Managing Errors and Uncertainty

It is inevitable that experiments will vary from their theoretical predictions. This may be due to natural variations, a lack of understanding of the process, or a simplified model in the theory. In this document, we discuss a few of these causes, and more importantly, how we can account for these problems and overcome them through careful collection, handling, and analysis.

Types of Uncertainty

There are three types of limitations to measurements:

1) **Instrumental limitations**
   Any measuring device is limited by the fineness of its manufacturing. Measurements can never be better than the instruments used to make them.

2) **Systematic errors**
   These are caused by a factor **that does not change** during the measurement. For example, if the balance you used was calibrated incorrectly, all your subsequent measurements of mass would be wrong. Systematic errors do not enter into the uncertainty. They can either be identified and eliminated, or lurk in the background, producing a shift from the true value.

3) **Random errors**
   These arise from unnoticed variations in measurement technique, tiny changes in the experimental environment, etc. Random variations affect precision. Truly random effects average out if the results of a large number of trials are combined.

   Note: "Human error" is a euphemism for doing a poor quality job. It is an admission of guilt.
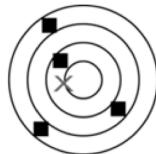
Precision vs. Accuracy

- A <u>precise</u> measurement is one where independent measurements of the same quantity closely cluster about a single value that may or may not be the correct value.

- An <u>accurate</u> measurement is one where independent measurements cluster about the true value of the measured quantity.

> Systematic errors are not random and therefore can never cancel out.
>
> They affect the <u>accuracy</u> but not the <u>precision</u> of a measurement.

A. **Low-precision, Low-accuracy:**

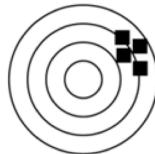   The average (the X) is not close to the center

B. **Low-precision, High-accuracy:**

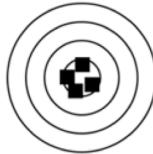   The average is close to the true value, but data points are far apart

A. Low Precision, Low Accuracy   B. Low Precision, High Accuracy

C. **High-precision, Low-accuracy:**

Data points are close together, but he average is not close to the true value

D. **High-precision, High-accuracy**

All data points close to the true value

C. High Precision, Low Accuracy   D. High Precision, High Accuracy

Writing Experimental Numbers

Uncertainty of Measurements

Measurements are quantified by associating them with an uncertainty. For example, the best estimate of a length L is 2.59cm, but due to uncertainty, the length might be as small as 2.57cm or as large as 2.61cm. L can be expressed with its uncertainty in two different ways:

1. **Absolute Uncertainty**
   Expressed in the units of the measured quantity: $L = 2.59 \pm 0.02 \text{cm}$

2. **Percentage Uncertainty**
   Expressed as a percentage which is independent of the units
   Above, since $0.02/2.59 \approx 1\%$ we would write $L = 7.7\text{cm} \pm 1\%$

Significant Figures

Experimental numbers must be written in a way consistent with the precision to which they are known. In this context one speaks of **significant figures** or digits that have physical meaning.

1. All definite digits and the first doubtful digit are considered significant.

2. Leading zeros are **not** significant figures.
   Example: $L = 2.31\text{cm}$ has 3 significant figures. For $L = 0.0231\text{m}$, the zeros serve to move the decimal point to the correct position. Leading zeros are not significant figures.

3. Trailing zeros are significant figures: they indicate the number's precision.

4. One significant figure should be used to report the uncertainty or occasionally two, especially if the second figure is a five.

Rounding Numbers

To keep the correct number of significant figures, numbers must be rounded off. The discarded digit is called the **remainde**r. There are three rules for rounding:

✓ **Rule 1**: If the remainder is less than 5, drop the last digit.
   Rounding to one decimal place: 5.3<u>46</u> → 5.3

✓ **Rule 2**: If the remainder is greater than 5, increase the final digit by 1.
   Rounding to one decimal place: 5.7<u>98</u> → 5.8

✓ **Rule 3**: If the remainder is exactly 5 then round the last digit to the <u>closest even number</u>. This is to prevent rounding bias. Given a large data set, remainders of 5 are rounded down half the time and rounded up the other half.

