our next step is to move on to the pure Yang Mills theory. This is achieved by smoothly varying the $\tilde{\mu}^2$ coefficient in the Lagrangian so that the coeficient of the mass term of the Higgs field becomes positive and eventually arbitrarily large. It is well known that in this model the weakly coupled Higgs regime and strongly coupled confining regime are not separated by a phase transition[6]. The pure Yang Mills limit in this model is therefore smooth.

In the pure Yang Mills limit the expressions eq.(9,12,17) have to be taken with care. When the mass of the Higgs field is very large, the configurations that dominate the path integral are those with very small value of the modulus of the Higgs field $|H| \propto 1/M$. The modulus of the Higgs field in turn controls the fluctuations of the unit vector n^a, since the kinetic term for n in the Lagrangian is $|H|^2(D_\mu n)^2$. Thus as the mass of the Higgs field increases the fluctuations of n grow in both, amplitude and frequency and the magnetic field operator B as defined in eq.(9) fluctuates wildly. This situation is of course not unusual. It happens whenever one wants to consider in the effective low energy theory an operator which explicitly depends on fast, high energy variables. The standard way to deal with it is to integrate over the fast variables. There could be two possible outcomes of this procedure. Either the operator in question becomes trivial (if it depends strongly on the fast variables), or its reduced version is well defined and regular on the low energy Hilbert space. The "magnetic field" operator B in eq.(9) is obviously of the first type. Since in the pure Yang Mills limit all the orientation of n^a are equally probable, integrating over the Higgs field at fixed A_μ will lead to vanishing of B. However what interests us is not so much the magnetic field but rather the generator of the magnetic Z_2 transformation U_C of eq.(17). In the pure Yang-Mills limit we are thus lead to consider the operator

$$U_C = lim_{H\to 0}\int Dn^a \quad \exp\Big\{-|H|^2(\vec{D}n_a)^2\Big\} \\ \exp\Big\{i\frac{g}{4}\int_C d^2x\big(\epsilon_{ij}F^a_{ij}n^a - \frac{1}{g}\epsilon^{ij}\epsilon^{abc}n_a(\mathcal{D}_i n)^b(\mathcal{D}_j n)^c\big)\Big\} \tag{19}$$

The weight for the integration over n is the kinetic term for the isovector n_a. As was noted before the action does not depend on n^a in the YM limit. This term however regulates the integral and we keep it for this reason. This operator may look somewhat unfamiliar at first sight. However in a remarkable paper [9] Diakonov and Petrov showed that eq.(20) is equal to the trace of the fundamental Wilson loop along the contour C[2].

$$U_C = W_C \equiv \mathrm{Tr}\mathcal{P}\exp\Big\{\mathrm{ig}\int_{\mathrm{C}}\mathrm{dl^i A^i}\Big\} \tag{20}$$

We conclude, that in the pure Yang-Mills theory the generator of the magnetic Z_2 symmetry is the fundamental spatial Wilson loop along the boundary of the spatial plane.

There is a slight subtlety here that may be worth mentioning. The generator of a unitary transformation should be a unitary operator. The trace of the fundamental Wilson loop on the other hand is not unitary. One should therefore strictly speaking consider instead a unitarized Wilson loop $\tilde{W} = \frac{W}{\sqrt{WW^\dagger}}$. However the factor between the two operators $\sqrt{WW^\dagger}$ is an operator that is only sensitive to behaviour of the fields at infinity. It commutes with all physical

[2]We note that Dyakonov and Petrov had to introduce a regulator to define the path integral over n. The regulator they required was precisely of the same form as in eq.(20). It is pleasing to see that this regulator appears naturally in our approach as the remnant of the kinetic term of the Higgs field.

local operators $O(x)$ unless $x \to \infty$. In this it is very different from the Wilson loop itself, which has a nontrivial commutator with vortex operators $V(x)$ at all values of x. Since the correlators of all gauge invariant local fields in the pure Yang Mills theory are massive and therefore short range, the operator $\sqrt{WW^\dagger}$ must be a constant operator on all finite energy states. The difference between W and $\tilde{W}$ is therefore a trivial constant factor and we will not bother with it in the following. Perhaps of more concern is the difference between W_C and $\tilde{W}_C$ when the contour C is not at infinity. However here again the factor between the two operators $\sqrt{W_C W_C^\dagger}$ is only sensitive to physical degrees of freedom on the contour C and not inside it. Due to its presence the vacuum averages of W_C and $\tilde{W}_C$ may differ at most by a factor which has a perimeter behaviour $< W_C >= \exp\{mP(C)\} < \tilde{W}_C >$ where $P(C)$ is a perimeter of C. The question we will be interested in is whether $< W_C >$ has a perimeter or area behaviour. As far as the answer to this question is concerned W_C and $\tilde{W}_C$ are completely equivalent, and we will not make distinction between them. In the rest of this paper we will therefore refer to W as the generator of Z_2 keeping this little caveat in mind.

Next we consider the vortex operator eq.(12). Again we have to integrate it over the orientations of the unit vector n^a. This integration in fact is equivalent to averaging over the gauge group. Following [9] one can write n_a in terms of the SU(2) gauge transformation matrix Ω.

$$\vec{n} = \frac{1}{2}\mathrm{Tr}\Omega\tau\Omega^\dagger\tau_3 \tag{21}$$

The vortex operator in the pure gluodynamics limit then becomes

$$\tilde{V}(x) = \int D\Omega \exp\frac{2\pi i}{g}\int_C dy_i \epsilon_{ij}\mathrm{Tr}\Omega E_j \Omega^\dagger \tau_3 \tag{22}$$

This form makes it explicit that $\tilde{V}(x)$ is defined as the gauge singlet part of the following, apparently non gauge invariant operator

$$V(x) = \exp\frac{2\pi i}{g}\int_C dy^i \epsilon_{ij} E^3_i(y) \tag{23}$$

The integration over Ω obviously projects out the gauge singlet part of V. In the present case however this projection is redundant. This is because even though V itself is not gauge invariant, when acting on a physical state it transforms it into another physical state[3]. By physical states we mean the states which satisfy the Gauss' constraint in the pure Yang-Mills theory. This property of V was noticed by 't Hooft [2]. To show this let us consider $V(x)$ as defined in eq.(23) and its gauge transform $V_\Omega = \Omega^\dagger V \Omega$ where Ω is an arbitrary nonsingular gauge transformation operator. The wave functional of any physical state depends

