Manifolds, Geometry, and Robotics

Frank C. Park
Seoul National University

Abstract

Ideas and methods from differential geometry and Lie groups have played a crucial role in establishing the scientific foundations of robotics, and more than ever, influence the way we think about and formulate the latest problems in robotics. In this talk I will trace some of this history, and also highlight some recent developments in this geometric line of inquiry. The focus for the most part will be on robot mechanics, planning, and control, but some results from vision and image analysis, and human modeling, will be presented. I will also make the case that many mainstream problems in robotics, particularly those that at some stage involve nonlinear dimension reduction techniques or some other facet of machine learning, can be framed as the geometric problem of mapping one curved space into another, so as to minimize some notion of distortion. A Riemannian geometric framework will be developed for this distortion minimization problem, and its generality illustrated via examples from robot design to manifold learning.

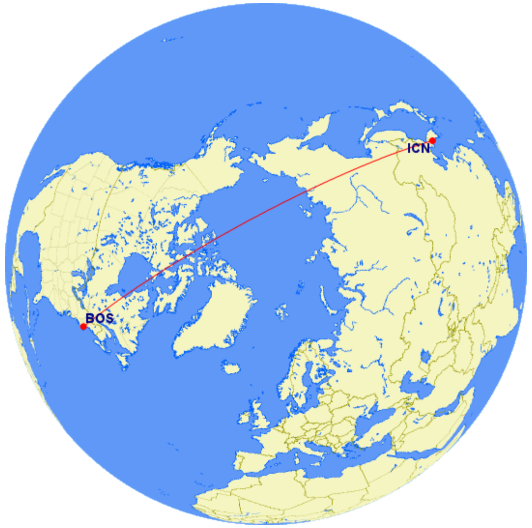
Outline

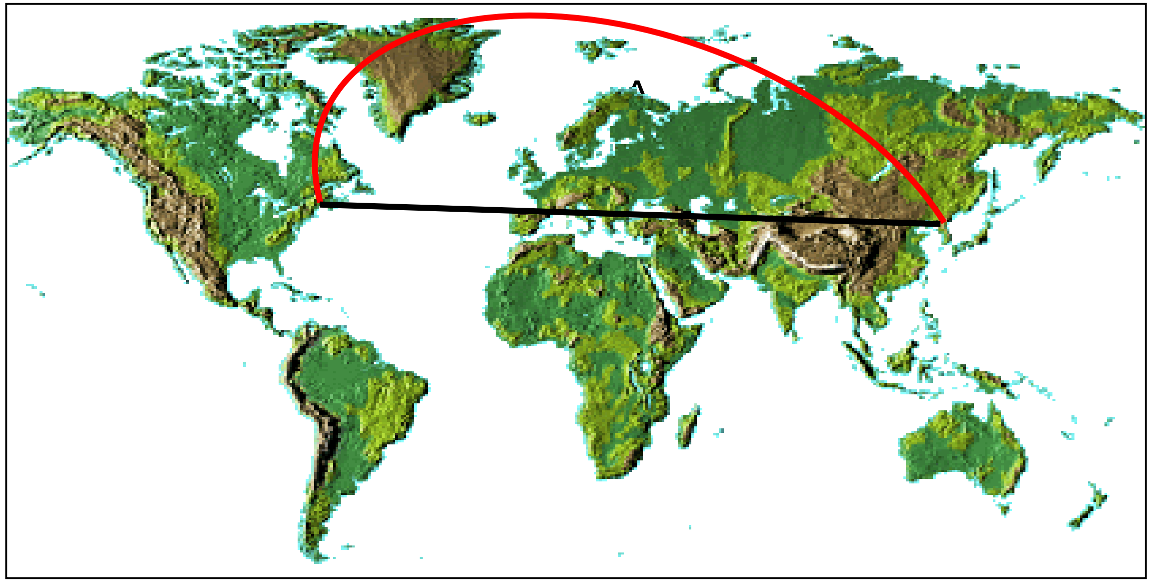
- A survey of differential geometric methods in robotics:
  - A retrospective critique
  - Some recent results and open problems
- Problems and case studies drawn from:
  - Kinematics and path planning
  - Dynamics and motion optimization
  - Vision and image analysis
  - Human and robot modeling
  - Machine learning

Contributors

Roger Brockett, Shankar Sastry, Mark Spong, Dan Koditschek, Matt Mason, John Baillieul, PS Krishnaprasad, Tony Bloch, Josip Loncaric, David Montana, Brad Paden, Vijay Kumar, Mike McCarthy, Richard Murray, Zexiang Li, Jean-Paul Laumond, Antonio Bicchi, Joel Burdick, Elon Rimon, Greg Chirikjian, Howie Choset, Kevin Lynch, Todd Murphey, Andreas Mueller, Milos Zefran, Claire Tomlin, Andrew Lewis, Francesco Bullo, Naomi Leonard, Kristi Morgansen, Rob Mahony, Ross Hatton, ***and many more***…

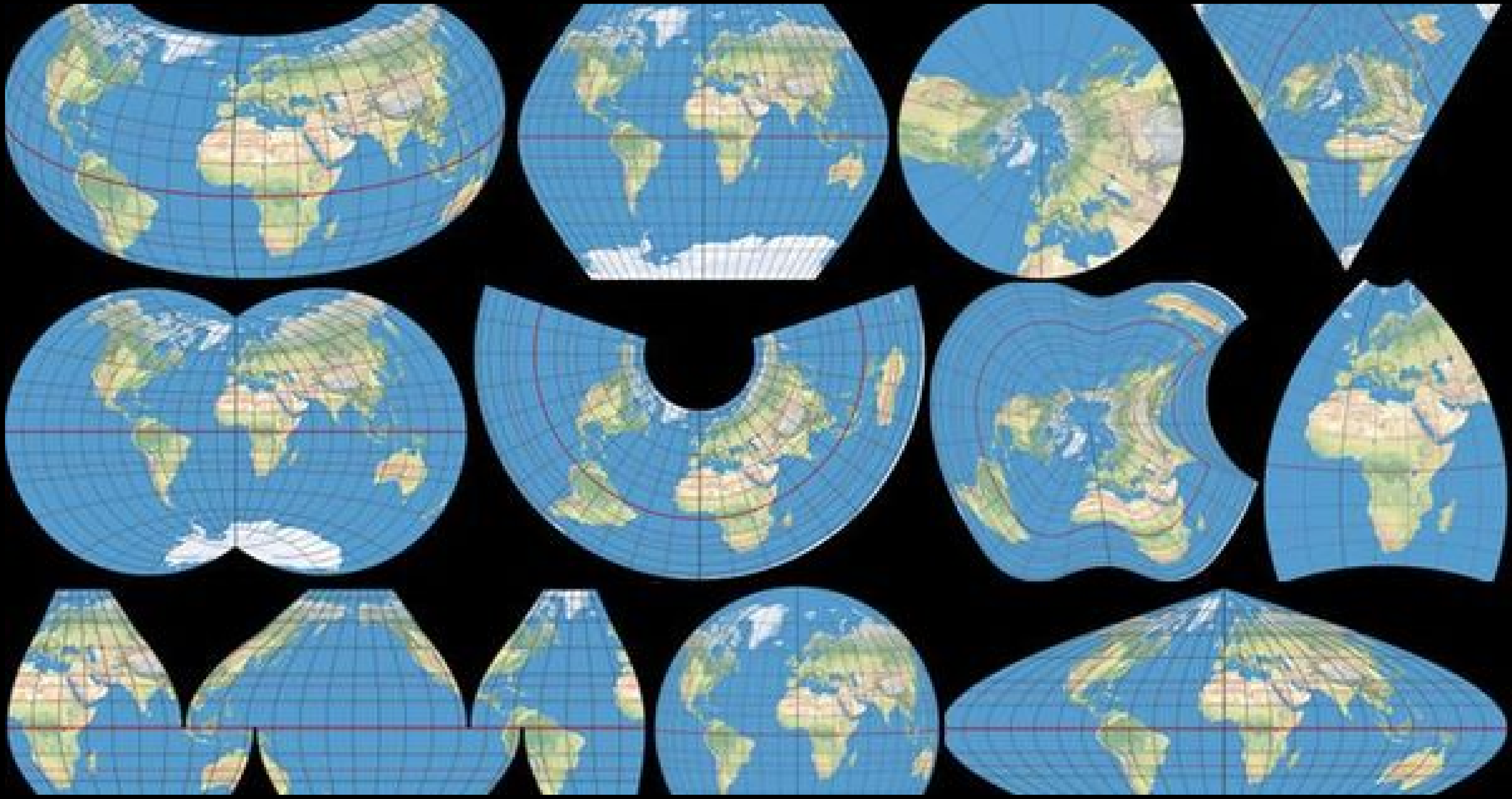
Why geometry matters





Shortest path on globe ≠ shortest path on map
North and south poles map to lines of latitude

2-D maps galore



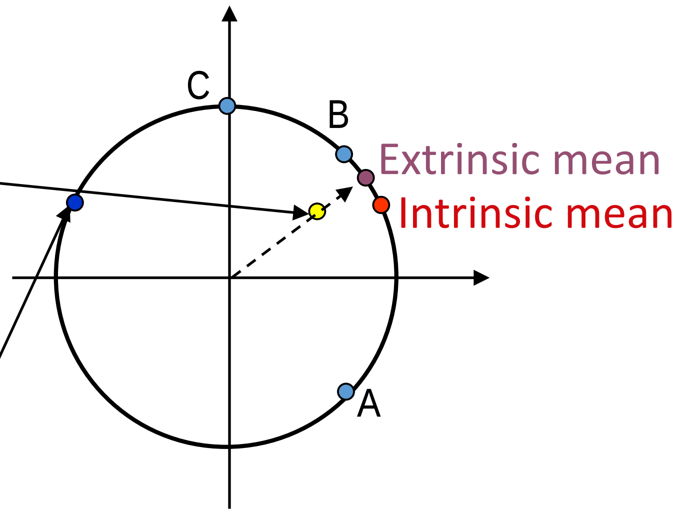
Why geometry matters

[Example] The average of three points on a circle

- Cartesian coordinates:

$$\mathrm{A}:\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right),\ \mathrm{B}:\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right),\ \mathrm{C}:(0,1)$$

$$\text{Mean}:\left(\frac{\sqrt{2}}{3},\frac{1}{3}\right)$$



- Polar coordinates:

$$\mathrm{A}:\left(1,\tfrac{7}{4}\pi\right),\ \mathrm{B}:\left(1,\tfrac{1}{4}\pi\right),\ C:\left(1,\tfrac{1}{2}\pi\right)$$

$$\text{Mean}:\left(1,\tfrac{5}{6}\pi\right)$$

Result depends on the local coordinates used

Why geometry matters

[Example] The average of two symmetric positive-definite matrices:

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 7 & 0 \\ 0 & 1 \end{bmatrix}$$

- Arithmetic mean

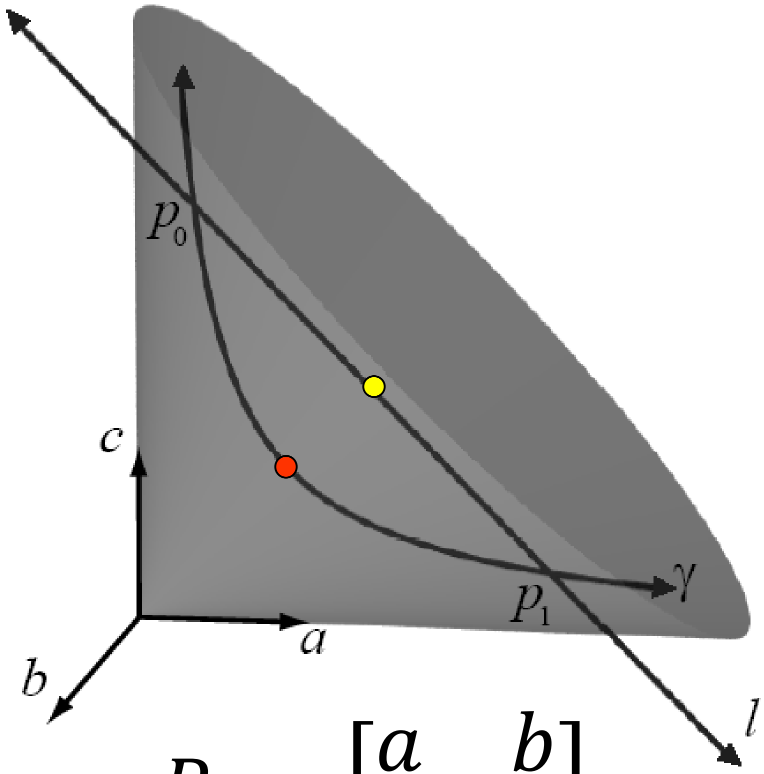
$$P = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$$

Determinant not preserved

- Intrinsic mean

$$P' = \begin{bmatrix} \sqrt{7} & 0 \\ 0 & \sqrt{7} \end{bmatrix}$$

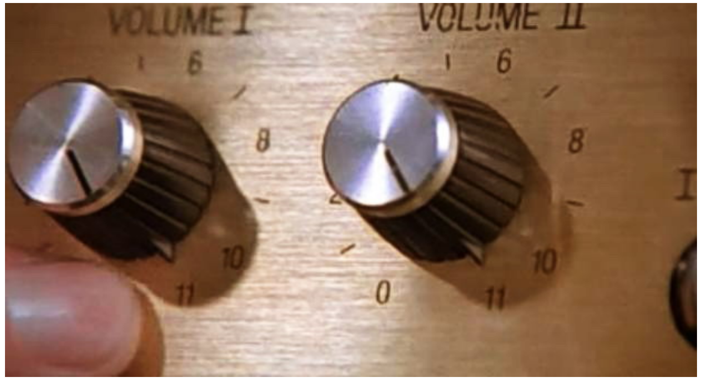
Determinant preserved



$$P = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

This is Spinal Tap (1984, Rob Reiner)

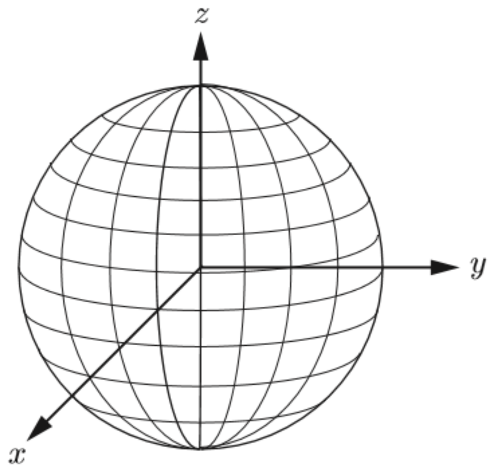
Nigel: The numbers all go to eleven. Look, right across the board, eleven, eleven, eleven and...
Marty: Oh, I see. And most amps go up to ten?
Nigel: Exactly.
Marty: Does that mean it's louder? Is it any louder?
Nigel: Well, it's one louder, isn't it? It's not ten. You see, most blokes, you know, will be playing at ten. You're on ten here, all the way up, all the way up, all the way up, you're on ten on your guitar. Where can you go from there? Where?
Marty: I don't know.
Nigel: Nowhere. Exactly. What we do is, if we need that extra push over the cliff, you know what we do?
Marty: Put it up to eleven.
Nigel: Eleven. Exactly. One louder.
Marty: Why don't you make ten a little louder, make that the top number and make that a little louder?
Nigel: These go to eleven.