Rounding to one decimal place: $3.5\underline{5} \rightarrow 3.6$, also $3.6\underline{5} \rightarrow 3.6$

Examples
The period of a pendulum is given by $T = 2\pi\sqrt{l/g}$.   Here, $l = 0.24$m is the pendulum length and $g = 9.81$m/s^2 is the acceleration due to gravity.   ✖**WRONG:** $T = 0.983269235922$s   ✓**RIGHT:** $T = 0.98$s   Your calculator may report the first number, but there is no way you know T to that level of precision. When no uncertainties are given, report your value with the same number of significant figures as the value with the smallest number of significant figures.
The mass of an object was found to be 3.56g with an uncertainty of 0.032g.   ✖**WRONG:** $m = 3.56 \pm 0.032$g   ✓**RIGHT:** $m = 3.56 \pm 0.03$g   The first way is wrong because the uncertainty should be reported with one significant figure
The length of an object was found to be 2.593cm with an uncertainty of 0.03cm.   ✖**WRONG:** $L = 2.593 \pm 0.03$cm   ✓**RIGHT:** $L = 2.59 \pm 0.03$cm   The first way is wrong because it is impossible for the third decimal point to be meaningful since it is smaller than the uncertainty.
The velocity was found to be 2.45m/s with an uncertainty of 0.6m/s.   ✖**WRONG:** $v = 2.5 \pm 0.6$m/s   ✓**RIGHT:** $v = 2.4 \pm 0.6$m/s   The first way is wrong because the first discarded digit is a 5. In this case, the final digit is rounded to the closest even number (i.e. 4)
The distance was found to be 45600m with an uncertainty around 1m   ✖**WRONG:** $d = 45600$m   ✓**RIGHT:** $d = 4.5600 \times 10^4$m   The first way is wrong because it tells us nothing about the uncertainty. Using scientific notation emphasizes that we know the distance to within 1m.

Statistical Analysis of Small Data Sets

Repeated measurements allow you to not only obtain a better idea of the actual value, but also enable you to characterize the uncertainty of your measurement. Below are a number of quantities that are very useful in data analysis. The value obtained from a particular measurement is x. The measurement is repeated N times. Oftentimes in lab N is small, usually no more than 5 to 10. In this case we use the formulae below:

Mean (x_{avg})	The average of all values of x (the "best" value of x)	$x_{avg} = \dfrac{x_1 + x_2 + \cdots + x_N}{N}$
Range (R)	The "spread" of the data set. This is the difference between the maximum and minimum values of x	$R = x_{max} - x_{min}$
Uncertainty in a measurement (Δx)	Uncertainty in a single measurement of x. You determine this uncertainty by making multiple measurements. You know from your data that x lies somewhere between x_{max} and x_{min}.	$\Delta x = \dfrac{R}{2} = \dfrac{x_{max} - x_{min}}{2}$
Uncertainty in the Mean (Δx_{avg})	Uncertainty in the mean value of x. The actual value of x will be somewhere in a neighborhood around x_{avg}. This neighborhood of values is the uncertainty in the mean.	$\Delta x_{avg} = \dfrac{\Delta x}{\sqrt{N}} = \dfrac{R}{2\sqrt{N}}$
Measured Value (x_m)	The final reported value of a measurement of x contains both the average value and the uncertainty in the mean.	$x_m = x_{avg} \pm \Delta x_{avg}$

The average value becomes **more and more precise** as the number of measurements N increases. Although the uncertainty of any single measurement is always Δx, the **uncertainty in the mean Δx_{avg}** becomes smaller (by a factor of $1/\sqrt{N}$) as more measurements are made.

Example

You measure the length of an object five times.
You perform these measurements twice and obtain the two data sets below.

Measurement	Data Set 1 (cm	Data Set 2 (cm)
x_1	72	80
x_2	77	81
x_3	8	81
x_4	85	81
x_5	88	82

Quantity	Data Set 1 (cm)	Data Set 2 (cm)
x_{avg}	81	81
R	16	2
Δx	8	1
Δx_{avg}	4	0.4

For Data Set 1, to find the best value, you calculate the mean (i.e. average value):

$$x_{\text{avg}} = \frac{72\text{cm} + 77\text{cm} + 82\text{cm} + 86\text{cm} + 88\text{cm}}{5} = 81\text{cm}$$

The range, uncertainty and uncertainty in the mean for Data Set 1 are then:

$$R = 88\text{cm} - 72\text{cm} = 16\text{cm}$$

$$\Delta x = \frac{R}{2} = 8\text{cm}$$

$$\Delta x_{\text{avg}} = \frac{R}{2\sqrt{5}} \approx 4\text{cm}$$

Data Set 2 yields the same average but has a much smaller range.