[3]This is not a trivial statement, since a generic nongauge invariant operator has nonvanishing matrix elements between the physical and an unphysical sectors.

only on gauge invariant characteristics of the vector potential, i.e. only on the values of Wilson loops over all possible contours.

$$\Psi[A_i] = \Psi[\{W(C)\}] \tag{24}$$

Acting on this state by the operators V and V_Ω respectively we obtain

$$\begin{aligned} V|\Psi> &= \Psi_V[A_i] = \Psi[\{VW(C)V^\dagger\}] \\ V_\Omega|\Psi> &= \Psi_V^\Omega[A_i] = \Psi[\{V_\Omega W(C)V_\Omega^\dagger\}] \end{aligned} \tag{25}$$

It is however easy to see that the action of $V(x)$ and $V_\Omega(x)$ on the Wilson loop is identical - they both multiply it by the centergroup phase (which stays unaffected by Ω) if x is inside C and do nothing otherwise. Therefore we see that

$$V|\Psi>= V_\Omega|\Psi> \tag{26}$$

for any physical state Ψ. Thus we have

$$\Omega V|\Psi>= \Omega V\Omega^\dagger|\Psi>= V|\Psi> \tag{27}$$

where the first equality follows from the fact that a physical state is invariant under action of any gauge transformation Ω and the second equality follows from eq.(26). But this equation is nothing but the statement that the state $V|\Psi>$ is physical, i.e. invariant under any nonsingular gauge transformation.

We have therefore proved that when acting on a physical state the vortex operator creates another physical state. For an operator of this type the gauge invariant projection only affects its matrix elements between unphysical states. Since we are only interested in calculating correlators of V between physical states, the gauge projection is redundant and we can freely use V rather than $\tilde{V}$ to represent the vortex operator.

It is instructive to note that this propery is not shared by the Wilson loop. One can in fact represent the Wilson loop as a singlet gauge projection of a simple Abelian loop operator. The second exponential in eq.(20) can be written as

$$\exp\left\{i\frac{g}{2}\int_C dl^i A^i_a n^a - \frac{i}{2}\int d^2x\epsilon_{ij}\epsilon^{abc}n_a\partial_i n_b\partial_j n_c\right\} \tag{28}$$

Using eq.(21) we can rewrite the integral in eq.(20) -omitting the regulating kinetic piece- as:

$$W_C = \int D\Omega \exp\left\{i\frac{g}{2}\int \mathrm{Tr}\tau_3\big(\Omega A^i\Omega^\dagger + i\Omega\partial^i\Omega^\dagger\big)dl^i\right\} \tag{29}$$

The Wilson loop is therefore the gauge singlet part of the Abelian loop

$$U_C^A = \exp i\frac{g}{2}\int \mathrm{Tr}A^i\tau_3 dl^i \tag{30}$$

The matrix elements of W_C and U^A_C on physical subspace therefore are the same. However U^A_C as opposed to V does have nonvanishing nondiagonal matrix elements, that is matrix elements between the physical and the unphysical sectors. It therefore can *not* be used instead of W_C in gauge theory calculations. For example non gauge invariant states will contribute as intermediate states in the calculation of quantities like the correlation function $< U^A_{C1} U^A_{C_2} >$, while their contribution vanishes in similar correlators which involve the Wilson loop.

The generalization of the preceding discussion to $SU(N)$ gauge theories is straightforward. Once again one can start with the Georgi-Glashow like model, where the $SU(N)$ is higgsed to $U(1)^{(N-1)}$[4]. The construction of the vortex operator and the generator of Z_N in this case is very similar and the details are given in [5]. Taking the mass of the Higgs field to infinity again projects the generator onto the trace of the fundamental Wilson loop. The vortex operator can be taken as

$$V(x) = \exp\{\frac{4\pi i}{gN}\int_C dy^i \epsilon_{ij} \mathrm{Tr(YE_i(y))} \tag{31}$$

where the hypercharge generator Y is defined as

$$Y = \mathrm{diag}\,(1, 1, ..., -(N-1)) \tag{32}$$

and the electric field is taken in the matrix notation $E_i = \lambda^a E^a_i$ with λ^a - the $SU(N)$ generator matrices in the fundamental representation.

2.4 Generalization to 3+1 dimensions

To conclude this section we discuss how the magnetic symmetry structure generalizes to four dimensions. The conserved Z_N generator in the Georgi-Glashow model is defined through

$$U_S = \exp\left\{i\frac{g}{2}\int_S d^2S^i \big(B^a_i n^a - \frac{1}{g}\epsilon^{ijk}\epsilon^{abc} n_a(\mathcal{D}_j n)^b(\mathcal{D}_k n)^c\big)\right\} \tag{33}$$

Although the definition of U contains explicitly the surface S through which the abelian magnetic flux is integrated, the operator in fact does not depend on S but is specified completely by its boundary. This is because changing S changes the phase of U by the magnetic flux through the closed surface. The only dynamical objects that carry magnetic flux in the theory are 't Hooft-Polyakov monopoles. Since their flux is quantized in units of $4\pi/g$ the change in the phase is always a multiple integer of 2π. In the pure Yang-Mills limit the operator U_S again reduces to the trace of the fundamental Wilson loop

[4]In $SU(N)$ theories with $N > 2$ there in principle can be phases separated from each other due to spontaneous breaking of some global symmetries. For instance the $SU(3)$ gauge theory with adjoint matter has a phase with spontaneously broken charge conjugation invariance [17]. Still even in this phase the confining properties are the same as in the strongly coupled pure Yang-Mills theory, with the Wilson loop having an area law.