Freshman calculus revisited

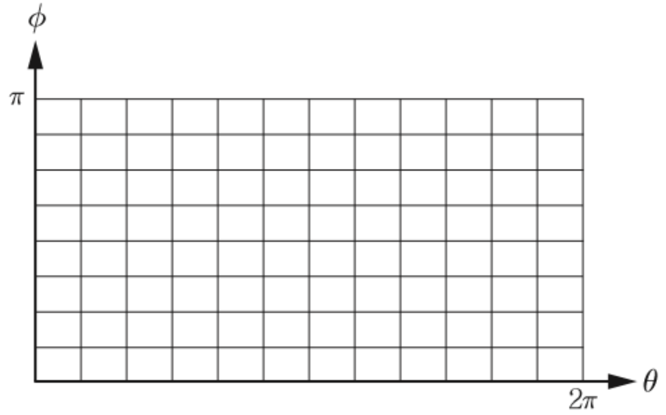
The unit two-sphere is parametrized as $x^2 + y^2 + z^2 = 1$. Spherical coordinates:

$$x = \cos\theta \sin\phi$$
$$y = \sin\theta \sin\phi$$
$$z = \cos\phi$$



Given a curve $(x(t), y(t), z(t))$ on the sphere, its incremental arclength is

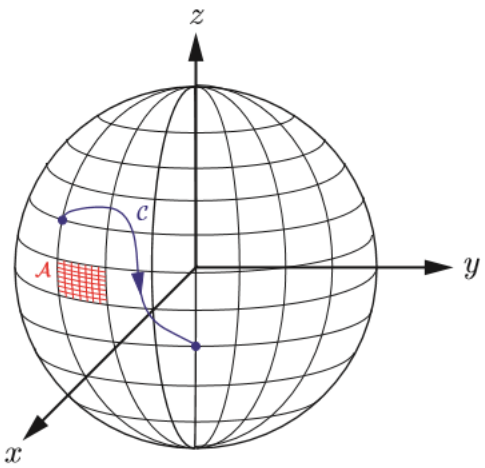
$$ds^2 = dx^2 + dy^2 + dz^2 = d\phi^2 + \sin^2\phi\, d\theta^2$$

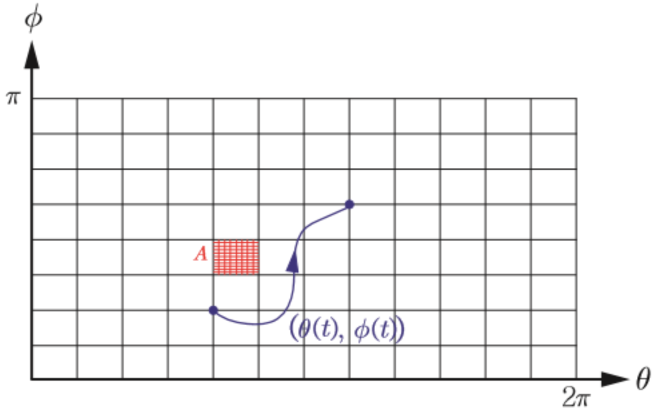


Freshman calculus revisited

Calculating lengths and areas on the sphere using spherical coordinates:

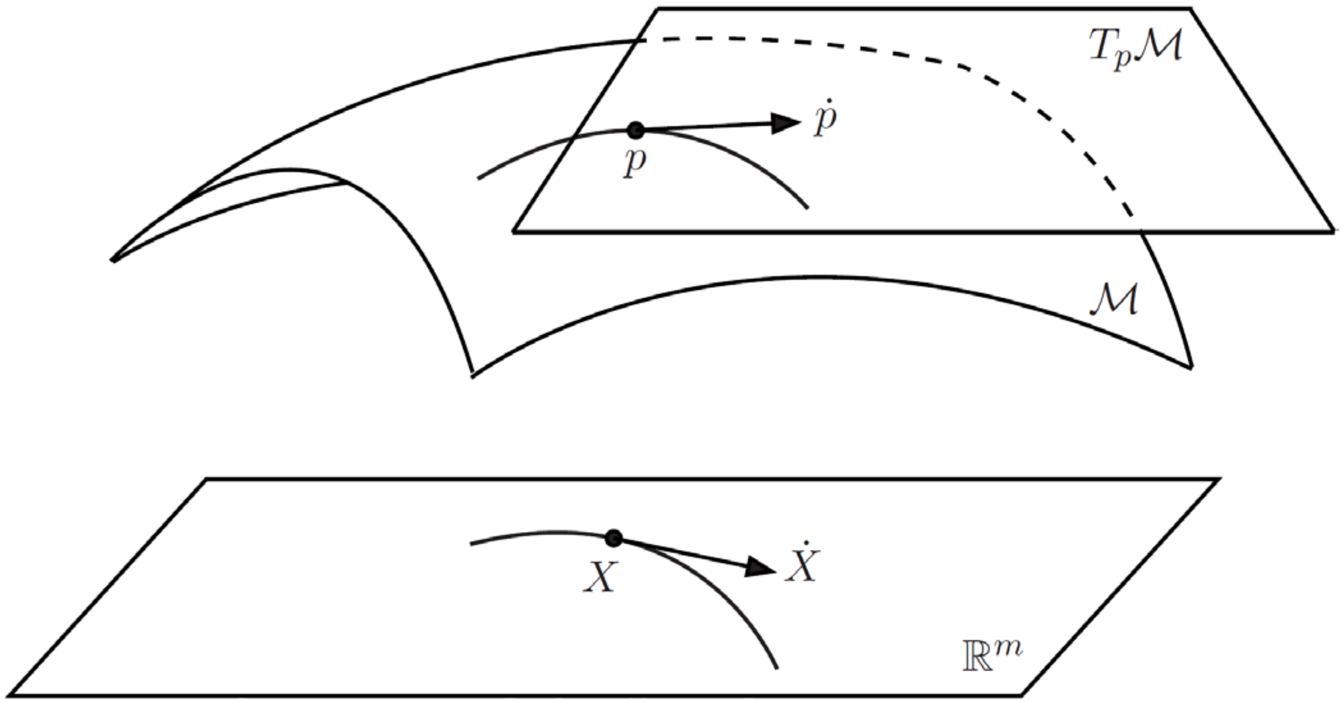
- Length of $\mathcal{C} = \int_0^T \sqrt{\dot{\phi}^2 + \dot{\theta}^2 \sin\phi}\ dt$
- Area of $\mathcal{A} = \iint_A |\sin\phi|\, d\phi\, d\theta$





Manifolds

A **differentiable manifold** is a space that is locally diffeomorphic* to Euclidean space (e.g., a multidimensional surface)



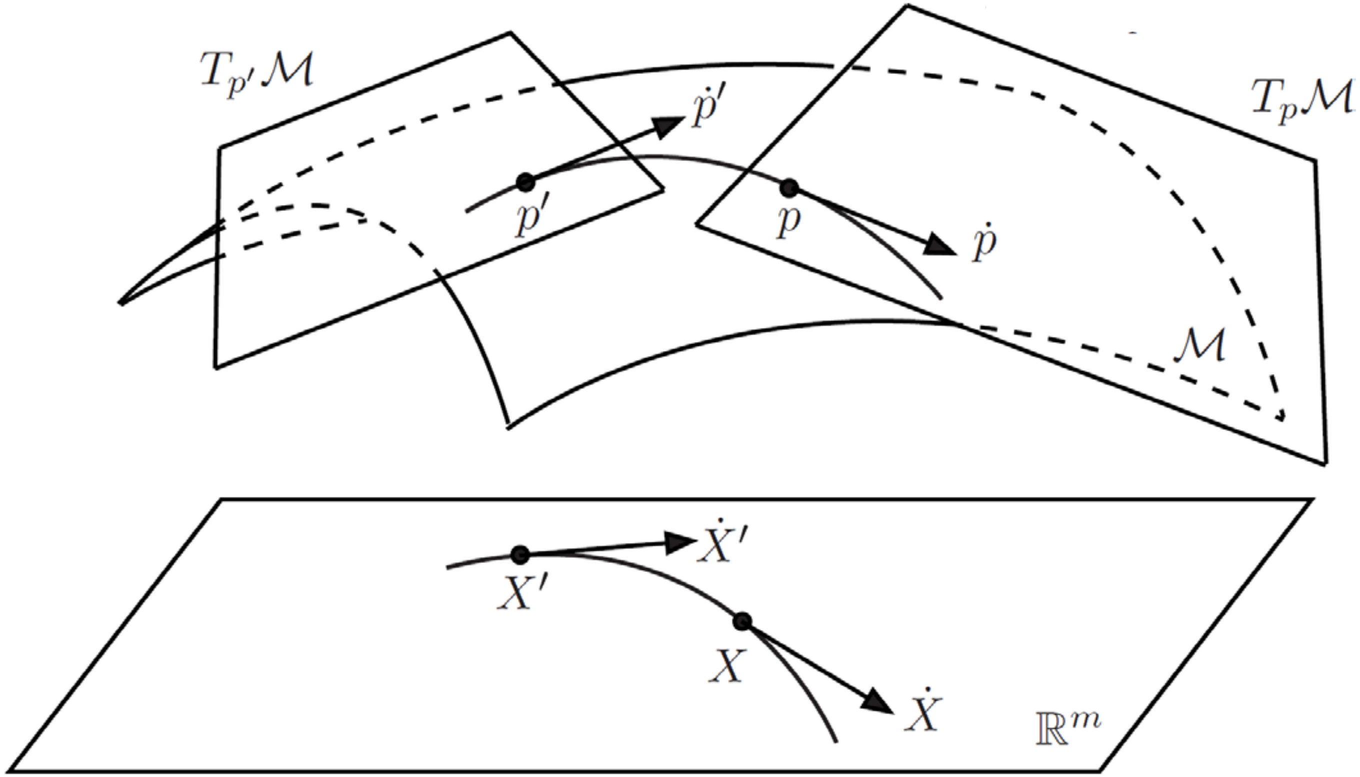
Manifold $\mathcal{M}$

local coordinates x

*Invertible with a differentiable inverse. Essentialy, one can be smoothly deformed into the other.

Riemannian metrics

A **Riemannian metric** is an inner product defined on each tangent space that varies smoothly over $\mathcal{M}$



$$ds^2 = \sum_i \sum_j g_{ij}(x)\, dx^i dx^j$$
$$= dx^T G(x) dx$$

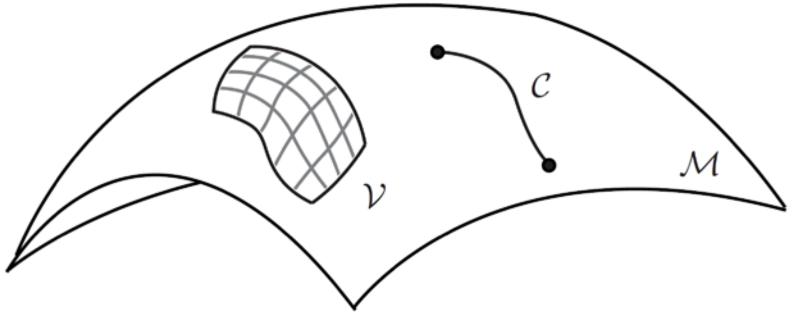
$G(x) \in \mathbb{R}^{m \times m}$
symmetric positive-definite

Calculus on Riemannian manifolds

- Length of a curve C on $\mathcal{M}$:

$$\text{Length} = \int_C ds$$

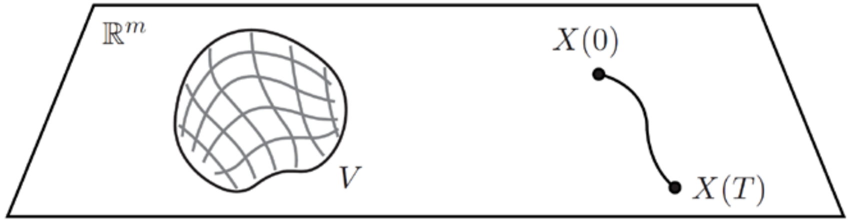
$$= \int_0^T \sqrt{\dot{x}(t)^T G\big(x(t)\big)\dot{x}(t)}\, dt$$



- Volume of a subset $\mathcal{V}$ of $\mathcal{M}$:

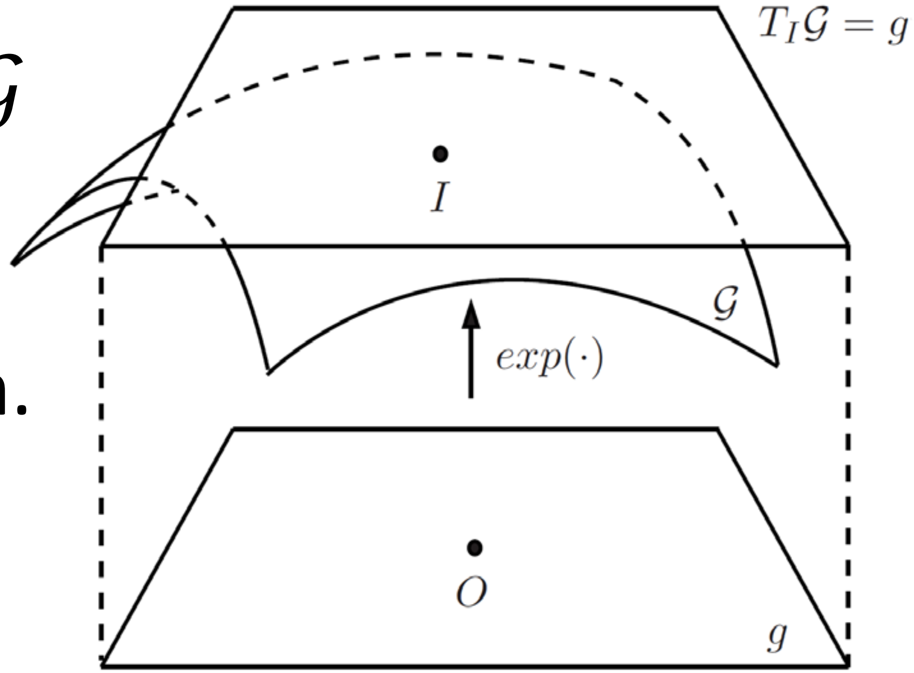
$$\text{Volume} = \int_{\mathcal{V}} dV$$

$$= \int \cdots \int_{\mathcal{V}} \sqrt{\det G(x)}\, dx_1 \cdots dx_m$$

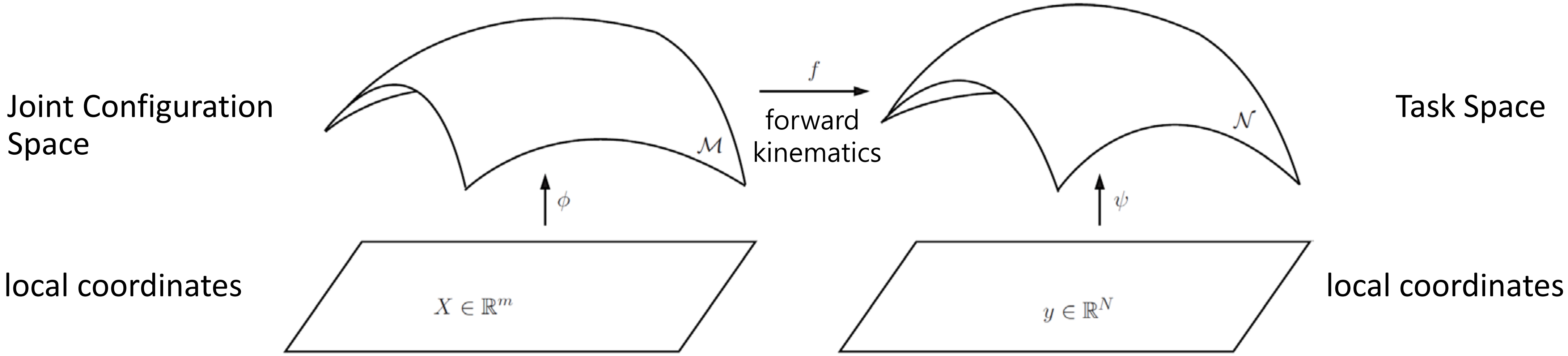


Lie groups

- A manifold that is also an algebraic group is a **Lie group**
- The tangent space at the identity is the **Lie algebra** $\mathcal{g}$.
- The **exponential map** exp: $\mathcal{g} \rightarrow \mathcal{G}$ acts as a set of local coordinates for $\mathcal{G}$
- [Example] GL(n), the set of $n \times n$ nonsingular real matrices, is a Lie group under matrix multiplication.
- Its Lie algebra gl(n) is $\mathbb{R}^{n \times n}$.



Robots and manifolds



$$ds^2 = \sum_i \sum_j g_{ij}(x) dx^i dx^j \qquad ds^2 = \sum_\alpha \sum_\beta h_{\alpha\beta}(y) dy^\alpha dy^\beta$$

- Mappings may be in complicated parametric form
- Sometimes the manifolds are unknown or changing
- Riemannian metrics must be specified on one or both spaces
- Noise models on manifolds may need to be defined

Kinematics and Path Planning

Minimal geodesics on Lie groups

Let $\mathcal{G}$ be a matrix Lie group with Lie algebra $\mathscr{g}$, and let $\langle \cdot , \cdot \rangle$ be an inner product on $\mathscr{g}$. The (left-invariant) minimal geodesic between $X_0, X_1 \in \mathcal{G}$ can be found by solving the following optimal control problem:

$$\min_{U(t)} \int_0^1 \langle U(t), U(t) \rangle dt$$

Subject to $\dot{X} = XU, X(t) \in \mathcal{G}, U(t) \in \mathscr{g}$, with boundary condition $X(0) = X_0, X(1) = X_1$.