We report the measured lengths x_{m} as:

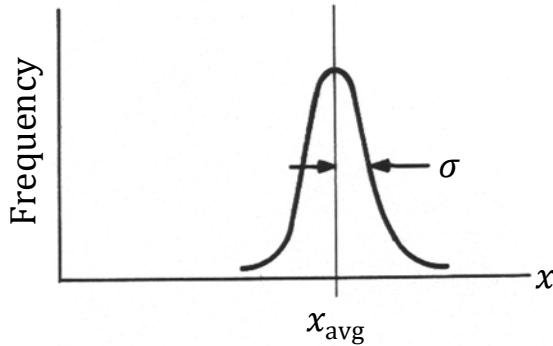Data Set 1: $x_{\text{m}} = \mathbf{81 \pm 4\text{cm}}$

Data Set 2: $x_{\text{m}} = \mathbf{81.0 \pm 0.4\text{cm}}$

Notice that for Data Set 2, Δx_{avg} is so small we had to add another significant figure to x_{m}.

Statistical Analysis of Large Data Sets

If only random errors affect a measurement, it can be shown mathematically that in the limit of an **infinite** number of measurements ($N \to \infty$), the distribution of values follows a **normal distribution** (i.e. the bell curve on the right). This distribution has a peak at the mean value x_{avg} and a width given by the standard deviation σ.

Obviously, we never take an infinite number of measurements. However, for a large number of measurements, say, $N \sim 10^2$ or more, measurements may be approximately normally distributed. In that event we use the formulae below:
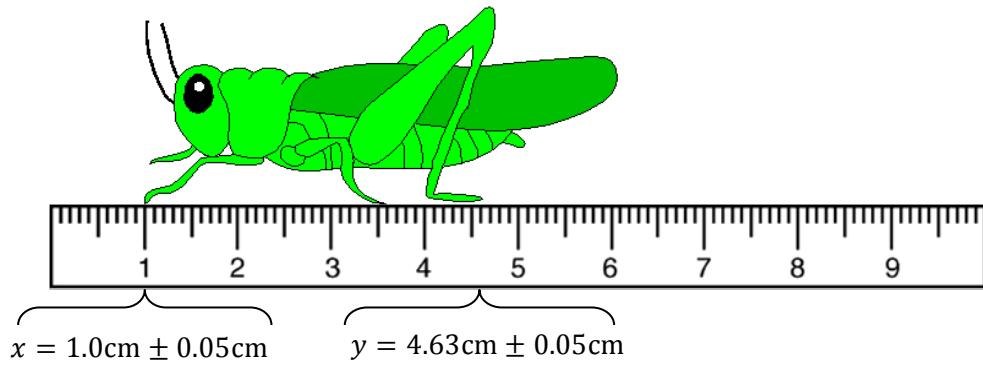

Mean (x_{avg})	The average of all values of x (the "best" value of x). This is the same as for small data sets.	$x_{\text{avg}} = \dfrac{\sum_{i=1}^{N} x_i}{N}$
Uncertainty in a measurement (Δx)	Uncertainty in a single measurement of x. The vast majority of your data lies in the range $x_{\text{avg}} \pm \sigma$	$\Delta x = \sigma = \sqrt{\dfrac{\sum_{i=1}^{N}(x_i - x_{\text{avg}})^2}{N}}$
Uncertainty in the Mean (Δx_{avg})	Uncertainty in the mean value of x. The actual value of x will be somewhere in a neighborhood around x_{avg}. This neighborhood of values is the uncertainty in the mean.	$\Delta x_{\text{avg}} = \dfrac{\sigma}{\sqrt{N}}$
Measured Value (x_{m})	The final reported value of a measurement of x contains both the average value and the uncertainty in the mean.	$x_{\text{m}} = x_{\text{avg}} \pm \Delta x_{\text{avg}}$

Most of the time we will be using the formulae for small data sets. However, occasionally we perform experiments with enough data to compute a meaningful standard deviation. In those cases we can take advantage of software that has programmed algorithms for computing x_{avg} and σ.