along the boundary of S. Taking the contour to infinity defines the generator of magnetic Z_N. As we have already noted, this charge is a little unusual in that it is defined as a surface integral rather than a volume integral. As a result the order parameter for this symmetry transformation is not a local but rather a stringy field. This is of course just a restatement of the fact that magnetic vortices in 3+1 dimensions are stringlike objects. The operator that creates a vortex can still be defined in a way similar to 2+1 dimensions. Skipping the intermediate steps which we went through in the previous discussion we give the final result for the pure Yang Mills $SU(N)$ gauge theory. The magnetic vortex along the curve C is created by the following operator of the "singular gauge transformation"[5]

$$V(C) = \exp\{\frac{i}{gN}\int d^3x \mathrm{Tr}(\mathrm{D^i}\omega_\mathrm{C}\mathrm{Y})\mathrm{E^i}\} = \exp\{\frac{4\pi \mathrm{i}}{\mathrm{gN}}\int_\mathrm{S} \mathrm{d^2S^i Tr(YE^i)}\} \tag{34}$$

with $\omega_C(x)$, the singular gauge function which is equal to the solid angle subtended by C as seen from the point x. The function ω is continuos everywhere, except on a surface S bounded by C, where it jumps by 4π. Other than the fact that S is bounded by C, its location is arbitrary. The vortex loop and the spatial Wilson loop satisfy the 't Hooft algebra

$$V^\dagger(C)W(C')V(C) = e^{\frac{2\pi i}{N}n(C,C')}W(C') \tag{35}$$

where $n(C,C')$ is the linking number of the curves C and C'. One can consider closed contours C or infinite contours that run through the whole system. For an infinite contour C and the Wilson loop along the spatial boundary of the system the linking number is always unity. The $V(C)$ for an infinite loop is therefore an eigenoperator of the Z_N magnetic symmetry and is the analog of the vortex operator $V(x)$ in 2+1 dimensions. Any closed vortex loop of fixed size commutes with the Wilson loop if the contour C' is very large. Such a closed loop is thus an analog of the vortex-antivortex correlator $V(x)V^\dagger(y)$, which also commutes with the global symmetry generator, but has a nontrivial commutator with U_C if C encloses only one of the points x or y.

To summarize this section, we have shown that pure Yang Mills theory in 2+1 and 3+1 dimensions has a global Z_N magnetic symmetry. The generator of the symmetry group in both cases is the trace of the fundamental Wilson loop along the spatial boundary of the system. The order parameter for this symmetry in 2+1 dimensions is a local scalar field $V(x)$, while in 3+1 dimensions a stringlike field $V(C)$. In both cases the field V is gauge invariant on physical states and is a *bona fide* canonical order parameter which distinguish in gauge invariant way the phases of the theory. In the next section we discuss the realization of the magnetic symmetry in the confined and the deconfined phases.

[5]The derivative term $\partial^i\omega$ in this expression should be understood to contain only the smooth part of the derivative and to exclude the contribution due to the discontinuity of ω on the surface S.

3 Hot and cold realizations of the magnetic Z_N.

As with any global symmetry, it is important to understand what is the mode of realization of magnetic Z_N in the ground state of the theory. This mode of realization depends of course on the parameters of the theory as well as on the temperature. The situation at zero temperature is well understood.

3.1 2+1 dimensions.

Again we start with three dimensions. There is a very general argument due to 'tHooft[2][6] stating that if the theory does not have zero mass excitations the area law of the Wilson loop implies the nonvanishing expectation value of the vortex operator $V(x)$. Conversely if the Wilson loop has a perimeter law the expectation value of $V(x)$ must vanish and the correlation function $V(x)V^\dagger(y)$ must have an exponential falloff with $|x-y|$. It follows that in the pure Yang Mills theory the vacuum expectation value of the vortex operator does not vanish and therefore the Z_N magnetic symmetry is spontaneously broken. The same is true in the partially broken Higgs phase of the Georgi-Glashow model. As mentioned in the last section the confining and the Higgs regimes in this model are analytically connected and therefore the realization of all global symmetries in the two regimes is the same.

In fact in the weakly coupled Higgs phase this can be verified by the direct calculation of the expectation value of V [5]. This calculation maps very simply into the classic monopole plasma calculation of Polyakov and was discussed in detail in [5]. One can also explicitly construct the low energy effective Lagrangian in terms of the field V which realizes the spontaneously broken Z_N symmetry and describes the low energy spectrum of the Georgi - Glashow vacuum.

$$\mathcal{L} = \partial_\mu V^* \partial^\mu V - \lambda(V^*V - \mu^2)^2 - \zeta(V^2 + V^{*2}) \tag{36}$$

Similar effective Lagrangian with some quantitative differences was argued to be valid also for the pure Yang-Mills theory in [10].

The application of the 't Hooft argument at finite temperature is somewhat less straightforward. Since at finite temperature the Lorentz invariance is broken, the temporal and spatial Wilson loops do not necessarily have the same behaviour and one has to be more careful. The original argument relates the behaviour of the vortex operator and the temporal Wilson loop. At finite temperature in the Euclidean formalism the extent of the system in the temporal direction is finite. As a result it is not possible to distinguish between the area and perimeter law for "asymptotically" large temporal loops. Instead the role of the temporal Wilson loop is taken over by the Polyakov line - the loop that

[6] The original argument as stated in [2] is formulated for 3+1 dimensional theories, however its generalization to 2+1 dimensions requires only linguistic changes.

winds around the total volume of the system in the temporal direction. Thus one expects that in the deconfining phase where the Polyakov line has a nonvanishing vacuum average, the vortex operator should have vanishing expectation value. Indeed this can be easily confirmed by the explicit calculation of the VEV of the vortex operator using the method of [1]. In [1] the calculation was performed in 3+1 dimensions, but adapting it to 2+1 dimensional case is trivial. We give below a brief outline.

Consider the equal time vortex-antivortex correlation function. At finite temperature is it given by the following expression

$$< V(x)V^\dagger(y) >= \mathrm{Tr} e^{-\frac{\beta}{2}(E^2+B^2)} e^{i\frac{2\pi i}{g}\int_x^y \epsilon_{ij} dl^i E^j_3} \tag{37}$$

The line integral can be taken along the straight line L connecting the points x and y. For definiteness we take x and y to be separated in the direction of the first axis. Introducing the imaginary time axis and the Lagrange multiplier field A_0 in the standard way this expression can be transformed to

$$< V(x)V^\dagger(y) >= \int DA_i DA_0 \exp\{-\frac{1}{2}\int_0^\beta dt \int d^3x\, (\partial_0 A^a_i - (D_i A_0)^a - T a^a_i)^2 + (B^a)^2\} \tag{38}$$

where the "external field " a^i is given by

$$a^a_i(\mathbf{x}) = \delta^{a3}\delta_{i2}\frac{2\pi}{g}\delta(\mathbf{x}-L) \tag{39}$$

The nonzero Matsubara modes are integrated out in precisely the same way as in the standard calculation of the finite temperature effective potential [12],[11] and the effective action in the presence of the external field is easily calculated to one loop order

$$S_{eff} = \frac{2T^2}{g^2}(\partial_i q + \frac{g}{2}a_i)^2 + U(q) \tag{40}$$

Here q is defined ([11]) as the average value of the first eigenvalue of the matrix $A_0 = \frac{A^a_0\tau^a}{gT}$ at zero Matsubara frequency. The matrices τ^a are the generators of $SU(2)$ in the fundamental representation and are normalized according to $\mathrm{tr}\tau^a\tau^b = \frac{1}{2}\delta^{ab}$. The effective potential U to one loop is related to Bernoulli polynomial and can be read off the expressions in [11] [18]. The only property of U which is important to us is that it has two degenerate minima at $q = 0, \pi$.