Minimal geodesics on Lie groups

For the choice $\langle U, V \rangle = \mathrm{Tr}(U^T V)$, the solution must satisfy

$$\dot{X} = XU$$

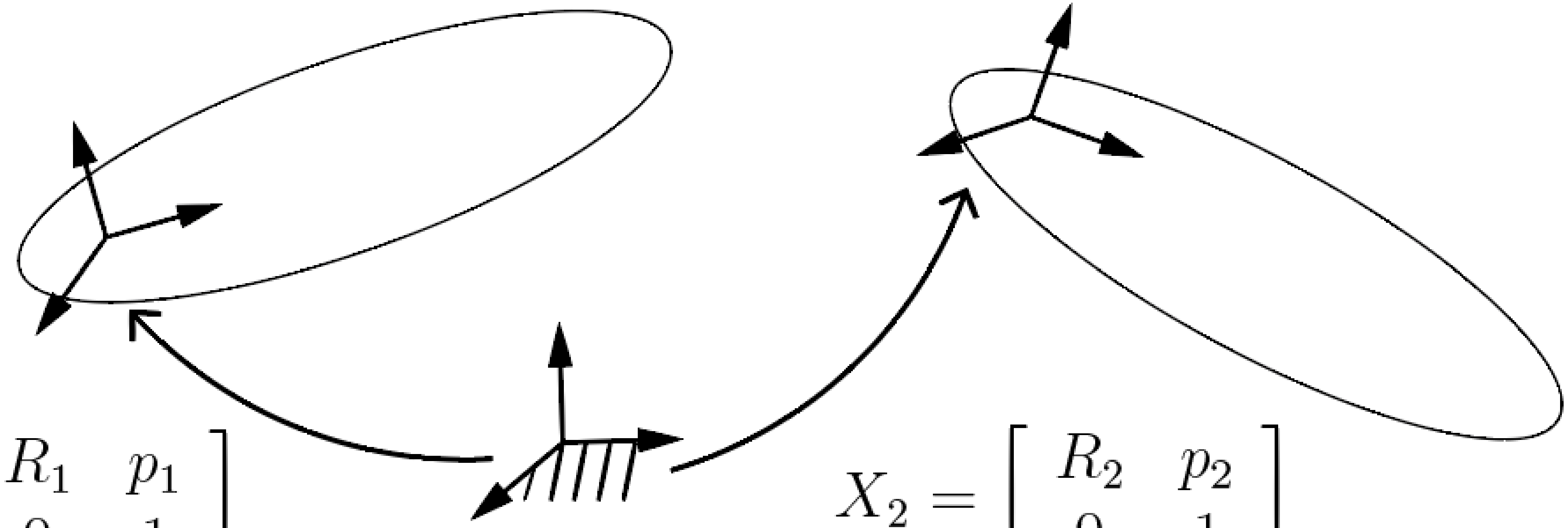
$$\dot{U} = U^T U - U U^T = [U^T, U].$$

If the objective function is replaced by $\int_0^1 \langle \dot{U}, \dot{U} \rangle \, dt$, the solution must satisfy $\dot{X} = XU$ and

$$\ddot{U} = [U^T, \ddot{U}].$$

Minimal geodesics, and minimum acceleration paths, can be found on various Lie groups by solving the above two-point boundary value problems. **For SO(n) and SE(n) the minimal geodesics are particularly simple to characterize.**

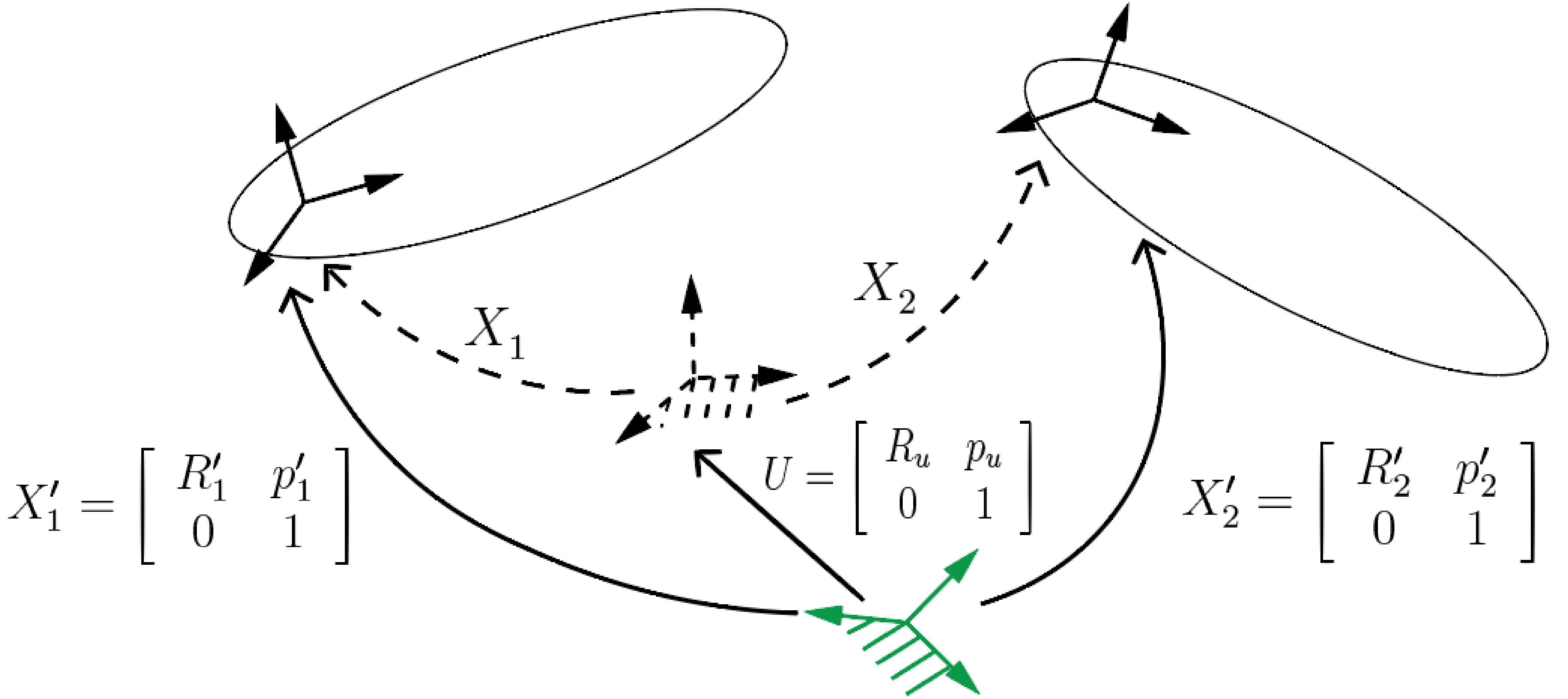
Distance metrics on SE(3)



$$X_1 = \begin{bmatrix} R_1 & p_1 \\ 0 & 1 \end{bmatrix} \qquad X_2 = \begin{bmatrix} R_2 & p_2 \\ 0 & 1 \end{bmatrix}$$

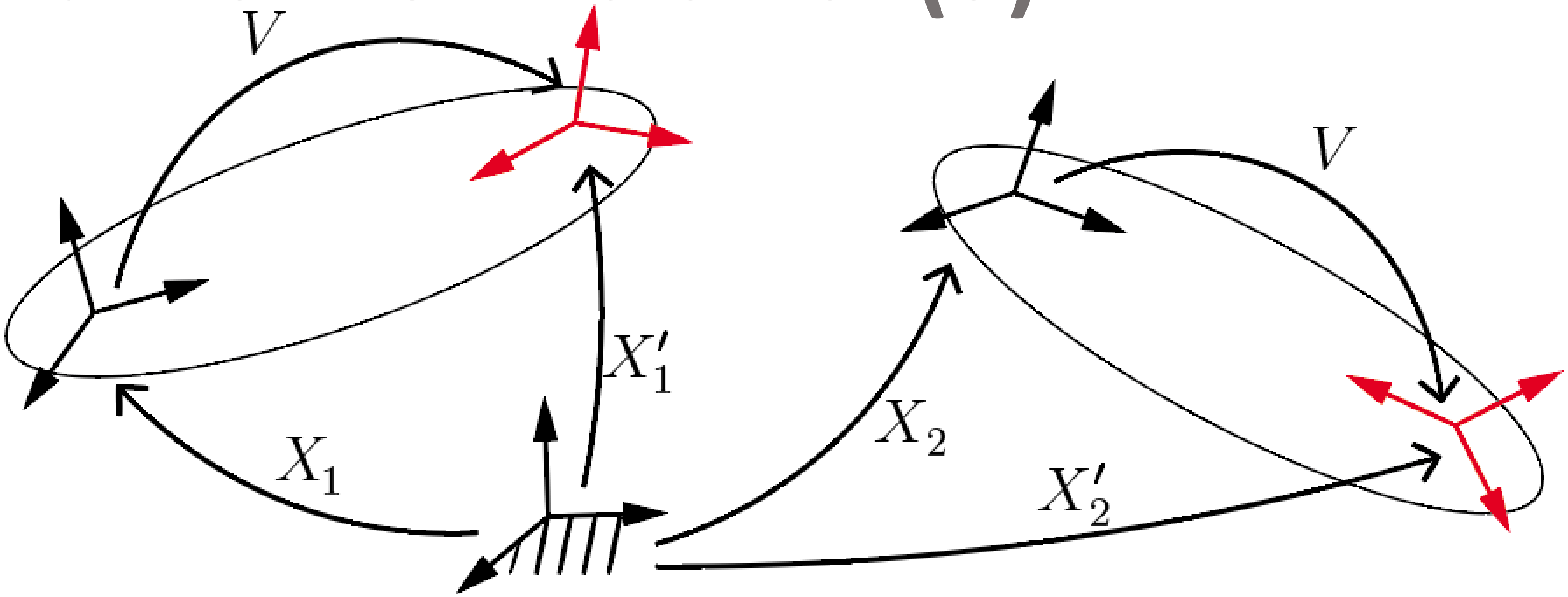
$d(X_1, X_2) =$ distance between X_1 and X_2

Distance metrics on SE(3)



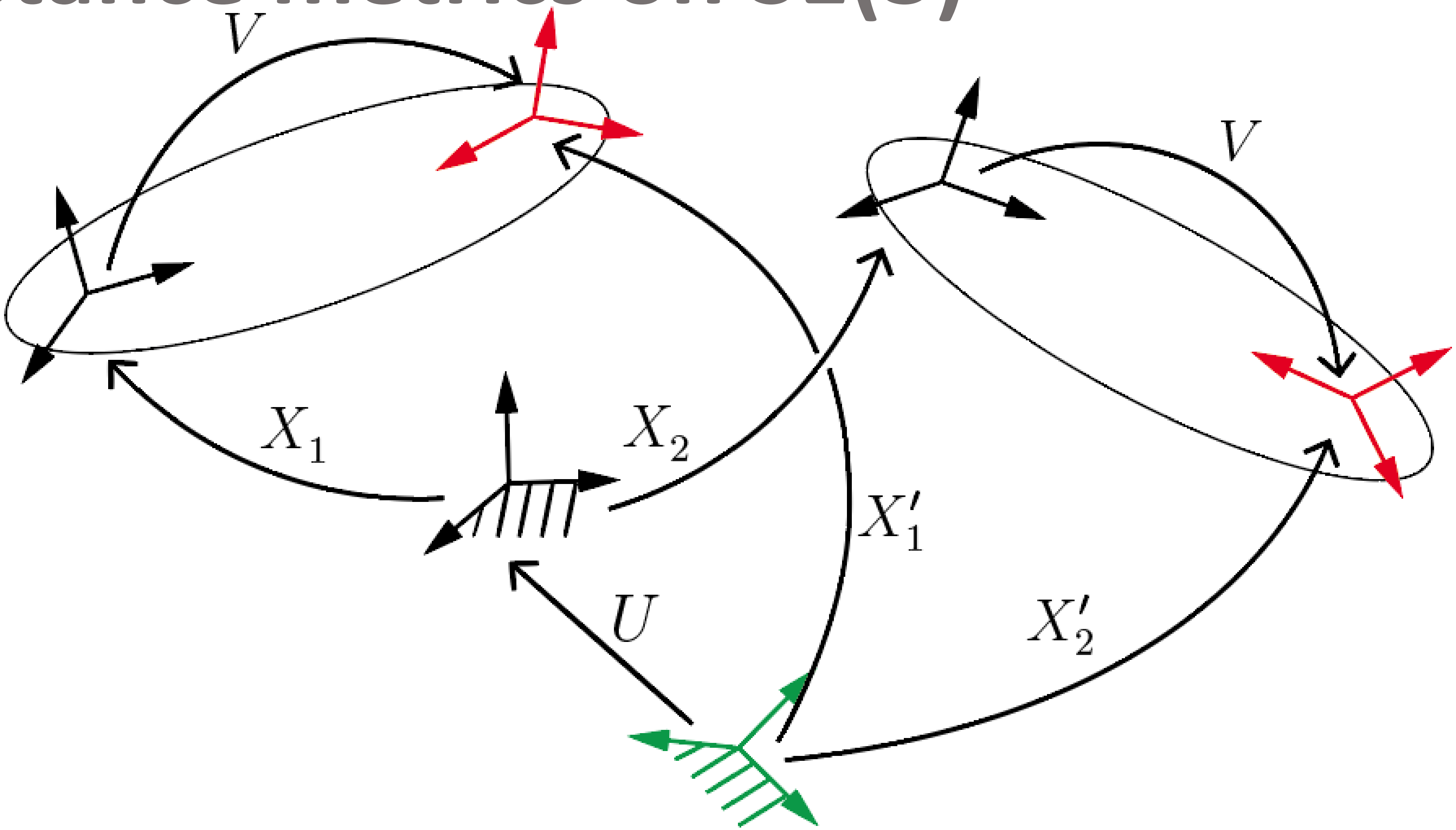
If $d({X_1}', {X_2}') = d(UX_1, UX_2) = d(X_1, X_2)$ for all U ∈ $SE(3)$,
$d(\cdot,\cdot)$ is a **left-invariant distance metric**.

Distance metrics on SE(3)



If $d(X_1{}', X_2{}') = d(X_1 V, X_2 V) = d(X_1, X_2)$ for all V $\in SE(3)$,
$d(\cdot\,,\cdot)$ is a **right-invariant** distance metric.

Distance metrics on SE(3)



If $d(X_1{}', X_2{}') = d(UX_1V, UX_2V) = d(X_1, X_2)$ for all U, V $\in SE(3)$, $d(\cdot, \cdot)$ is a **bi-invariant** distance metric.

Some facts about distance metrics on SE(3)

- **Bi-invariant metrics on SO(3) exist:** some simple ones are

$$d(R_1, R_2) = ||\log(R_1^T R_2)|| = \phi \in [0, \pi]$$

$$d(R_1, R_2) = \left||R_1 - R_2|\right| = \sqrt{3 - Tr(R_1^T R_2)} = 1 - \cos\phi$$

- **No bi-invariant metric exists on SE(3)**
- **Left- and right-invariant metrics exist on SE(3):** a simple left-invariant metric is $d(X_1, X_2) = d(R_1, R_2) + \left||p_1 - p_2|\right|$
- Any distance metric on SE(3) depends on the **choice of length scale for physical space**

Remarks

- (Too) many papers on SE(3) distance metrics have been written!
- Robots are doing fine even without bi-invariance (but make sure the metric you use is left-invariant)
- Notwithstanding J. Duffy's claims about "The fallacy of modern hybrid control theory that is based on "orthogonal complements" of twist and wrench spaces," *J. Robotic Systems*, 1989, hybrid force-position control seems to be working well.

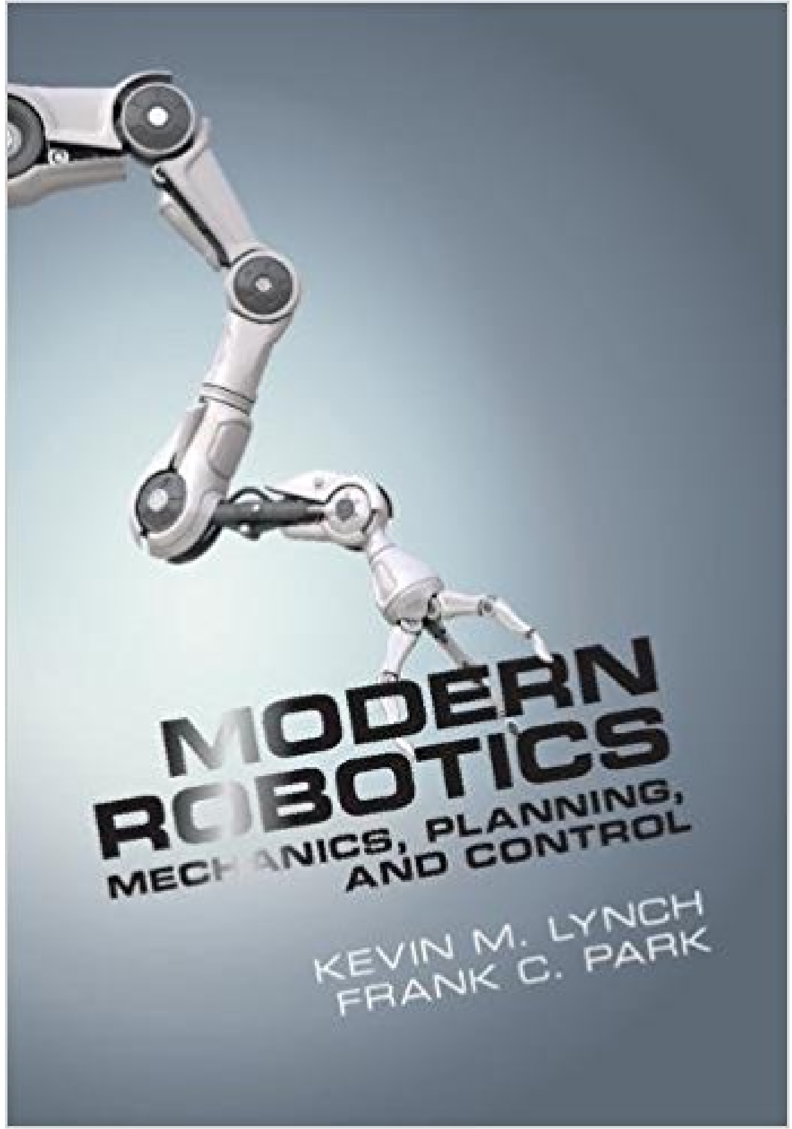
Robot kinematics: modern screw theory

- The **product-of-exponentials (PoE) formula** for open kinematic chains (Brockett 1989) puts on a more sure footing the classical screw-theoretic tools for kinematic modeling and analysis:

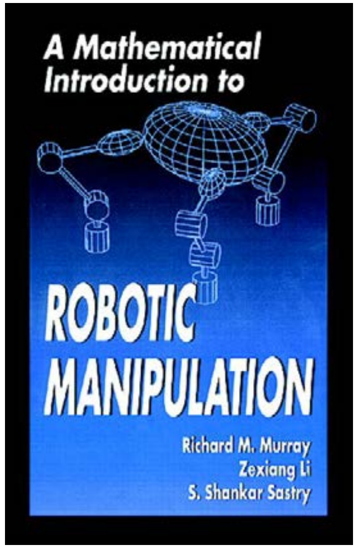
$$T = e^{[S_1]\theta_1} \cdots e^{[S_n]\theta_n} M$$

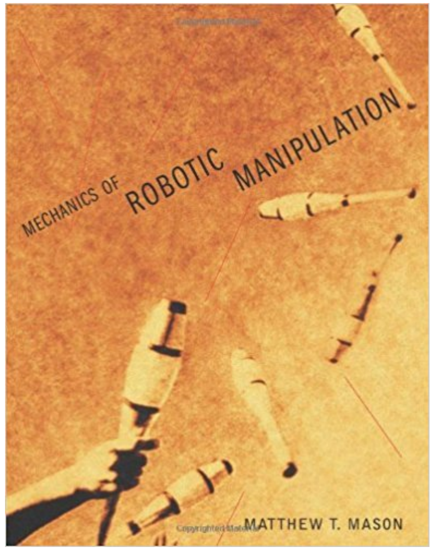
- Some advantages: no link frames needed, intuitive physical meaning, easy differentiation, can apply well-known machinery and results of general matrix Lie groups, etc.
- It is mystifying to me why the PoE formula is not more widely taught and used, and why people still cling to their Denavit-Hartenberg parameters.