Propagation of Uncertainties

Oftentimes we combine multiple values, each of which has an uncertainty, into a single equation. In fact, we do this every time we measure something with a ruler. Take, for example, measuring the distance from a grasshopper's front legs to his hind legs. For rulers, we will assume that the uncertainty in all measurements is one-half of the smallest spacing.

$x = 1.0\text{cm} \pm 0.05\text{cm}$ $y = 4.63\text{cm} \pm 0.05\text{cm}$

The measured distance is $d_\text{m} = d \pm \Delta d$ where $d = 4.63\text{cm} - 1.0\text{cm} = 3.63\text{cm}$. What is the uncertainty in d_m? You might think that it is the sum of the uncertainties in x and y (i.e. $\Delta d = \Delta x + \Delta y = 0.1\text{cm}$). However, statistics tells us that if the uncertainties are independent of one another, the uncertainty in a sum or difference of two numbers is obtained by quadrature: $\Delta d = \sqrt{(\Delta x)^2 + (\Delta y)^2} = 0.07\text{cm}$. The way these uncertainties combine depends on how the measured quantity is related to each value. Rules for how uncertainties propagate are given below.

Addition/Subtraction	$z = x \pm y$	$\Delta z = \sqrt{(\Delta x)^2 + (\Delta y)^2}$
Multiplication	$z = xy$	$\Delta z = \lvert xy \rvert \sqrt{\left(\dfrac{\Delta x}{x}\right)^2 + \left(\dfrac{\Delta y}{y}\right)^2}$
Division	$z = \dfrac{x}{y}$	$\Delta z = \left\lvert\dfrac{x}{y}\right\rvert \sqrt{\left(\dfrac{\Delta x}{x}\right)^2 + \left(\dfrac{\Delta y}{y}\right)^2}$
Power	$z = x^n$	$\Delta z = \lvert n \rvert x^{n-1} \Delta x$
Multiplication by a Constant	$z = cx$	$\Delta z = \lvert c \rvert \Delta x$
Function	$z = f(x, y)$	$\Delta z = \sqrt{\left(\dfrac{\partial f}{\partial x}\right)^2 (\Delta x)^2 + \left(\dfrac{\partial f}{\partial y}\right)^2 (\Delta y)^2}$

Examples
Addition
The sides of a fence are measured with a tape measure to be 124.2cm, 222.5cm, 151.1cm and 164.2cm. Each measurement has an uncertainty of 0.07cm. Calculate the measured perimeter P_m including its uncertainty. $$P = 124.2\text{cm} + 222.5\text{cm} + 151.1\text{cm} + 164.2\text{cm} = 662.0\text{cm}$$ $$\Delta P = \sqrt{(0.07\text{cm})^2 + (0.07\text{cm})^2 + (0.07\text{cm})^2 + (0.07\text{cm})^2} = 0.14\text{cm}$$ $$\boxed{P_m = 662.0 \pm 0.1\text{cm}}$$
Multiplication
The sides of a rectangle are measured to be 15.3cm and 9.6cm. Each length has an uncertainty of 0.07cm. Calculate the measured area of the rectangle A_m including its uncertainty. $$A = 15.3\text{cm} \times 9.6\text{cm} = 146.88 cm^2$$ $$\Delta A = 15.3\text{cm} \times 9.6\text{cm} \sqrt{\left(\frac{0.07}{15.3}\right)^2 + \left(\frac{0.07}{9.6}\right)^2} = 1.3\text{cm}^2$$ $$\boxed{A_m = 147 \pm 1\text{cm}^2}$$
Power/Multiplication by Constant
A ball drops from rest from an unknown height h. The time it takes for the ball to hit the ground is measured to be $t = 1.3 \pm 0.2s$. The height is related to this time by the equation $h = \frac{1}{2}gt^2$ where $g = 9.81\text{m/s}^2$. Assume that the value for g carries no uncertainty and calculate the height h including its uncertainty. $$h = \frac{1}{2}(9.8\text{m/s}^2)(1.3\text{s})^2 \approx 8.281\text{m}$$ $$\Delta h = \frac{1}{2}(9.8\text{m/s}^2)(2 \times 1.3\text{s} \times 0.2\text{s}) \approx 2.5\text{m}$$ $$\boxed{h_m = 8 \pm 3\text{m}}$$