To calulate the correlator we have to find the configuration of q which minimizes the action eq.(40). Qualitatively the form of the solution is clear. The considerations identical to those in [1] tell us that it must be the "broken" electric Z_2 domain wall: half a wall ($q \to_{x_2\to\infty} 0$, $q(x_2 = 0) = \frac{\pi}{2}$) above the line L and half a wall ($q(x_2 = 0) = -\frac{\pi}{2}$, $q \to_{x_2\to-\infty} 0$,) below the line L separated by a discontinuity $\delta q = \pi$. The action of such a configuration is $S_{eff} = \tilde{\alpha}|x-y|$ where $\tilde{\alpha}$ is the "Z_2 domain wall tension". The vortex correlator is thus given by

$$< V(x)V^\dagger(y) >= \exp\{-\tilde{\alpha}|x-y|\} \tag{41}$$

As $|x-y|$ become large the correlation function decreases exponentially, and thus the expectation value of the vortex operator vanishes. For the $SU(N)$ group this calculation trivially generalizes and gives the same result. The exponential decay is also obtained for the correlator of V^m with any power $m < N$.

Recall that the vortex operator is the order parameter for the magnetic Z_N symmetry. Moreover the powers of V exhaust all possible local order parameters[7]. Their vanishing is therefore an unambiguous indication that the magnetic Z_N is restored in the high temperature deconfined phase.

In hindsight this is not very surprising. Indeed, we are dealing with physical discrete symmetry which is spontaneously broken at zero temperature. When the system is heated it is unavoidable that entropy effects take over and at some sufficiently high temperature the symmetry must be restored. A good qualitative guide here is the effective Lagrangian eq.(36). It describes a simple Z_2 invariant scalar theory. There is very little doubt that a system described by this Lagrangian indeed undergoes a symmetry restoring phase transition at some T_c. Moreover the effective Lagrangian approach also suggests that this phase transition has deconfining character. As shown in [5, 8, 10] the charged states in the effective theory eq.(36) are represented by solitonic configurations of the vortex field V with unit winding number. The energy of any such state is linearly divergent in the infrared. The reason is that due to finite degeneracy of vacuum states, the minimum energy configuration looks like a quasi onedimensional strip across which the phase of V winds. The energy density inside this "electric flux tube" is proportinal to the vacuum expectation value of V. When the VEV vanishes, so does the string tension. Stated in other words, when V vanishes, the phase fluctuations are large and the winding number is not a sharp observable. An external charge is thus screened easily by regions of space around it with vanishing V . The phase with $< V >= 0$ is therefore not confining. In the theory with several Higgs fields this phase exists even at zero temperature and corresponds to a completely Higgsed phase - where the gauge group is broken completely. In such a Higgs phase indeed the colour is screened rather than confined. In the pure Yang-Mills theory this phase is absent at zero temperature, but is realized as the deconfined phase at $T > T_c$. We thus see that the behaviour of the vortex operator at high temperature does indeed match the simple intuition coming from a Z_N invariant effective Lagrangian very well.

3.2 Extension to 3+1 dimensions

The 't Hooft argument now states that the vanishing vacuum average of the Polyakov line is incompatible with the area law behaviour of the spatial 't Hooft loop and vice versa. This means that in the confining phase the 't Hooft loop

[7]The latter statement is correct modulo multiplication of V^m by local gauge invariant and Z_N invariant operators. These possible factors do not change the fact of the exponential decay of the corelators and are therefore unimportant for our discussion.

has perimeter law. In the high temperature deconfined phase the behaviour of the spatial 't Hooft loop must become area since the average of the Polyakov line is finite. Again this is confirmed by explicit calculation in [1].

A more subtle question is how the behaviour of the 't Hooft loop relates to the realization of the magnetic Z_N symmetry. The Z_N symmetry does not have an order parameter which is a local field defined at a point. The only order parameters in the strict sense (an eigenoperator with a nonvanishing eigenvalue) is a 't Hooft line $V(C)$ which runs through the whole system [13].

In a system which is finite in the direction of the loop, but is infinite in the perpendicular directions everything is clear cut. In this case there are two possibilities:

a) $< V >\neq 0$ and the magnetic Z_N broken, or

b). $< V >= 0$ and the nagnetic Z_N restored.

In the system infinite in all directions C is necessarily an infinite line, and the expectation value $< V(C) >$ clearly vanishes irrespective of whether Z_N is broken or not. The 't Hooft loop along a closed contour on the other hand is never zero, since it is globally invariant under the Z_N transformation. It is therfore impossible to find an operator whose VEV distinguishes between the two possible realizations of the magnetic symmetry by vanishing in one phase and not vanishing in the other. Nevertheless the behaviour of the closed loop does indeed reflect the realization mode of the symmetry, since it is qualitatively different in the two possible phases. Namely vacuum expectation value of a large closed 'tHooft loop (by large, as usual we mean that the linear dimensions of the loop are much larger than the correlation length in the theory) has an area law decay if the magnetic symmetry is spontaneously broken, and perimeter law decay if the vacuum state is invariant.