Some relevant robotics textbooks



Targeted to upper-level undergraduates, can be complemented by more advanced textbooks like *A Mathematical Introduction to Robotic Manpulation* (Murray, Li, Sastry) and *Mechanics of Robot Manipulation* (Mason). Free PDF available at ***http://modernrobotics.org***

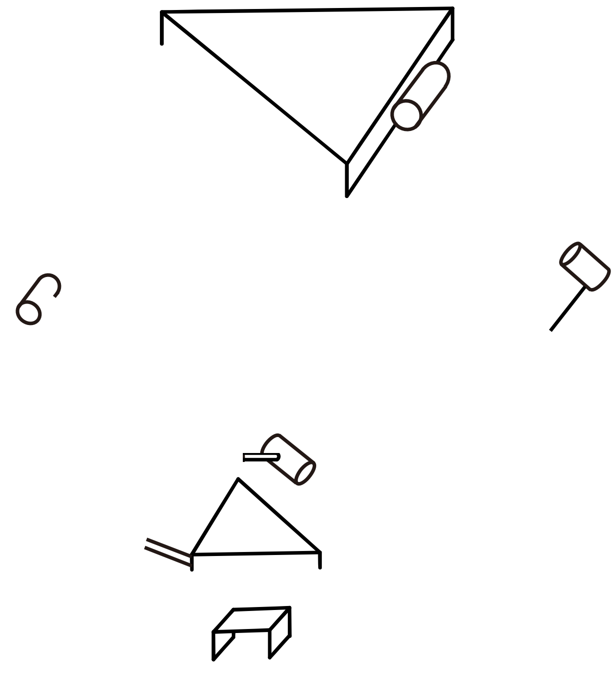




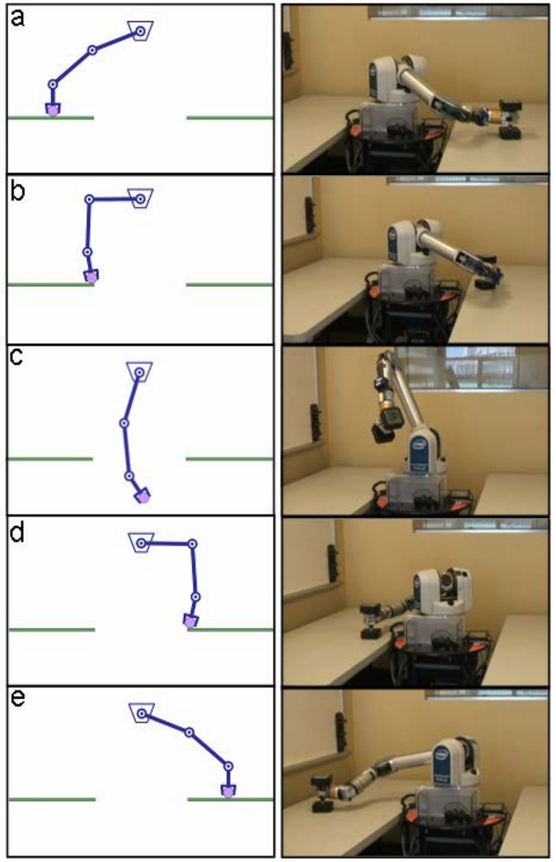
Closed chain kinematics

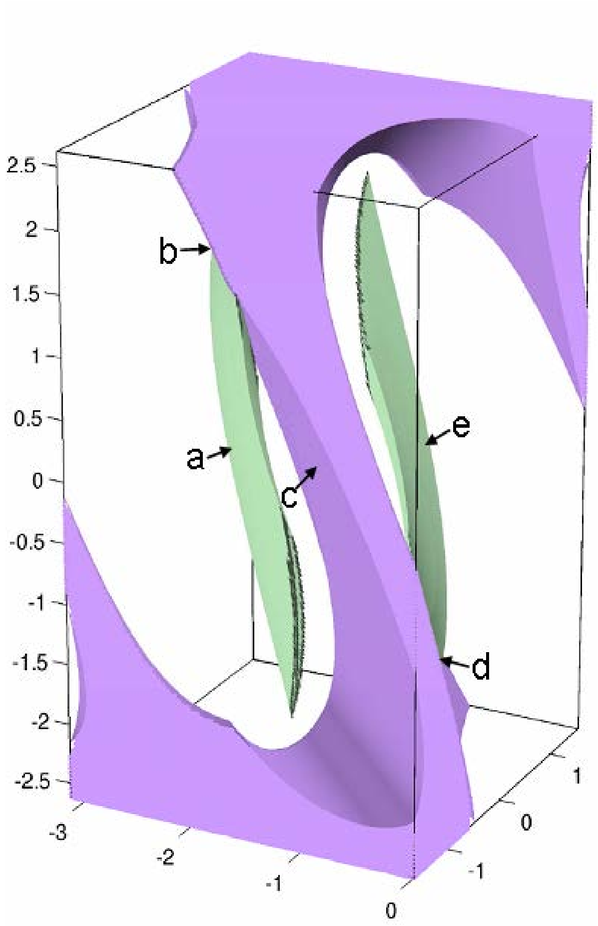
- Closed chains typically have curved configuration spaces, and can be under- or over-actuated. Their singularity behavior is also more varied and subtle.

- Differential geometric methods have been especially useful in their analysis: representing the forward kinematics $f(M,g) \rightarrow (N,h)$as a mapping between Riemannian manifolds, **manipulability** and **singularity analysis** can be performed via analysis of the pullback form (in local coordinates, $J^T HJG^{-1}$).



Path planning on constraint manifolds

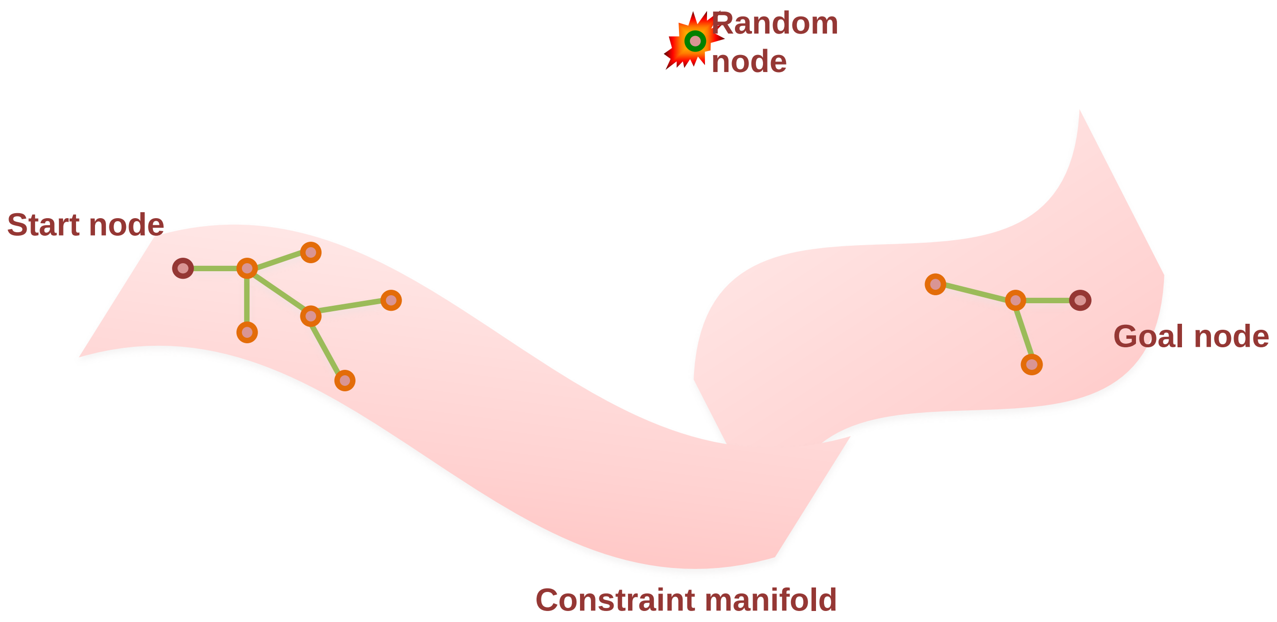




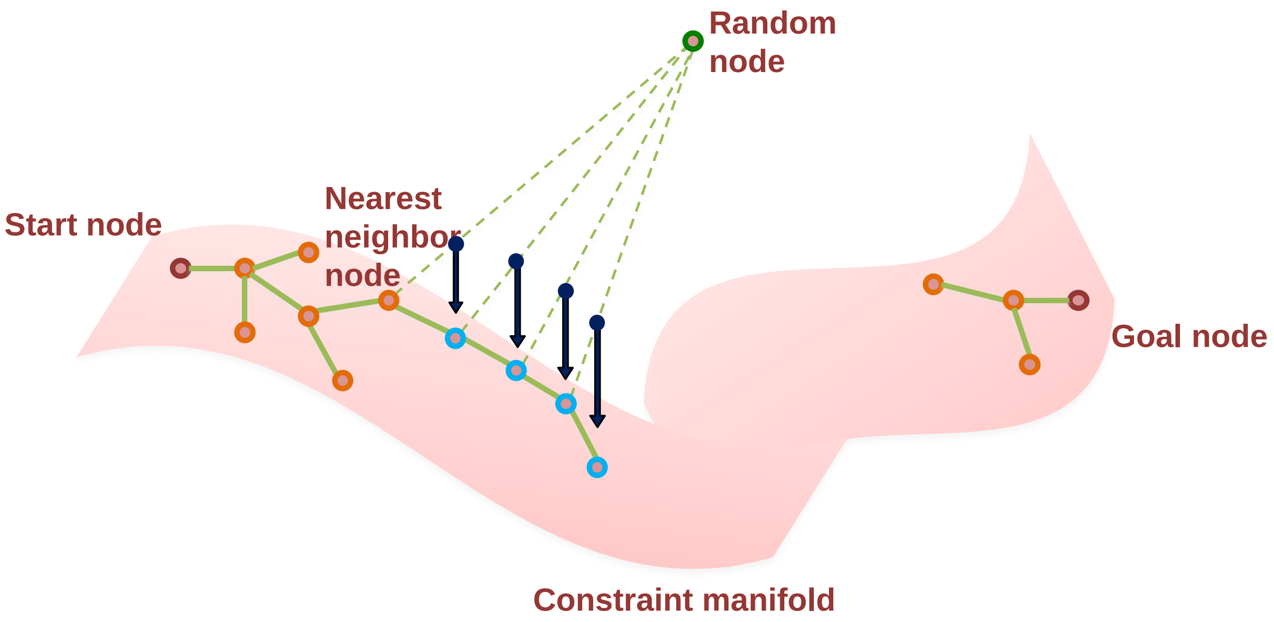
Path planning for robots subject to holonomic constraints (e.g., closed chains, contact conditions). The configuration space is a curved manifold whose structure we do not exactly know in advance.

Dmitry Berenson et al, "Manipulation planning on constraint manifolds," *ICRA 2009.*