To understand the physics of this behaviour it is useful to think of the 't Hooft line as built of "local" operators - little "magnetic dipoles". Consider eq.(34) with the contour C running along the x- axis and the surface S chosen as the (x, y) plane. Let us mentally divide the line into (short) segments of length 2Δ centered at x_i. Each one of these segments is a little magnetic dipole and the 't Hooft loop is a product of the operators that create these dipoles. The definition of these little dipole operators is somewhat ambiguous but since we only intend to use them here for the purpose of an intuitive argument any reasonable definition will do. It is convenient to define a single dipole operator in the following way

$$D_\Delta(x) = \exp\{i \int d^3y[a_i^+(x+\Delta-y) + a_i^-(x-\Delta-y)]\mathrm{Tr}(\mathrm{YE^i(y)})\} \qquad (42)$$

where $a_i^\pm(x-y)$ is the c-number vector potential of the abelian magnetic monopole (antimonopole) of strength $4\pi/gN$. The monopole field correpsonding to a_i contains both, the smooth x_i/x^3 part as well as the Dirac string contribution. The Dirac string of the monopole - antimonopole pair in eq.(42) is chosen so that is connects the points $x - \Delta$ and $x + \Delta$ along the straight line. The dipole operators obviously have the property

$$D_\Delta(x)D_\Delta(x + 2\Delta) = D_{2\Delta}(x + \Delta) \tag{43}$$

This is because in the product the smooth field contribution of the monopole in $D_\Delta(x)$ cancels the antimonopole contribution in $D_\Delta(x + 2\Delta)$, while the Dirac string now stretches between the points $(x-\Delta)$ and $(x+3\Delta)$. When multiplied over the closed contour, the smooth fields cancel out completely, while the surviving Dirac string is precisely the magnetic vortex created by a closed 't Hooft loop operator. The 't Hooft loop can therefore be written as

$$V(C) = \Pi_{x_i} D_\Delta(x_i) \tag{44}$$

The dipole operator $D(x_i)$ is an eigenoperator of the magnetic flux defined on a surface that crosses the segment $[x_i - \Delta, x_i + \Delta]$. Suppose the magnetic symmetry is broken. Then we expect the dipole operator to have a nonvanishing expectation value[8] $< D >= d(\Delta)$. If there are no massless excitations in the theory, the operators $D(x_i)$ and $D(x_j)$ should be decorrelated if the distance $x_i - x_j$ is greater than the correlation length l. Due to eq.(44), the expectation value of the 't Hooft loop should therefore roughly behave as

$$< V(C) >= d(l)^{L/l} = \exp\{-\ln\left(\frac{1}{d(l)}\right)\frac{L}{l}\} \tag{45}$$

where L is the perimeter of the loop. In the system of finite length L_x, the vacuum expectation value of the vortex line which winds around the system in x-direction is therefore finite as in eq.(45) with $L \to L_x$.

On the other hand in the unbroken phase the VEV of the dipole operator depends on the size of the system in the perpendicular plane L_y. For large L_y it must vanish exponentially as $d = \exp\{-aL_y\}$. So the expectation value of V behaves at finite L_y in the unbroken phase as:

$$< V(C) >= \exp\{-aL_yL_x\} \tag{46}$$

and vanishes as $L_y \to \infty$. Thus in a system which is finite in x direction, but infinite in y direction, the 't Hooft line in the x direction has a finite VEV in the broken phase and vanishing VEV in the unbroken phase.

In the limit of the infinite system size $L_x \to \infty$ the VEV obviously vanishes in both phases. This is of course due to the fact that V is a product of infinite number of dipole operators, and this product vanishes even if individual dipole

[8]The magnetic dipole operators defined above are strictly speaking not local, since they carry the long range magnetic field of a dipole. However, the dipole field falls off with distance very fast. Therefore even though this fall off is not exponential the slight nonlocality of D should not affect the following qualitative discussion.

operators have finite VEV[9]. However one can avoid any reference to finite size system and infinite vortex lines by considering closed 't Hooft loops.

For a closed loop with long sides along x axis at $y = 0$ and $y = R$ the above argument leads to the conclusion that in the broken phase V must have a perimeter law, eq.(45). In the unbroken phase the correlation between the dipoles at $y = 0$ and dipoles at $y = R$ should decay exponentially $< D(0)D(R) >\propto \exp\{-\alpha\frac{R}{l}\}$ and thus

$$< V(C) >= \exp\{-\alpha\frac{LR}{l^2}\} = \exp\{-\alpha\frac{S}{l^2}\} \tag{47}$$

Thus the perimeter behaviour of $< V(C) >$ indicates vacuum state which breaks spontaneously the magnetic Z_N while the area behaviour means that the magnetic Z_N is unbroken.

The results of [1] then mean that in 3+1 dimensions as well as in 2+1 dimension the magnetic symmetry is restored above the deconfining phase transition, in the sense of eq.(46).

In the next section we discuss what is the implication of this conclusion on the behaviour of the spatial Wilson loop.

4 Spatial Wilson loop at high temperature.

As we have shown in Section 2 the spatial Wilson loop is the generator of the magnetic Z_N symmetry. We expect therefore that the mode of realization of the magnetic Z_N determines directly the behaviour of W. We will formulate the general argument shortly, but first let us consider a toy model which exemplifies the basic physics in a very simple setting.

Rather than talk about nonabelian gauge theory, consider a scalar theory of a complex field ϕ with global Z_N theory in 2+1 dimensions.

$$\mathcal{L} = \partial_\mu\phi\partial_\mu\phi^* + \lambda(\phi^*\phi - \mu^2)^2 + \zeta\Big(\phi^N + (\phi^*)^N\Big) \tag{48}$$

The generator of the Z_N symmetry is given by

$$U = \exp\Big\{i\frac{2\pi}{N}\int d^2xj_0(x)\Big\} = \exp\Big\{\frac{2\pi}{N}\int d^2x(\pi\phi - \pi^*\phi^*)\Big\} \tag{49}$$

where $\pi = \partial_0\phi^*$ is the momentum conjugate to the field ϕ. Obviously with the canonical commutation relations between π and ϕ one has

$$U\phi(x)U^\dagger = e^{i\frac{2\pi}{N}}\phi(x) \tag{50}$$

[9]The VEV of the dipole D must be smaller than one since D is defined as a unitary operator.

We will be interested in the behaviour of the operator which generates the Z_N transformation only inside some region S of the two dimensional plane.

$$\begin{aligned} U(S) &= \exp\Big\{\frac{2\pi}{N}\int_S d^2x(\pi\phi-\pi^*\phi^*)\Big\} \\ U(S)\phi(x)U^\dagger(S) &= e^{i\frac{2\pi}{N}}\phi(x) \quad x\in S \\ &= \phi(x), \quad x\notin S \end{aligned} \tag{51}$$

We will refer to this operator as the U-loop. Throughout this discussion we assume that there are no massless excitations in the spectrum of the theory and that the linear dimensions of the area S are much larger than the correlation length.