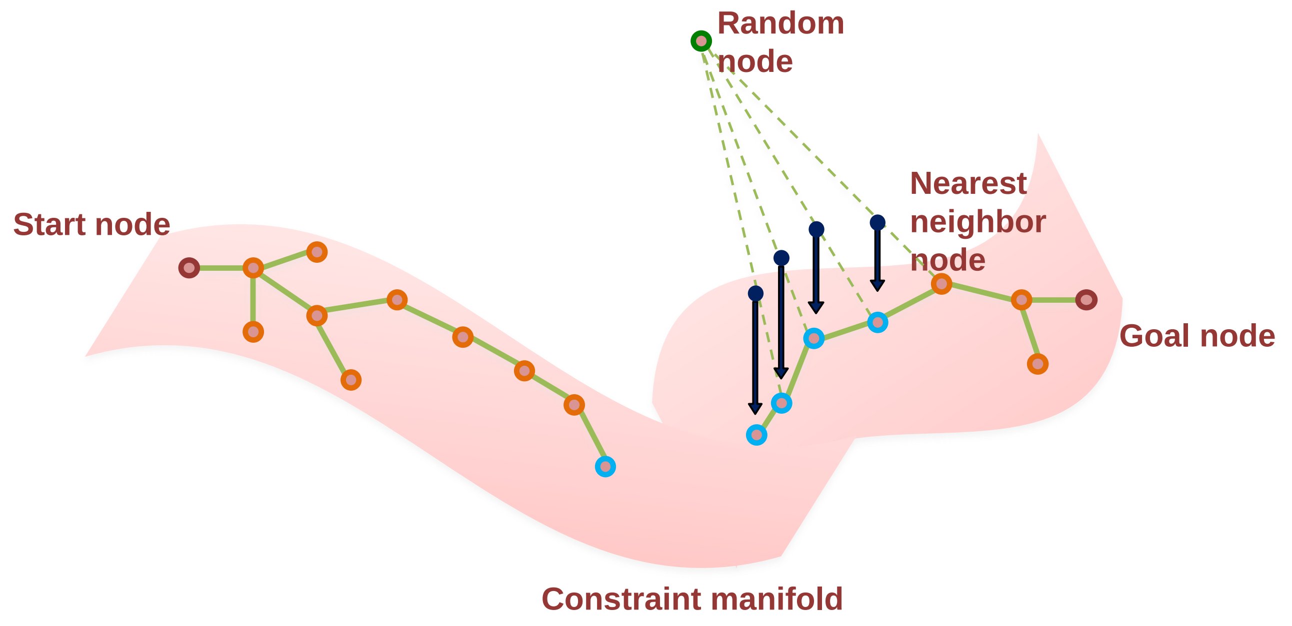
A simple RRT sampling-based algorithm



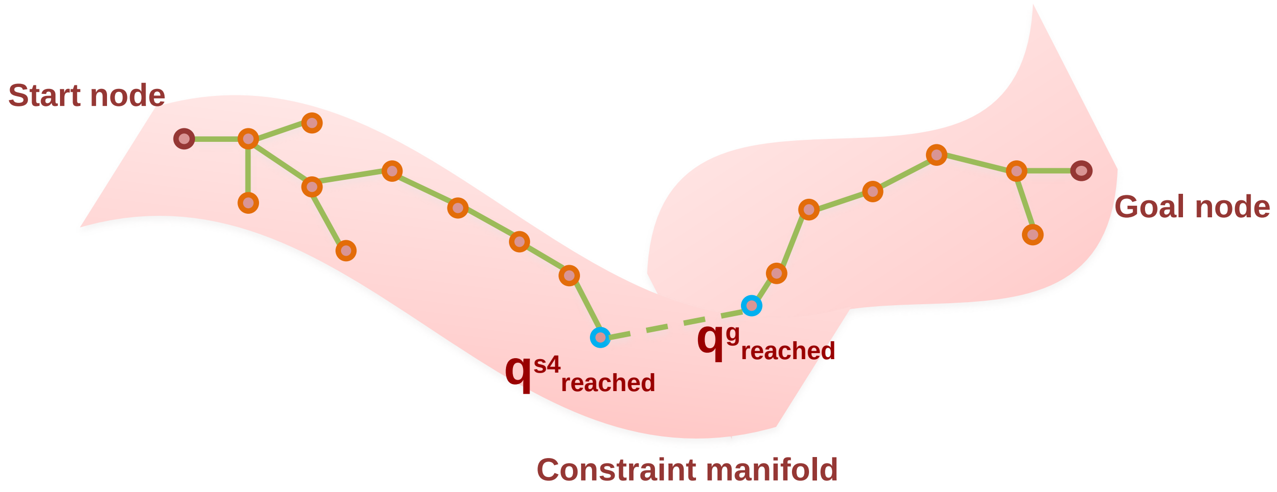
A simple RRT sampling-based algorithm



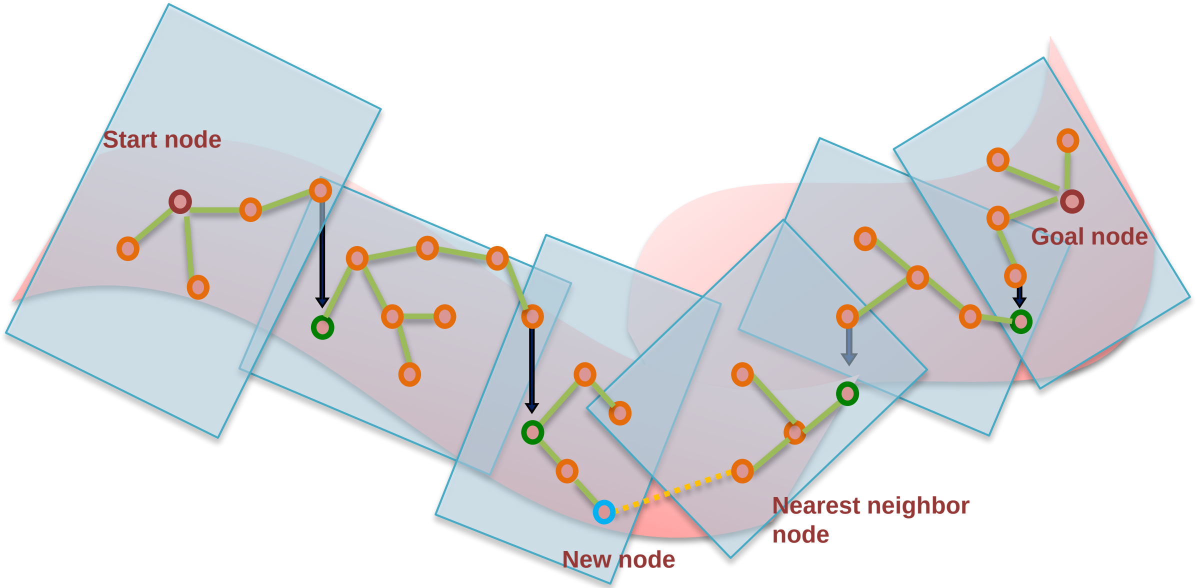
A simple RRT sampling-based algorithm



A simple RRT sampling-based algorithm



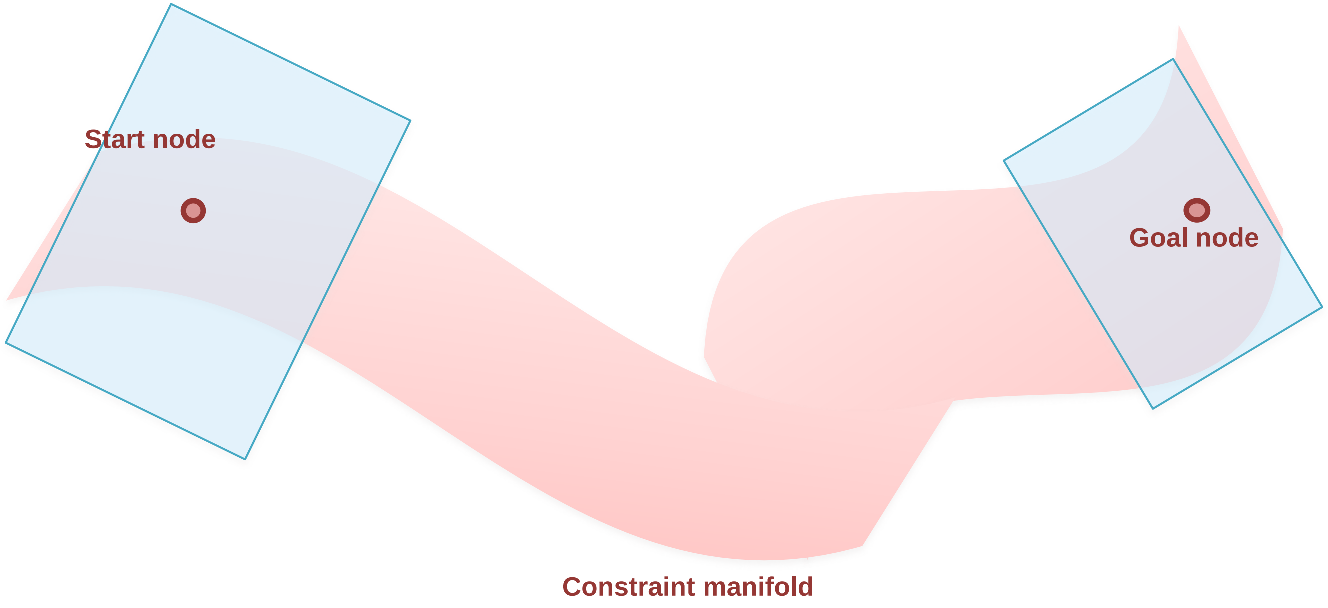
Tangent Bundle RRT (Kim et al 2016)



Tangent Bundle RRT: An RRT algorithm for planning on curved configuration spaces:

- Trees are first propagated on the tangent bundle
- Local curvature information is used to grow the tangent space trees to an appropriate size.

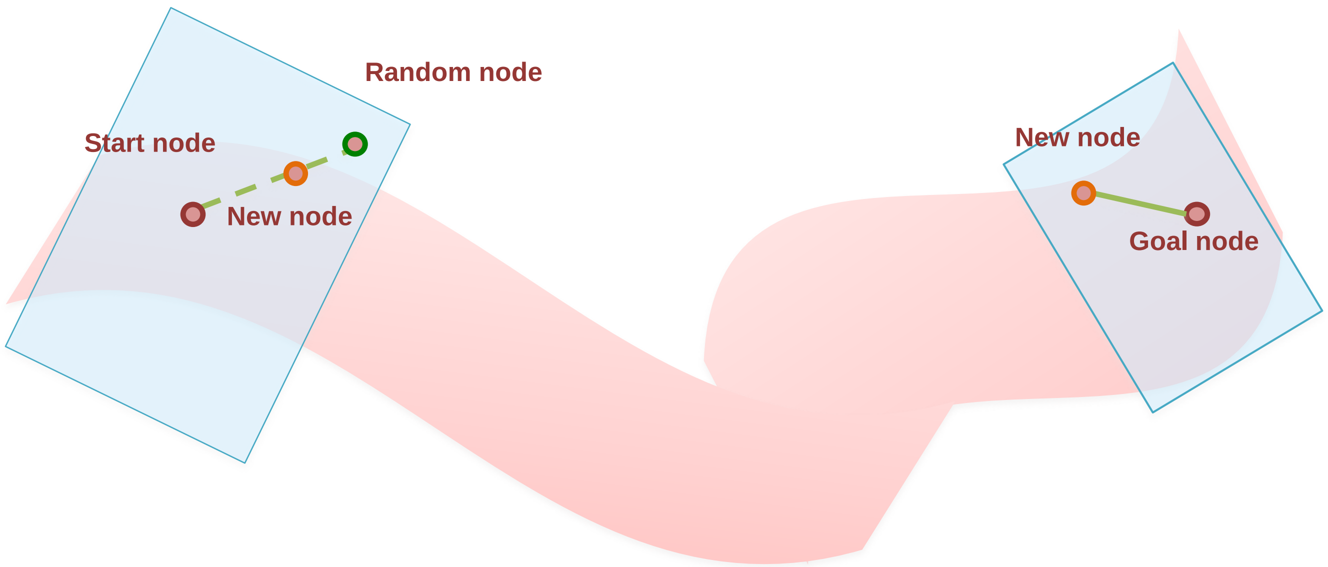
Tangent Bundle RRT



Initializing

- Start and goal nodes assumed to be on constraint manifold.
- Tangent spaces are constructed at start and goal nodes.

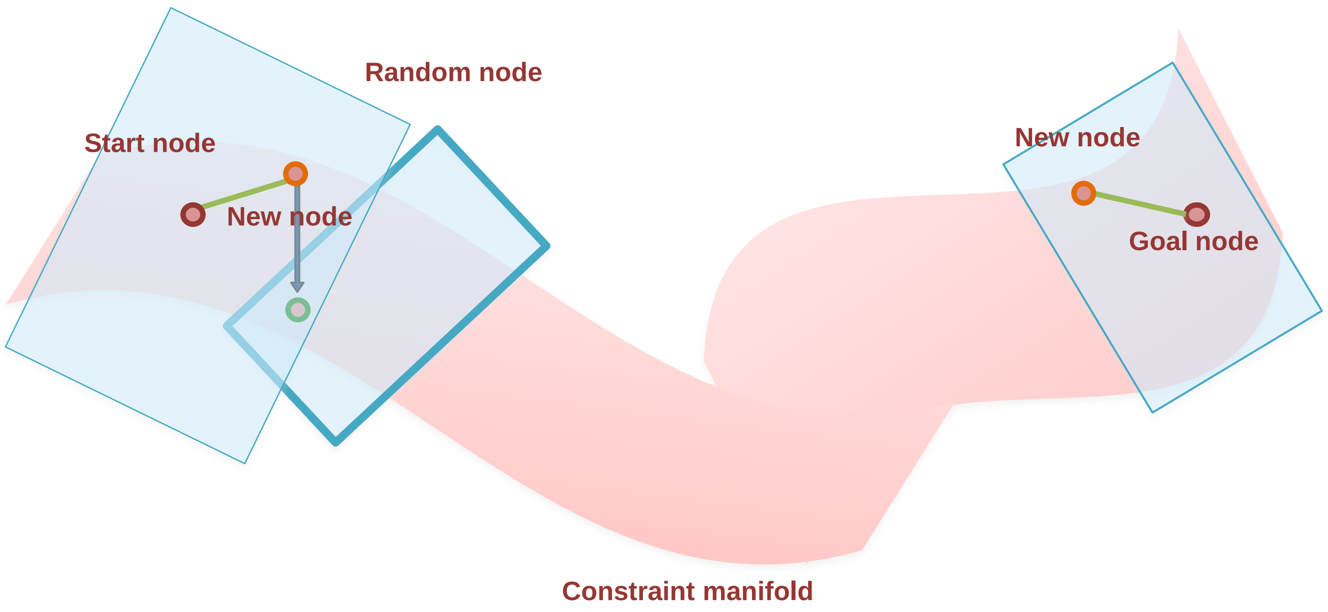
Tangent Bundle RRT



Random sampling on tangent spaces:

- Generate a random sample node on a tangent space.
- Find the nearest neighbor node in the tangent space and take a single step of fixed size toward random target node.
- Find the nearest neighbor node on the opposite tree and then extend tree via tangent space.

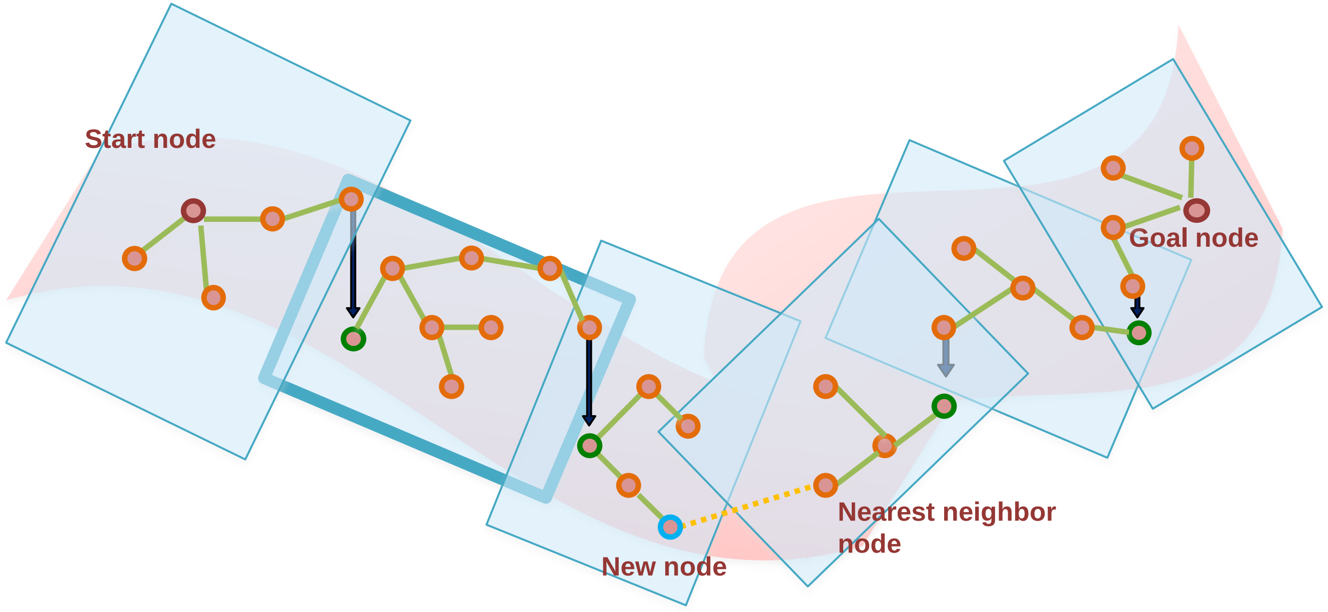
Tangent Bundle RRT



Creating a new tangent space:

- When the distance to the constraint manifold exceeds a certain threshold, project the extended node to the constraint manifold and create a new bounded tangent space.

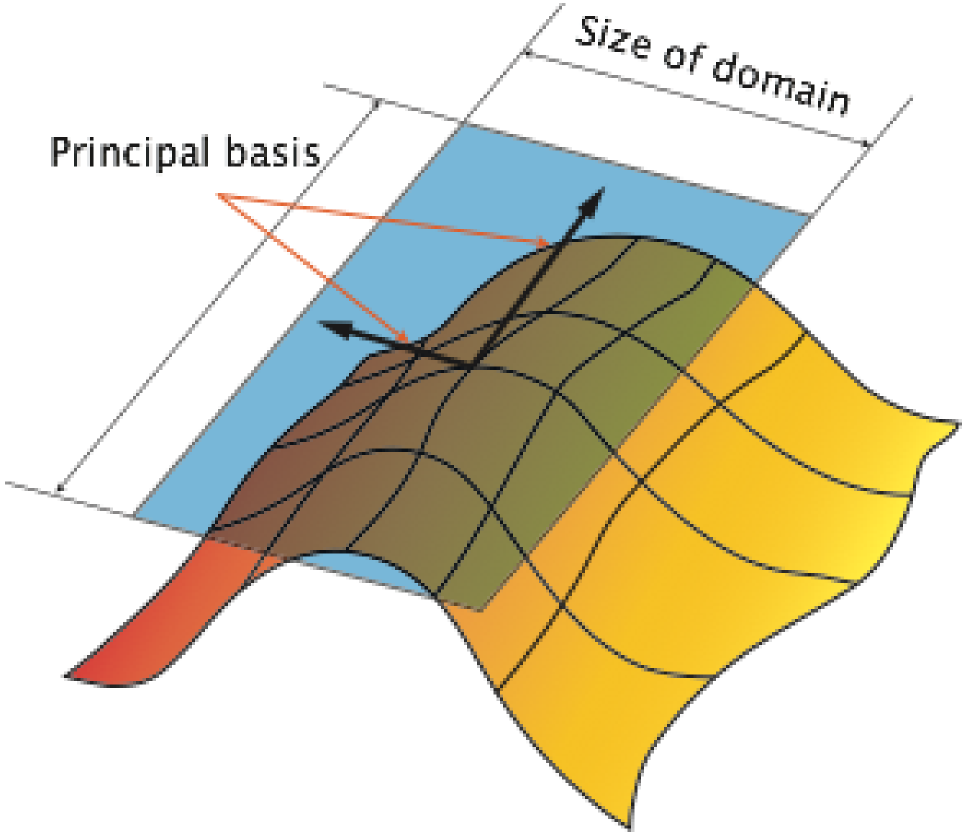
Tangent Bundle RRT



Select a tangent space using a size-biased function (roulette selection): For each tangent space, assigned a fitness value that is

- proportional to the size of the tangent space and,
- Inversely proportional to the number of nodes belonging to the tangent space.