The statement we are aiming at is that in the phase with broken Z_N the U-loop has an area law behaviour while in the phase with unbroken Z_N this changes into the perimeter law behaviour.

4.1 U-loop in the broken phase

Consider the broken phase first. We are interested in the vacuum expectation value of $U(S)$. This is nothing but the overlap of the vacuum state $<0|$ and the state which is obtained by acting with $U(S)$ on the vacuum state $|S>=U|0>$. If the symmetry is broken, the field ϕ in the vacuum state is pointing in some fixed direction in the internal space. In the state $|S>$ on the other hand its direction in the internal space is different - rotated by $2\pi/N$ - at points inside the area S. In the local theory with finite correlation length the overlap between the two states approximatelly factorises into the product of the overlaps taken over the region of space of linear dimension of order of the correlation length l

$$<0|S>=\Pi_x<0_x|S_x> \tag{52}$$

where the label x is the coordinate of the point in the center of a given small region of space. For x outside the area S the two states $|0_x>$ and $|S_x>$ are identical and therefore the overlap is unity. However for x inside S the states are different and the overlap is therefore some number $e^{-\gamma}$ smaller than unity. The number of such regions inside the area is obviously of order S/l^2 and we thus

$$<U(S)>=\exp\{-\gamma\frac{S}{l^2}\} \tag{53}$$

In a weakly coupled theory this argument is confirmed by explicit calculation. The expectation value of the U-loop in the theory eq.(48) is given by the following path integral

$$<U(C)>=\int d\phi d\phi^*\exp\left\{-\int(\partial_\mu\phi+i\phi\chi_\mu)(\partial_\mu\phi^*-i\phi^*\chi_\mu)+\lambda(\phi^*\phi-\mu^2)^2+\zeta\Big(\phi^N+(\phi^*)^N\Big)\right\} \tag{54}$$

with

$$\chi_\mu(x) = \frac{2\pi}{N}\delta^{\mu 0}\delta(x_0), \quad x \in S \tag{55}$$

$$= 0, \quad x \notin S \tag{56}$$

This expression directly follows from eq.(51) and integration over the canonical momentum in the phase space path integral. At weak coupling this path integral is dominated by a simple classical configuration. First, it is clear that the solution must be such that the phase of the field ϕ has a discontinuity of $2\pi/N$ when crossing the surface S since otherwise the action is UV divergent due to singular χ. Asymtotically at large distance from the surface the field should approach its vacuum expectation value. Since the source term χ vanishes outside S, eveywhere where ϕ is continuous it has to solve classical equations of motion. Also, for values of x_1 and x_2 which are well inside S the profile ϕ should not depend on these coordinates, but should only depend on x_0. It is easy to see that a solution with these properties exists: it is given by the "broken" domain wall solution. Recall that the vacuum is degenerate and so there certainly exists a classical solution of the equations of motion which interpolates between two adjacent vacuum states $\phi \rightarrow_{x_0\rightarrow\infty} \phi_0$ and $\phi \rightarrow_{x_0\rightarrow-\infty} e^{i\frac{2\pi}{N}}\phi_0$. Breaking this classical solution along the plane $x_0 = 0$ and rotating the piece $x_0 < 0$ by $2\pi/N$ produces precisely the configuration with the correct boundary conditions and the discontinuity structure. The path integral in eq.(54) is therefore dominated by this classical configuration. Its action (up to corrections associated with the boundary effects of S) is αS where α is the classical wall tension of the domain wall which separates two adjacent Z_N vacua. Thus we find that the expectation value of the U-loop is related to the domain wall tension of the Z_N domain wall by

$$< U(S) >= \exp\{-\alpha S\} \tag{57}$$

4.2 U-loop in the unbroken phase

Now consider the unbroken phase. Again the U-loop average has the form of the overlap of two states which factorizes as in eq.(52). Now however all observables noninvariant under Z_N vanish in the vacuum. The action of the symmetry generator does not affect the state $|0>$. The state $|S>$ is therefore locally exactly the same as the state $|0>$ except along the boundary of the area S. Therefore the only regions of space which contribute to the overlap are those which lay within one correlation length from the boundary. Thus

$$< U(S) >= \exp\{-\gamma P(S)\} \tag{58}$$

where $P(S)$ is the perimeter of the boundary of S. The absence of the area law is again easily verified by a perturbative calculation. In the unbroken phase the fluctuations of the field ϕ as well as the current density $j_0 = i(\pi\phi - \pi^*\phi^*)$ are small. To leading order in the coupling constant

$$< U(S) >= \exp\Big\{ -\frac{1}{2}\int_{x,y\in S} d^2x d^2y < j_0(x) j_0(y) > \Big\} \tag{59}$$

The possible area law contribution in the exponent is

$$S\int d^2x < j_0(0) j_0(x) >= S\lim_{p\to 0} G(p) \tag{60}$$

where $G(p)$ is the Fourier transform of the charge density correlation function. The correlator of the charge densities however vanishes at zero momentum. This is because in the leading perturbative order the symmetry of the theory is actually $U(1)$ and not just Z_N as seen in eq. 54. Since the vacuum state is invariant it follows that the total charge $Q = \int d^2x j_0(x) = j_0(p=0)$ on this state vanishes, and so does any correlation function that involves zero momentum component of the charge density. So the area contribution in eq.(60) is zero. Strictly speaking in the leading order in perturbation theory eq.(59) is not the complete result. The exact expression contains in the exponential also higher point correlators of the current density. Again however the possible area law contribution contains correlators of the total charge Q with powers of j_0 and therefore vanishes.

4.3 U-loop at high temperature

At nonzero temperature the argument has to be only slightly modified. The only difference now is that the vacuum is not a pure state but rather a statistical ensemble. The average of the U-loop is therefore not given by a single matrix element but rather by

$$< U >= \sum_i e^{-E_i T} < i|U|i > \tag{61}$$

Let us consider the theory in which the Z_N symmetry is broken at zero temperature. Then for temperatures below the critical temperature T_C the states that contribute in the trace are not invariant under Z_N. The average of U in each one of these states has an area law behaviour and so does the whole temperature average of U.