Tangent bundle RRT



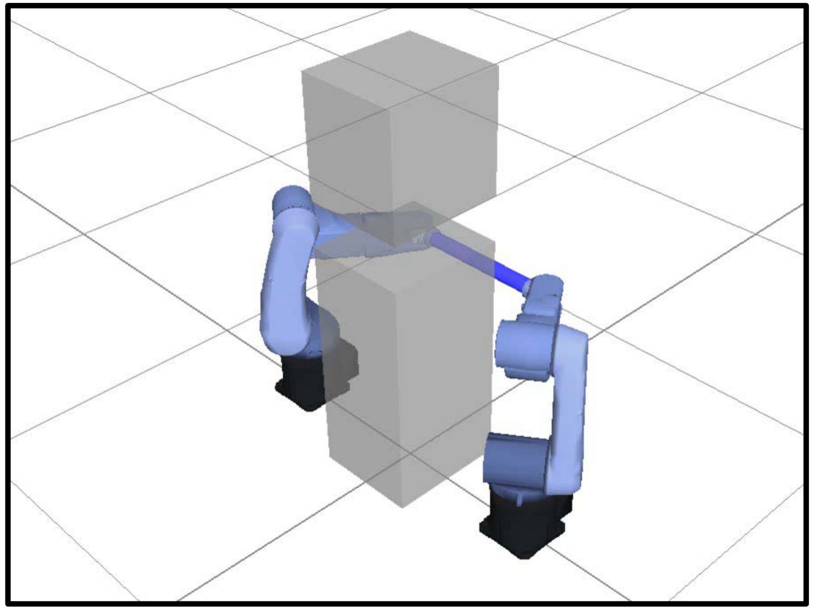
Constructing a bounded tangent space:

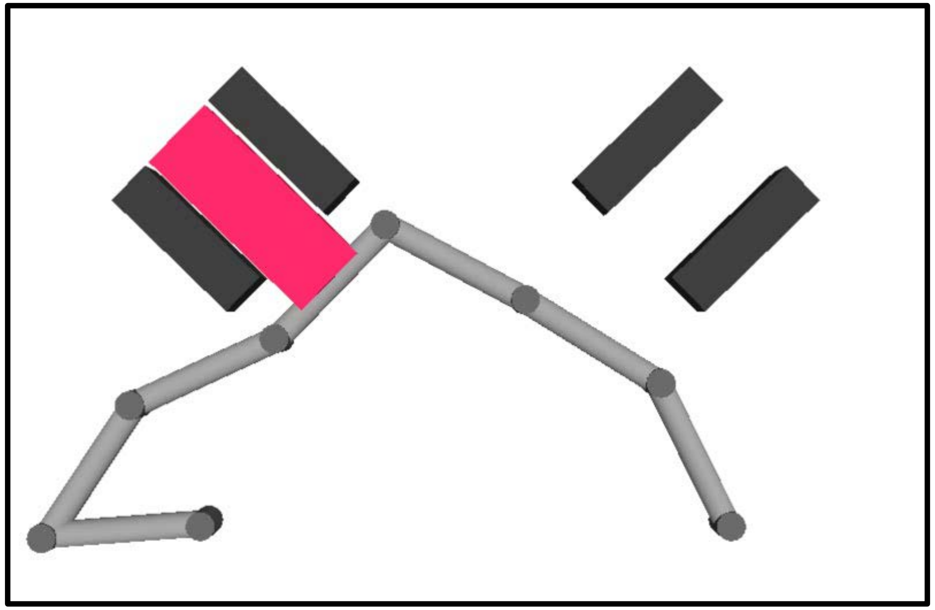
- Use local curvature information to find the principal basis of the tangent space, and to bound the tangent space domain.
- Principal curvatures and principal vectors can be computed from the second fundamental form of the constraint manifold (details need to be worked out if there is more than one normal direction)
- If the principal curvatures are close to zero, the manifold is nearly flat, and thus relatively larger steps can be taken along the corresponding principal directions

Tangent bundle RRT

Q: Is the extra computation and bookkeeping worth it?

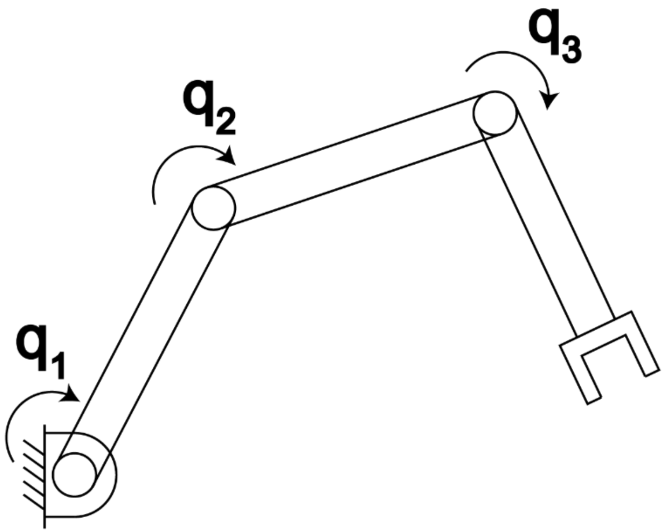
A: It's highly problem-dependent, but for higher-dimensional systems, it seems so.



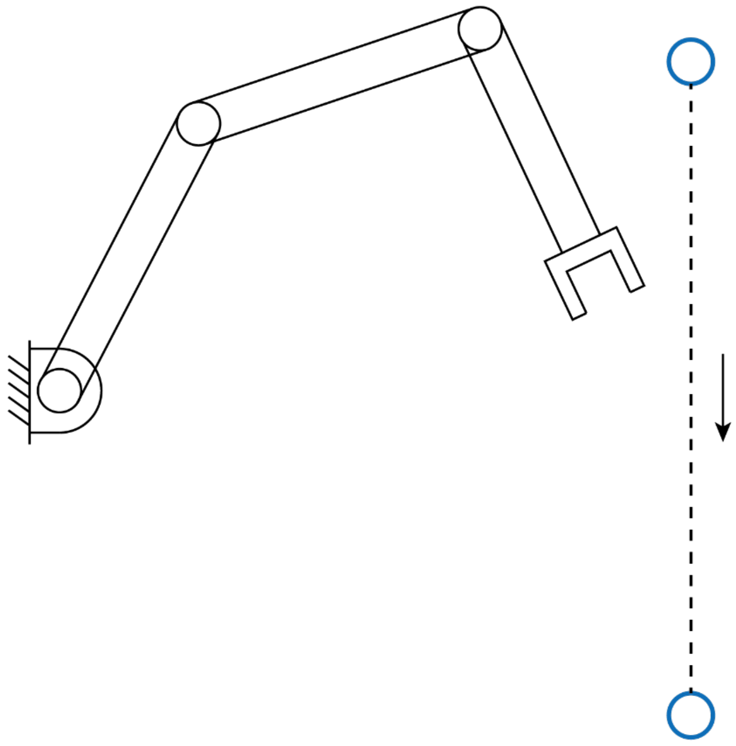


Planning on Foliations

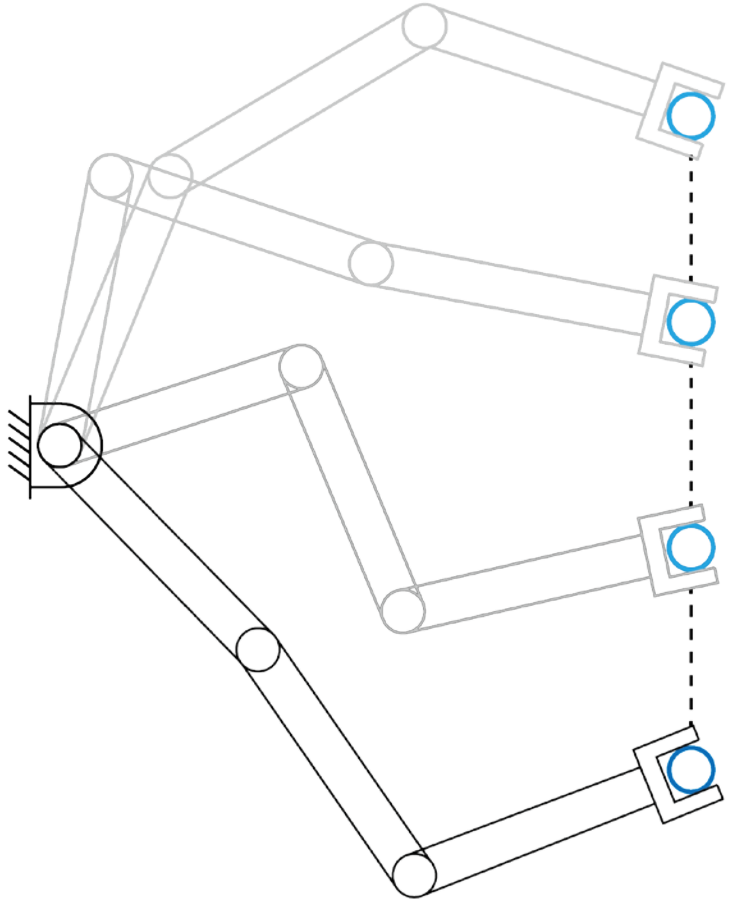
Planning on Foliations



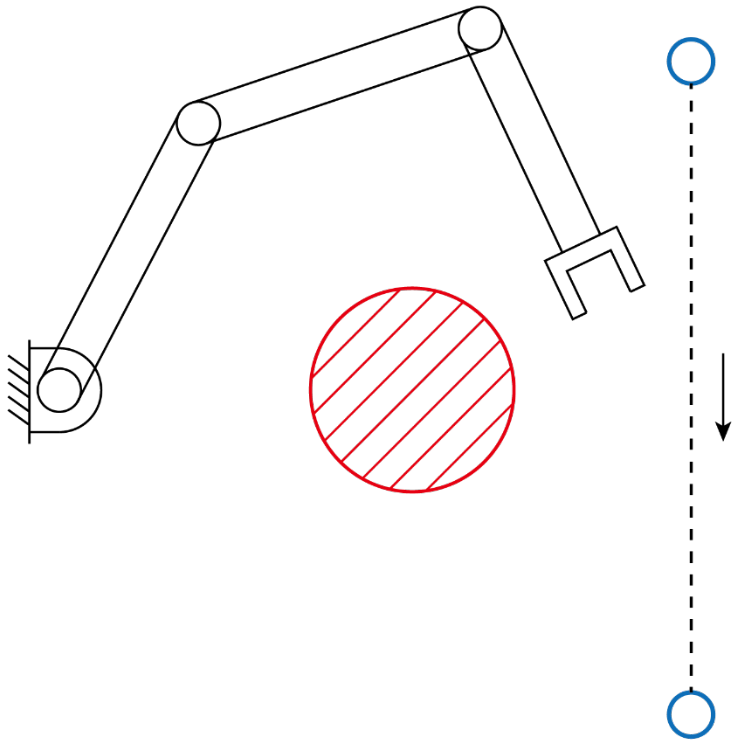
Planning on Foliations



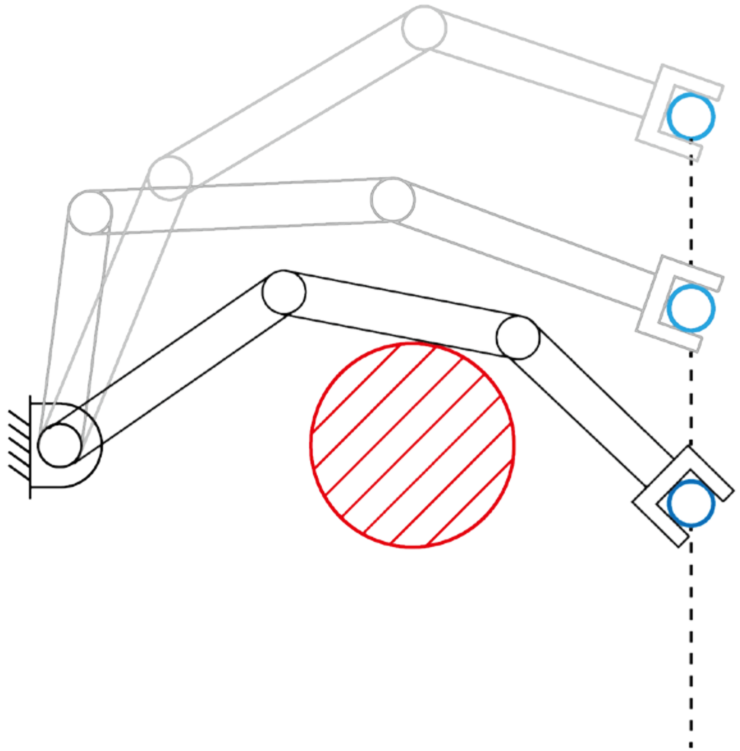
Planning on Foliations



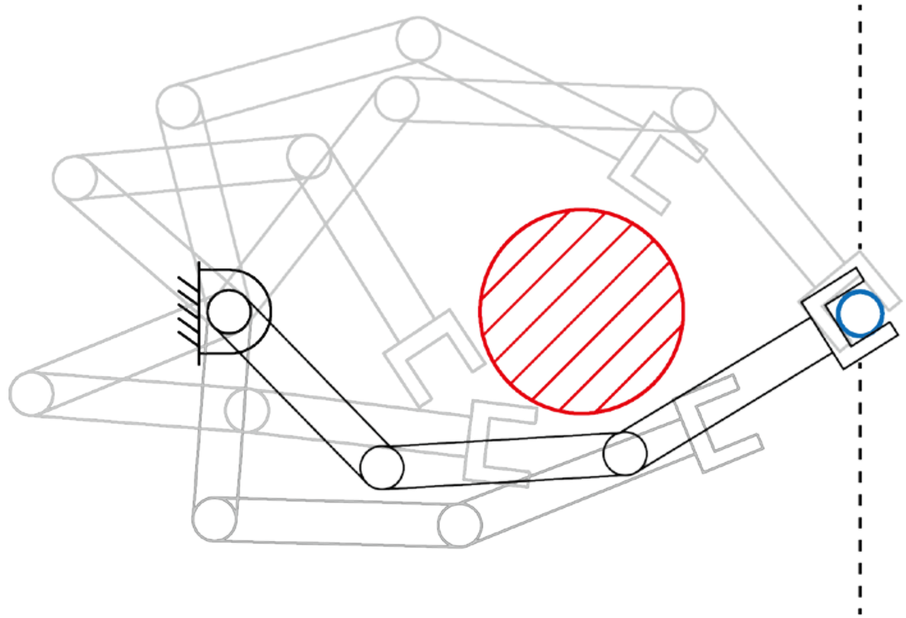
Planning on Foliations



Planning on Foliations

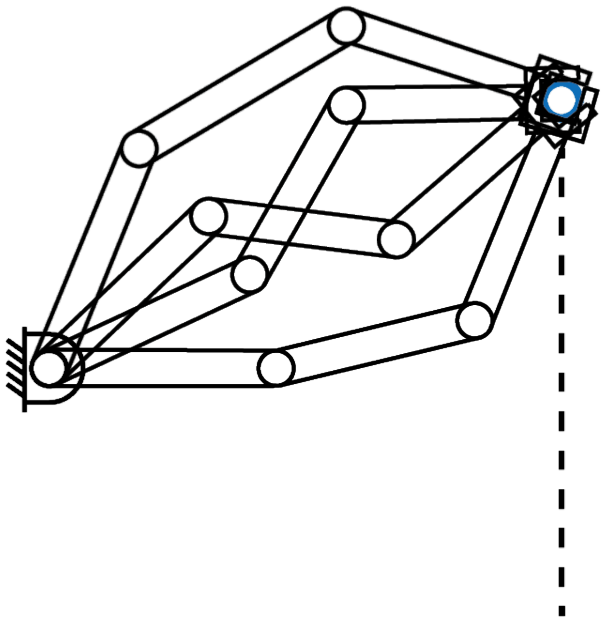


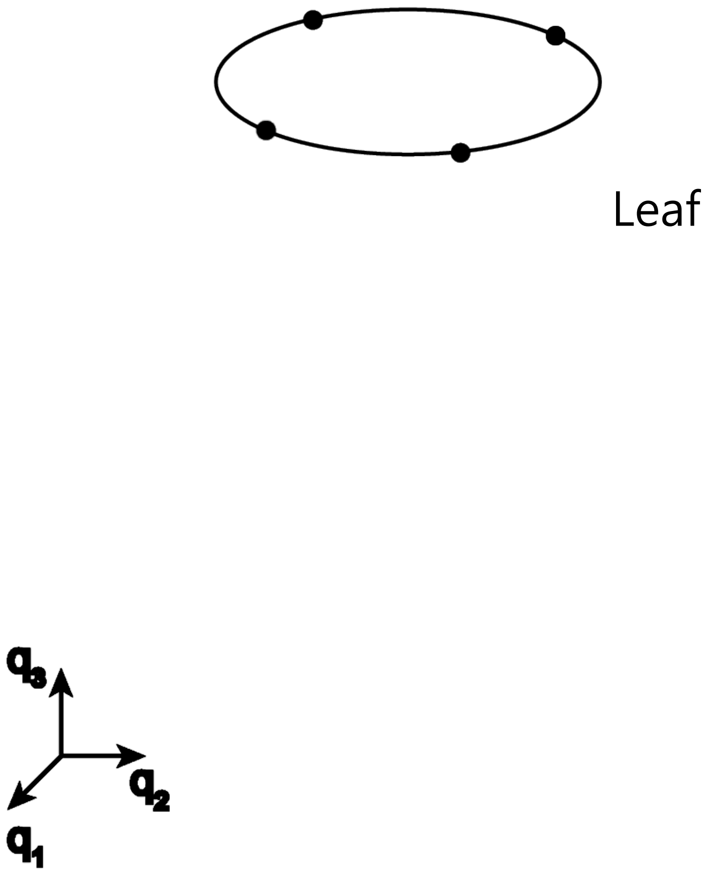
Planning on Foliations



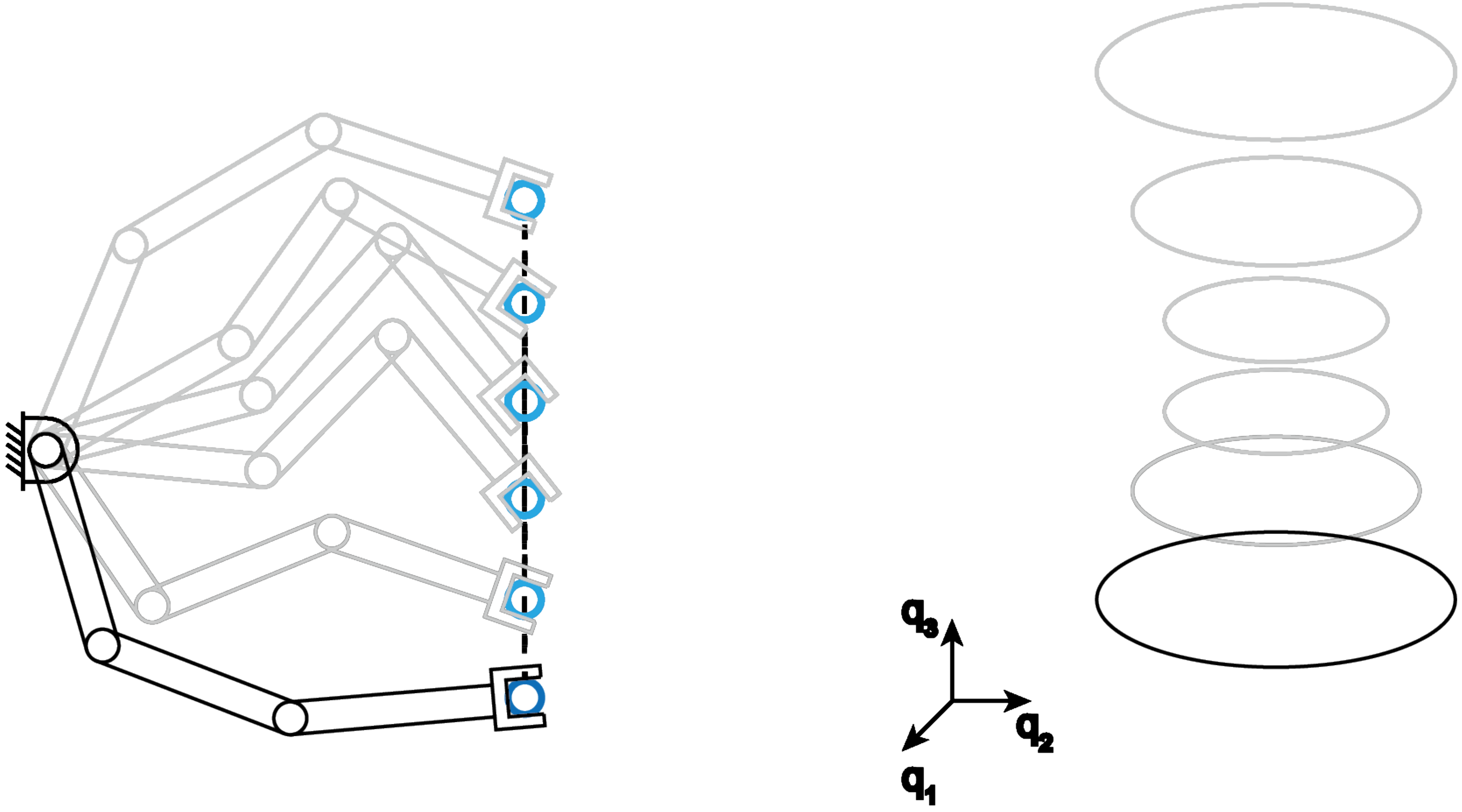
Jump motion

Planning on Foliations

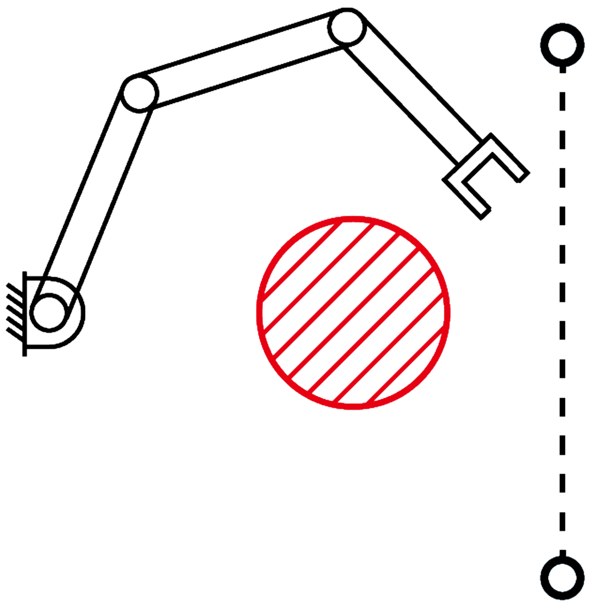


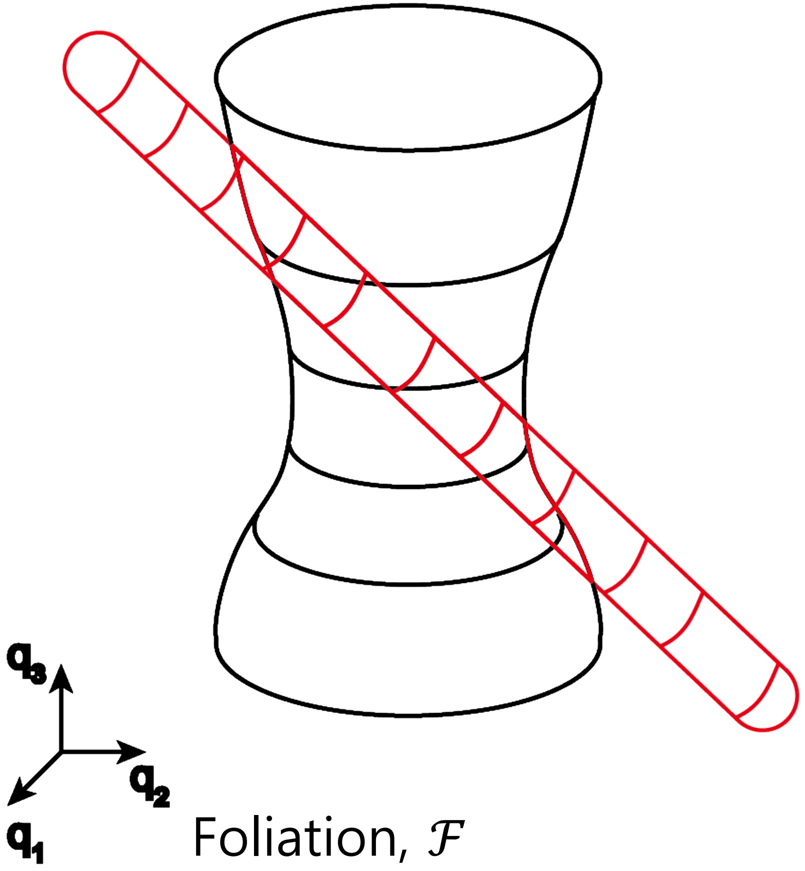


Planning on Foliations

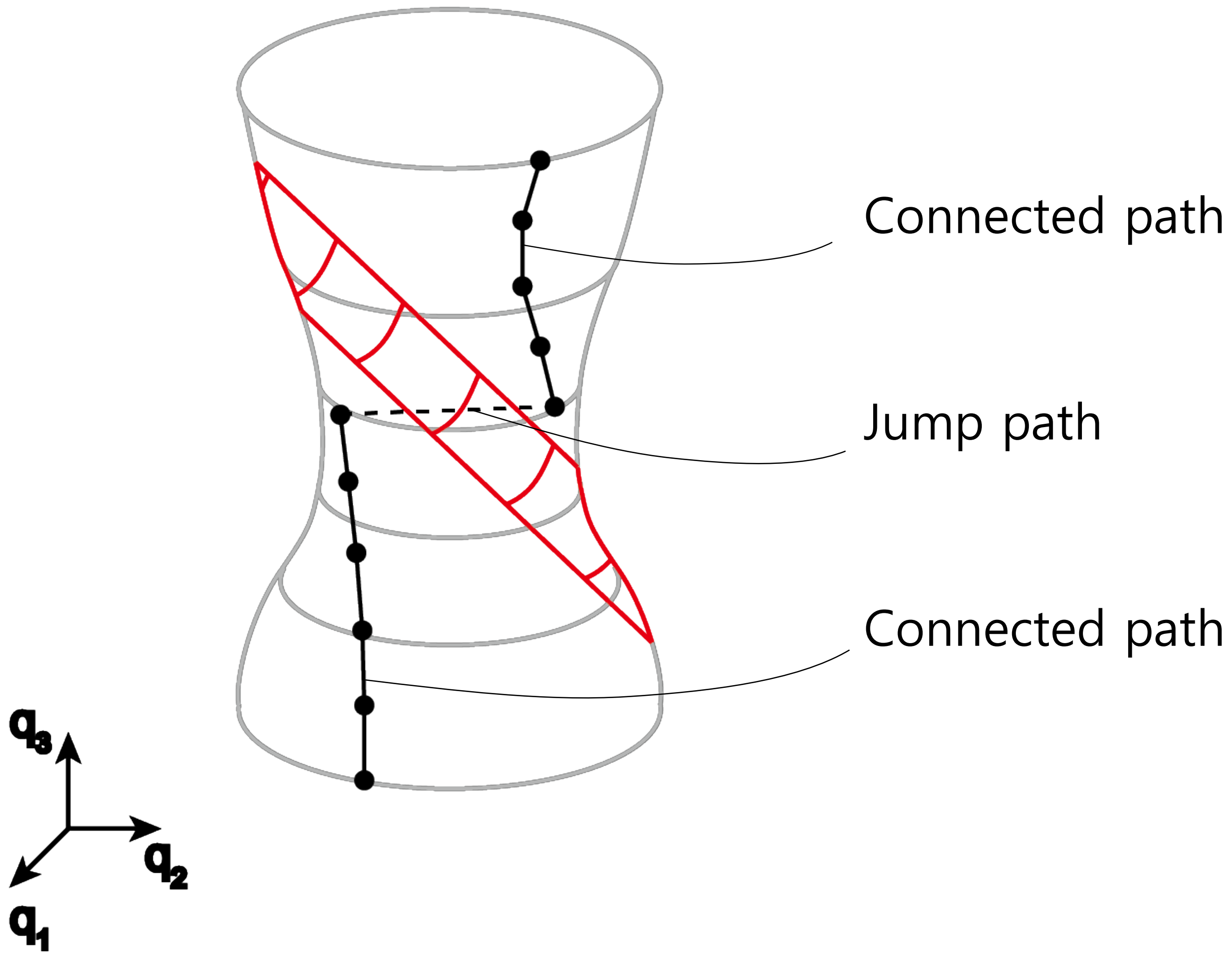


Planning on Foliations





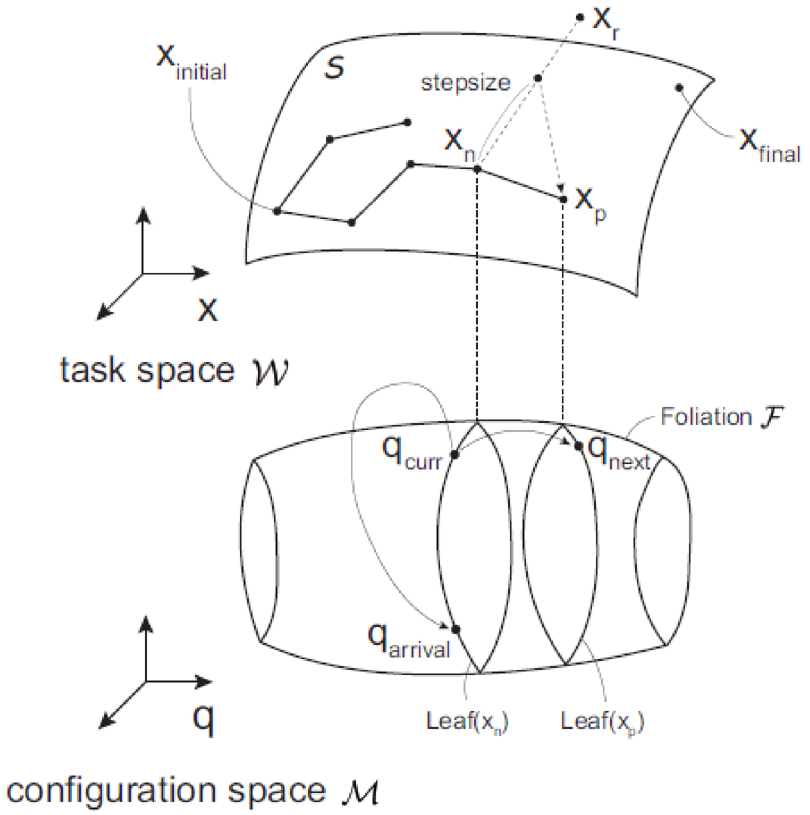
Planning on Foliations



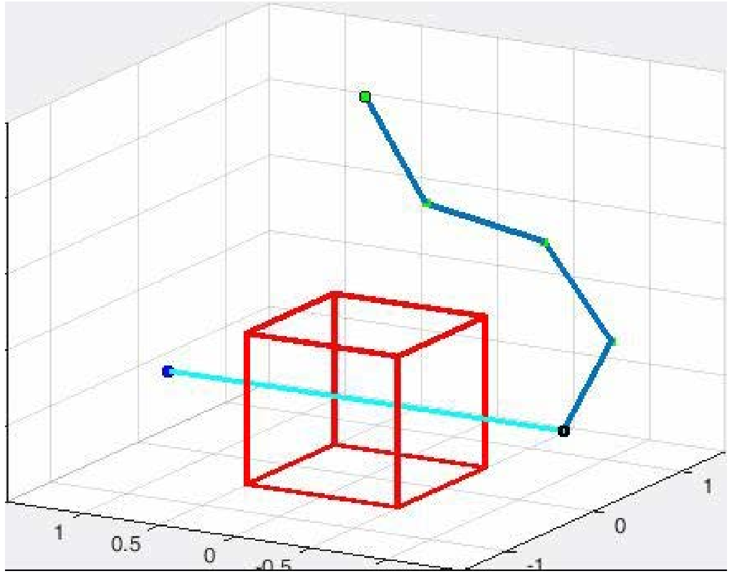
Planning on Foliations (J Kim et al 2016)

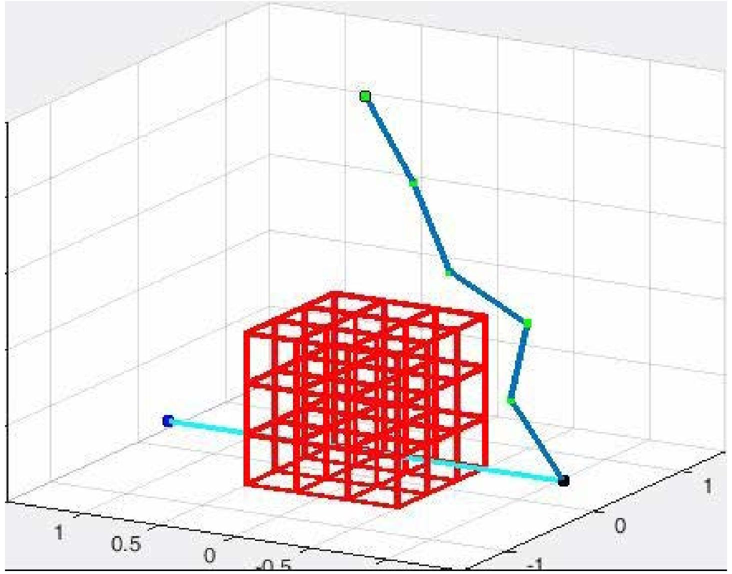
Algorithm 1 Planning Algorithm

Given: $x_{initial}, x_{final}$
Tree T_S.Init($x_{initial}$)
T_S.Node($x_{initial}$).CurrConfig $\leftarrow$ InverseKinematics($q_{rand}, x_{initial}$)
repeat
    $q_{curr}, x_n, x_p \leftarrow$ ExploreTree(T_S)
    $q_{next} \leftarrow$ ProjectToLeaf(q_{curr}, x_p)
    $\overline{q_{curr}q_{next}} \leftarrow$ FindPathOn$\mathcal{F}$(q_{curr}, q_{next})
    if CollisionFree($\overline{q_{curr}q_{next}}$) **then**
        T_S.AddNode(x_p)
        T_S.AddEdge(x_n, x_p)
        T_S.Node(x_p).Path $\leftarrow \overline{q_{curr}q_{next}}$
        T_S.Node(x_p).CurrConfig $\leftarrow q_{next}$
    else
        $\overline{q_{curr}q_{arrival}} \leftarrow$ Jump(q_{curr}, x_n, x_p)
        T_S.Node(x_n).Path $\leftarrow \overline{q_{curr}q_{arrival}}$
        T_S.Node(x_n).CurrConfig $\leftarrow q_{arrival}$
    end if
until x_{final} is added to T_S
$Q \leftarrow$ ExtractPath()
return final path Q



Planning on Foliations





Dynamics and Motion Optimization

PUMA 560 dynamics equations

B. Armstrong, O. Khatib, J. Burdick, "The explicit dynamic model and inertial parameters of the Puma 560 Robot arm," *Proc*. *ICRA*, 1986.