When the temperature reaches T_C the phase transition occurs. The reason for the onset of the phase transition is the following. The spectrum of the theory contains both Z_N invariant and noninvariant states. The Z_N invariant states have all finite energy density, the lowest one being the state which sits at the maximum of the potential between the adjacent Z_N invariant minima. As long as the equilibrium thermal energy density is lower than the energy density of this lowest invariant state, there is no contribution from the invariant states to any physical observable since the Boltzman factor for these states vanishes in the infinite volume. However when the energy density reaches the critical threshold value the invariant states become accesssible and they start contributing to the thermal ensemble. The sudden change in the entropy due to these new channels drives the phase transition. Above the phase transition therefore there are two kinds of states that contribute to thermal averages: the noninvariant and the invariant ones. One can then write

$$< U >_{T>T_c} = \sum_n e^{-E_n/T} < n|U|n > + \sum_s e^{-E_s/T} < s|U|s > \tag{62}$$

where the first term is due to averaging over the nonsymmetric states and the second - due to the symmetric states in the thermal ensemble. As discussed earlier each state in the first term gives an area contribution of the type $\exp\{\gamma_n S\}$ while each state in the second term gives a perimeter contribution $\exp\{\gamma_s P\}$. For large enough surfaces S the perimeter term dominates and therefore above the phase transition we again expect to have the perimeter law for the U-loop. A more subtle question is how large S should be for the perimeter term to dominate. We expect that the size is determined by the spatial correlation length, but in general this is a nontrivial dynamical question the answer to which depends on entropy considerations, and it is out of scope of our present discussion.

Our conclusion is that at finite as well as at zero temperature the mode of the realization of the Z_N symmetry is in one to one correspondence with the behaviour of the U-loop. The argument is very general and does not depend on the exact form of the Z_N invariant potential and more generally on the field content of the theory - we could have added any number of extra fields to the theory eq.(48) without changing the conclusions.

To reiterate, the physics involved is very simple. When acting on a state, the U-loop performs the Z_N transformation inside the loop. The only degrees of freedom that are changed by this operation inside the loop, are the Z_N - noninvariant fields. If the vacuum wavefunction depends on the configuration of the noninvariant degrees of freedom (the state in question is not Z_N invariant) the action of U-loop affects the state everywhere inside the loop. The VEV of U-loop then falls off as an area. If the vacuum is Z_N invariant, the wavefunction does not depend on the configuration of the noninvariant degrees of freedom. The action of U-loop then perturbs the state only along the perimeter, hence the perimeter law in the unbroken phase.

Clearly the same exact correspondence must exist between the mode of realization of the magnetic Z_N symmetry and the behaviour of the Wilson loop in the pure Yang-Mills theory. The direct analogs of the scalar field $\phi(x)$ in eq.(48) and the U-loop of the scalar theory are correspondingly the vortex field $V(x)$, and the spatial Wilson loop $W(C)$.

As we have shown in the previous section, the magnetic Z_N is restored at high temperature. It thus follows that the behaviour of the spatial Wilson loop must change at the deconfining phase transition - in the hot phase it must have a perimeter law behaviour.

Unfortunately the Yang-Mills theory is strongly coupled in the infrared both at low and high temperature. We therefore do not have a simple semiclassical perturbative method to actually calculate the expectation value of the Wilson loop. We are aware that our conclusion bluntly contradicts the common lore. Nevertheless we think that the argument presented above is very compelling, and since the common lore is based on indirect arguments and numerical calculations it is worthwhile to examine this question with greater care. In the next section we will discuss some possible loopholes in the standard argumentation.

4.4 Wilson loop in 3+1 dimensions

The previous considerations generalizes to the 3+1 dimensons. In the broken phase when acting with the Wilson loop $W(C)$ on the vacuum one changes the state of those magnetic vortices which loop through C. The number of such vortices which are present in a generic configuration in the broken phase is proportional to the minimal area subtending C. The number of the degrees of freedom that is changed by the action of W is thus proportional to the area S. Each of these degrees of freedom contributes a factor smaller than unity to the overlap with the vacuum state and so the VEV of W scales with the exponential of the area. In the unbroken phase the vacuum does not contain vortices of arbitrarily large size. The size of the vortices present in the vacuum is cutoff by the relevant correlation length. In the case of the hot Yang Mills theory this correlation length should be smaller, or of the order of the magnetic mass $1/g^2T$. Therefore for contours C of linear dimension much larger than this length, the action of $W(C)$ only disturbs degrees of freedom close to the contour C itself and the VEV must have the perimeter behaviour.

In this case we do not have a simple scalar toy model to use as an illustration for this argument like we did in the 2+1 dimensional case. The generality of the argument however does not depend on this. It is especially clear in the unbroken phase. Since the vacuum is invariant, there is no way that the action of the symmetry generator can have any nontrivial effect in the bulk, and thus only perimeter behaviour of W is possible.

To close this section we note that the considerations of this section do not apply to Abelian theories. The magnetic symmetry does exist in this case too, but here it is the continuos $U(1)$ group and the spectrum is massless at low and probably also at high temperature. In this case there is no reason to expect the local factorization of the overlap and generically therefore the arguments of this section do not hold. In particular in the presence of long range correlations it is perfectly possible that the Wilson loop has area law even though the state is perturbed only along the perimeter of the loop[10].

[10] In 2+1 dimensions it is actualy only the noncompact Abelian theories that are excluded from the consideration. Compact theories are massive and therefore should behave in the same way as the nonabelian Yang-Mills.

5 Discussion.

In this paper our aim was to point out two facts. First that the calculation of the VEV of the 't Hooft loop [1] implies the restoration of the magnetic Z_N symmetry above the deconfinement transition. Second, that this in turn implies perimeter law for the spatial Wilson loop in the hot phase. Both of these points raise interesting further questions.

In particular the second point is rather in contradiction with the accepted common wisdom. We however urge the reader not to discard it right away as nonsense based solely on this argument. It is unfortunate that we are unable to present an explicit analytical calculation of the spatial Wilson loop, but such is life - the infrared region of the Yang - Mills theory is strongly coupled even at high temperature. However the argument we presented is very simple and general and seems to us extremely compelling. In a nutshell: the "activity" of the magnetic flux loops in the hot phase is so small, that it can only bring about perimeter behaviour for the spatial Wilson loop.

It is therefore worthwhile to explore this question further by the available numerical methods. If they unambiguously show an area law behaviour, it may mean there is additional magnetic activity in some different form; this caveat in itself may teach us something interesting about the Yang - Mills theory. This however is far beyond the scope of this note. What we rather want to do in this section is to discuss possible loopholes in the arguments which underly the standard paradigm.

The standard argument is this. At very high temperature the nonzero frequency Matsubara modes are very heavy. In the vacuum therefore they can not be excited and so the physics in the static sector is completely dominated by the zero Matsubara modes. The Lagrangian for these modes is just the Yang - Mills Lagrangian in one lower dimension with an additional light adjoint scalar field - the zeroth component of the vector potential. This dimensionally reduced theory is certainly confining in 2 and 3 dimensions. Since the leading contribution to the Wilson loop comes from this confining theory, $W(C)$ must have an area law. If one wishes to be more careful, rather than discarding the nonzero Matsubara frequency modes one integrates them out thus generating corrections to the effective Lagrangian. These corrections renormalize the scalar potential and also add higher dimensional terms to the Yang-Mills part, but they are believed to be inessential for the qualitative behaviour of the reduced theory at long distances.

Although this argument may seem quite convincing it is certainly not waterproof. It may sometimes happen that heavy degrees of freedom are important for certain aspects of long distance physics. An example of this kind is heavy fundamental charges. The Wilson loop in a theory with fundamental charges is known to have a perimeter law at any temperature. However if one follows this logic - integrate out heavy fundamental charges to generate the effective potential for light fields, and then calculate the Wilson loop in the effective theory, the result will be an area law. This is bound to happen since the reduced theory does not contain fundamental charges and is confining. This conclusion is equally true at zero and high temperature. The reason for the breakdown of the reduced theory for this particular quantity is clear. Physically the perimeter law is the result of the snapping of the fundamental confining string, due to creation of a pair of heavy fundamental particles from the vacuum. If we insist on integrating out the heavy fields, we are allowing them to appear only as virtual states but not as real particles in the final state. The string breaking is then impossible and the area law follows. The core of the problem is that a state with a long confining string has large energy even though spatial momenta associated with light fields in this state are small. Thus we are trying to use the reduced Lagrangian where it is clearly not applicable: to describe the state with large energy. Such misguided application of low energy effective Lagrangians often leads to similar problems, see for example discussion in [14].

It is clear that the operators of the type of large Wilson loops are precisely the dangerous ones from this point of view. They are liable to excite high energy states and one therefore has to be very careful. It seems to us that it is entirely possible that for very large loops the high temperature path integral for the Yang - Mills theory will be dominated by configurations with large nonzero Matsubara frequency fields in some spatial regions (presumably close to the loop) which will screen the loop dynamically and thus produce perimeter law. Such large fields do not appear in the vacuum and in the reduced theory. This would mean that W has an area law behaviour at intermediate sizes, while the perimeter behaviour takes over for large loops. We note that this scenario is quite in line with our discussion in the previous section. In particular this precisely is suggested by eq.(62), where the area behaviour is due to the contribution of the nonsymmetric state in the thermal ensemble and is likely to dominate for not too large loops.

Another argument that has been brought up in favour [19] of the area behaviour of the spatial Wilson loop is that the action of a vortex winding through the periodic time direction is finite. These vortices then may be expected to contribute to the path integral and disorder the spatial Wilson loop. However for this to be the case the vortices must be uncorrelated. If strong correlations between the temporal vortices exist it is far from clear that they have an effect on the Wilson loop. In fact the result $< V >= 0$ in 2+1 dimensions rules out the existence of the condensate of the short vortices and therefore suggests that they are indeed correlated. In 3+1 dimensions the same should be true whenever $< V(C) >$ has an area law fall off.

We also note that there are several lattice gauge theory calculations which seem to support the area law behaviour of W in the hot phase [15]. However if the mechanism of the type we just mentioned in fact exists it would make it quite difficult to see the crossover into perimeter law on the lattice. The situation would be analogous to the (non)observation of the breaking of the string in

QCD with fundamental charges. Lattice calculations are unable to see this breaking without explicitly allowing the unbroken string state to mix with the broken string state which also contains the pair of charges created from the vacuum. This latter calculation has only been performed quite recently [16] and it is quite possible that the much earlier results of [15] are affected by a similar phenomenon.

It would be very interesting to perform additional lattice calculations which could clarify the situation.

In particular it will be very interesting to measure on the lattice the free energy of a magnetic vortex. In ref.([13]) the behaviour of the free energy of magnetic and electric fluxes has been discussed in the *low* temperature phase. To be able to do it in the lattice framework one has to define the theory in a finite volume. As discussed by 't Hooft this can be achieved by imposing on the potentials periodic boundary conditions modulo a gauge transformation. As discussed in ref. [13] this admits the presence of vortices in 2+1 and of the vortex lines in 3+1 dimensions. 't Hooft's discussion was based on a Euclidean rotation identity for the twisted 4d path integrals valid for any temperature, and a factorization property of magnetic and electric fluxes, whose validity at low T is very reasonable, but is inconsistent with the Euclidean rotation identity at high T. Based on this 't Hooft could prove ($N \leq 3$) that in the confining phase, where the free energy of an electric flux is linear with the length (with the string tension ρ), the free energy of magnetic flux vanishes exponentially in the infinite volume limit. For a magnetic flux in, say the z-direction it is $\exp -\rho L_x L_y$. Thus the free energy of a magnetic flux is related to the behaviour of the Wilson loop.

One can therefore reasonably ask how the magnetic flux free energy behaves in the hot phase. Our arguments presented above suggest that this free energy should be finite in the infinite volume limit and proportional to the length of the vortex. In 2+1 dimensions one similarly expects a finite free energy of a vortex. We note that an interesting recent lattice calculation [20] measures the monopole-antimonopole correlation. The results of [20] point to the screened behaviour of this correlation function just like its electric partner, the correlator of Polyakov loops. This via 't Hoofts argument, is consistent with the measured area behaviour of Wilson loops [15]. It is inconsistent with our prediction of finite free energy denisity per unit length for a magnetic vortex. We note however that this simulation [20] also points to the Coulomb behaviour for the spatial 't Hooft loop in the hot phase, in contradiction to analytic results [1][21] and lattice results [22]. We feel that also here more work should be done to clarify the situation.

Finally we note that the restoration of the Z_N symmetry at high temperature should have phenomenological consequences which are worth exploring. One obvious feature is that at high temperature the magnetic vortices must have finite free energy density per unit length and therefore are stable objects. The high temperature excitations should therefore be "made" out of the magnetic strings. This possibly may have implications for some characteristics of the final states in heavy ion collisions.

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