PUMA 560 dynamics equations

$I_2 = I_{zz2} + m_2 \cdot (r_{x2}^2 + r_{y2}^2) + (m_3 + m_4 + m_5 + m_6) \cdot a_2^2 ;$

$I_3 = -I_{xx2} + I_{yy2} + (m_3 + m_4 + m_5 + m_6) \cdot a_2^2$
$m_2 \cdot r_{x2}^2 - m_2 \cdot r_{y2}^2 ;$

$I_4 = m_2 \cdot r_{x2} \cdot (d_2 + r_{z2}) + m_3 \cdot a_2 \cdot r_{z3}$
$+(m_3 + m_4 + m_5 + m_6) \cdot a_2 \cdot (d_2 + d_3) ;$

$I_5 = -m_3 \cdot a_2 \cdot r_{y3} + (m_4 + m_5 + m_6) \cdot a_2 \cdot d_4 + m_4 \cdot a_2 \cdot r_{z4} ;$

$I_6 = I_{zz3} + m_3 \cdot r_{y3}^2 + m_4 \cdot a_3^2 + m_4 \cdot (d_4 + r_{z4})^2 + I_{yy4}$
$+m_5 \cdot a_3^2 + m_5 \cdot d_4^2 + I_{zz5} + m_6 \cdot a_3^2 + m_6 \cdot d_4^2$
$+m_6 \cdot r_{z6}^2 + I_{xx6} ;$

$I_7 = m_3 \cdot r_{y3}^2 + I_{xx3} - I_{yy3} + m_4 \cdot r_{z4}^2 + 2 \cdot m_4 \cdot d_4 \cdot r_{z4}$
$+(m_4 + m_5 + m_6) \cdot (d_4^2 - a_3^2) + I_{yy4} - I_{zz4} + I_{zz5}$
$-I_{yy5} + m_6 \cdot r_{z6}^2 - I_{zz6} + I_{xx6} ;$

$I_8 = -m_4 \cdot (d_2 + d_3) \cdot (d_4 + r_{z4}) - (m_5 + m_6) \cdot (d_2 + d_3) \cdot d_4$
$m_3 \cdot r_{y3} \cdot r_{z3} + m_3 \cdot (d_2 + d_3) \cdot r_{y3} ;$

$I_9 = m_2 \cdot r_{y2} \cdot (d_2 + r_{z2}) ;$

$I_{10} = 2 \cdot m_4 \cdot a_3 \cdot r_{z4} + 2 \cdot (m_4 + m_5 + m_6) \cdot a_3 \cdot d_4 ;$

$I_{11} = -2 \cdot m_2 \cdot r_{x2} \cdot r_{y2} ;$

$I_{12} = (m_4 + m_5 + m_6) \cdot a_2 \cdot a_3 ;$

$I_{13} = (m_4 + m_5 + m_6) \cdot a_3 \cdot (d_2 + d_3) ;$

$I_{14} = I_{zz4} + I_{yy5} + I_{xx6} ;$

$I_{15} = m_6 \cdot d_4 \cdot r_{z6} ;$

$I_{16} = m_6 \cdot a_2 \cdot r_{z6} ;$

$I_{17} = I_{zz5} + I_{xx6} + m_6 \cdot r_{z6}^2 ;$

$I_{18} = m_6 \cdot (d_2 + d_3) \cdot r_{z6} ;$

$I_{19} = I_{yy4} - I_{xx4} + I_{zz5} - I_{yy5} + m_6 \cdot r_{z6}^2 + I_{xx6} - I_{zz6} ;$

$I_{20} = I_{yy5} - I_{xx5} - m_6 \cdot r_{z6}^2 + I_{zz6} - I_{xx6} ;$

$I_{21} = I_{xx4} - I_{yy4} + I_{xx5} - I_{zz5} ;$

$I_{22} = m_6 \cdot a_3 \cdot r_{z6} ;$

$I_{23} = I_{zz6} ;$

Part II. Gravitional Constants

$g_1 = -g \cdot ((m_3 + m_4 + m_5 + m_6) \cdot a_2 + m_2 \cdot r_{x2}) ;$

$g_2 = g \cdot (m_3 \cdot r_{y3} - (m_4 + m_5 + m_6) \cdot d_4 - m_4 \cdot r_{z4}) ;$

$g_3 = g \cdot m_2 \cdot r_{y2} ;$

$g_4 = -g \cdot (m_4 + m_5 + m_6) \cdot a_3 ;$

$g_5 = -g \cdot m_6 \cdot r_{z6} ;$

Table A3. Computed Values for the Constants Appearing in the Equations of Forces of Motion.

(Inertial constants have units of kilogram meters-squared)

I_1	=	1.43	± 0.05	I_2	=	1.75	± 0.07
I_3	=	1.38	± 0.05	I_4	=	6.90×10^{-1}	$\pm 0.20 \times 10^{-1}$
I_5	=	3.72×10^{-1}	$\pm 0.31 \times 10^{-1}$	I_6	=	3.33×10^{-1}	$\pm 0.16 \times 10^{-1}$
I_7	=	2.98×10^{-1}	$\pm 0.29 \times 10^{-1}$	I_8	=	-1.34×10^{-1}	$\pm 0.14 \times 10^{-1}$
I_9	=	2.38×10^{-2}	$\pm 1.20 \times 10^{-2}$	I_{10}	=	-2.13×10^{-2}	$\pm 0.22 \times 10^{-2}$
I_{11}	=	-1.42×10^{-2}	$\pm 0.70 \times 10^{-2}$	I_{12}	=	-1.10×10^{-2}	$\pm 0.11 \times 10^{-2}$
I_{13}	=	-3.79×10^{-3}	$\pm 0.90 \times 10^{-3}$	I_{14}	=	1.64×10^{-3}	$\pm 0.07 \times 10^{-3}$
I_{15}	=	1.25×10^{-3}	$\pm 0.30 \times 10^{-3}$	I_{16}	=	1.24×10^{-3}	$\pm 0.30 \times 10^{-3}$
I_{17}	=	6.42×10^{-4}	$\pm 3.00 \times 10^{-4}$	I_{18}	=	4.31×10^{-4}	$\pm 1.30 \times 10^{-4}$
I_{19}	=	3.00×10^{-4}	$\pm 14.0 \times 10^{-4}$	I_{20}	=	-2.02×10^{-4}	$\pm 8.00 \times 10^{-4}$
I_{21}	=	-1.00×10^{-4}	$\pm 6.00 \times 10^{-4}$	I_{22}	=	-5.80×10^{-5}	$\pm 1.50 \times 10^{-5}$
I_{23}	=	4.00×10^{-5}	$\pm 2.00 \times 10^{-5}$				
I_{m1}	=	1.14	± 0.27	I_{m2}	=	4.71	± 0.54
I_{m3}	=	8.27×10^{-1}	$\pm 0.93 \times 10^{-1}$	I_{m4}	=	2.00×10^{-1}	$\pm 0.16 \times 10^{-1}$
I_{m5}	=	1.79×10^{-1}	$\pm 0.14 \times 10^{-1}$	I_{m6}	=	1.93×10^{-1}	$\pm 0.16 \times 10^{-1}$

(Gravitational constants have units of newton meters)

g_1	=	-37.2	± 00.5	g_2	=	-8.44	± 0.20
g_3	=	1.02	± 0.50	g_4	=	2.49×10^{-1}	$\pm 0.25 \times 10^{-1}$
g_5	=	-2.82×10^{-2}	$\pm 0.56 \times 10^{-2}$				

PUMA 560 dynamics equations (page 2)

Table A4. The expressions giving the elements of the kinetic energy matrix.
(The Abbreviated Expressions have units of kg-m².)

$$
\begin{aligned}
a_{11} = {} & I_{m1} + I_1 + I_3 \cdot CC2 + I_7 \cdot SS23 + I_{10} \cdot SC23 + I_{11} \cdot SC2 \\
& + I_{20} \cdot (SS5 \cdot (SS23 \cdot (1 + CC4) - 1) - 2 \cdot SC23 \cdot C4 \cdot SC5) \\
& + I_{21} \cdot SS23 \cdot CC4 + 2 \cdot \{ I_5 \cdot C2 \cdot S23 + I_{12} \cdot C2 \cdot C23 \\
& + I_{15} \cdot (SS23 \cdot C5 + SC23 \cdot C4 \cdot S5) \\
& + I_{16} \cdot C2 \cdot (S23 \cdot C5 + C23 \cdot C4 \cdot S5) \\
& + I_{18} \cdot S4 \cdot S5 + I_{22} \cdot (SC23 \cdot C5 + CC23 \cdot C4 \cdot S5) \} ; \\
\approx {} & 2.57 + 1.38 \cdot CC2 + 0.30 \cdot SS23 + 7.44 \times 10^{-1} \cdot C2 \cdot S23 .
\end{aligned}
$$

$$
\begin{aligned}
a_{12} = {} & I_4 \cdot S2 + I_8 \cdot C23 + I_9 \cdot C2 + I_{13} \cdot S23 - I_{15} \cdot C23 \cdot S4 \cdot S5 \\
& + I_{16} \cdot S2 \cdot S4 \cdot S5 + I_{18} \cdot (S23 \cdot C4 \cdot S5 - C23 \cdot C5) \\
& + I_{19} \cdot S23 \cdot SC4 + I_{20} \cdot S4 \cdot (S23 \cdot C4 \cdot CC5 + C23 \cdot SC5) \\
& + I_{22} \cdot S23 \cdot S4 \cdot S5 ; \\
\approx {} & 6.90 \times 10^{-1} \cdot S2 - 1.34 \times 10^{-1} \cdot C23 + 2.38 \times 10^{-2} \cdot C2 .
\end{aligned}
$$

$$
\begin{aligned}
a_{13} = {} & I_8 \cdot C23 + I_{13} \cdot S23 - I_{15} \cdot C23 \cdot S4 \cdot S5 + I_{19} \cdot S23 \cdot SC4 \\
& + I_{18} \cdot (S23 \cdot C4 \cdot S5 - C23 \cdot C5) + I_{22} \cdot S23 \cdot S4 \cdot S5 \\
& + I_{20} \cdot S4 \cdot (S23 \cdot C4 \cdot CC5 + C23 \cdot SC5) ; \\
\approx {} & -1.34 \times 10^{-1} \cdot C23 + -3.97 \times 10^{-3} \cdot S23 .
\end{aligned}
$$

$$
\begin{aligned}
a_{14} = {} & I_{14} \cdot C23 + I_{15} \cdot S23 \cdot C4 \cdot S5 + I_{16} \cdot C2 \cdot C4 \cdot S5 \\
& + I_{18} \cdot C23 \cdot S4 \cdot S5 - I_{20} \cdot (S23 \cdot C4 \cdot SC5 + C23 \cdot SS5) \\
& + I_{22} \cdot C23 \cdot C4 \cdot S5 ; \qquad \approx 0 .
\end{aligned}
$$

$$
\begin{aligned}
a_{15} = {} & I_{15} \cdot S23 \cdot S4 \cdot C5 + I_{16} \cdot C2 \cdot S4 \cdot C5 + I_{17} \cdot S23 \cdot S4 \\
& + I_{18} \cdot (S23 \cdot S5 - C23 \cdot C4 \cdot C5) + I_{22} \cdot C23 \cdot S4 \cdot C5 ; \\
\approx {} & 0 .
\end{aligned}
$$

$$a_{16} = I_{23} \cdot (C23 \cdot C5 - S23 \cdot C4 \cdot S5) ; \qquad \approx 0 .$$

$$
\begin{aligned}
a_{22} = {} & I_{m2} + I_2 + I_6 + I_{20} \cdot SS4 \cdot SS5 + I_{21} \cdot SS4 \\
& + 2 \cdot \{ I_5 \cdot S3 + I_{12} \cdot C3 + I_{15} \cdot C5 \\
& + I_{16} \cdot (S3 \cdot C5 + C3 \cdot C4 \cdot S5) + I_{22} \cdot C4 \cdot S5 \} ; \\
\approx {} & 6.79 + 7.44 \times 10^{-1} \cdot S3 .
\end{aligned}
$$

$$
\begin{aligned}
a_{23} = {} & I_5 \cdot S3 + I_6 + I_{12} \cdot C3 + I_{16} \cdot (S3 \cdot C5 + C3 \cdot C4 \cdot S5) \\
& + I_{20} \cdot SS4 \cdot SS5 + I_{21} \cdot SS4 + 2 \cdot \{ I_{15} \cdot C5 + I_{22} \cdot C4 \cdot S5 \} ; \\
\approx {} & .333 + 3.72 \times 10^{-1} \cdot S3 - 1.10 \times 10^{-2} \cdot C3 .
\end{aligned}
$$

$$
\begin{aligned}
a_{24} = {} & -I_{15} \cdot S4 \cdot S5 - I_{16} \cdot S3 \cdot S4 \cdot S5 + I_{20} \cdot S4 \cdot SC5 ; \\
\approx {} & 0 .
\end{aligned}
$$

$$
\begin{aligned}
a_{25} = {} & I_{15} \cdot C4 \cdot C5 + I_{16} \cdot (C3 \cdot S5 + S3 \cdot C4 \cdot C5) \\
& + I_{17} \cdot C4 + I_{22} \cdot S5 ; \qquad \approx 0 .
\end{aligned}
$$

$$a_{26} = I_{23} \cdot S4 \cdot S5 ; \qquad \approx 0 .$$

$$
\begin{aligned}
a_{33} = {} & I_{m3} + I_6 + I_{20} \cdot SS4 \cdot SS5 + I_{21} \cdot SS4 \\
& + 2 \cdot \{ I_{15} \cdot C5 + I_{22} \cdot C4 \cdot S5 \} ; \qquad \approx 1.16 .
\end{aligned}
$$

$$
\begin{aligned}
a_{34} = {} & -I_{15} \cdot S4 \cdot S5 + I_{20} \cdot S4 \cdot SC5 ; \\
\approx {} & -1.25 \times 10^{-3} \cdot S4 \cdot S5 .
\end{aligned}
$$

$$
\begin{aligned}
a_{35} = {} & I_{15} \cdot C4 \cdot C5 + I_{17} \cdot C4 + I_{22} \cdot S5 ; \\
\approx {} & 1.25 \times 10^{-3} \cdot C4 \cdot C5 .
\end{aligned}
$$

$$a_{36} = I_{23} \cdot S4 \cdot S5 ; \qquad \approx 0 .$$

$$a_{44} = I_{m4} + I_{14} - I_{20} \cdot SS5 ; \qquad \approx 0.20 .$$

$$a_{45} = 0.$$

$$a_{46} = I_{23} \cdot C5 ; \qquad \approx 0 .$$

$$a_{55} = I_{m5} + I_{17} ; \qquad \approx 0.18 .$$

$$a_{56} = 0.$$

$$a_{66} = I_{m6} + I_{23} ; \qquad \approx 0.19 .$$

Table A5. The expressions giving the elements of the Coriolis matrix
(The Abbreviated Expressions have units of kg-m².)

$$
\begin{aligned}
b_{112} = {} & 2 \cdot \{ -I_3 \cdot SC2 + I_5 \cdot C223 + I_7 \cdot SC23 - I_{12} \cdot S223 \\
& + I_{15} \cdot (2 \cdot SC23 \cdot C5 + (1 - 2 \cdot SS23) \cdot C4 \cdot S5) \\
& + I_{16} \cdot (C223 \cdot C5 - S223 \cdot C4 \cdot S5) + I_{21} \cdot SC23 \cdot CC4 \\
& + I_{20} \cdot ((1 + CC4) \cdot SC23 \cdot SS5 - (1 - 2 \cdot SS23) \cdot C4 \cdot SC5) \\
& + I_{22} \cdot ((1 - 2 \cdot SS23) \cdot C5 - 2 \cdot SC23 \cdot C4 \cdot S5) \} \\
& + I_{10} \cdot (1 - 2 \cdot SS23) + I_{11} \cdot (1 - 2 \cdot SS2) ; \\
\approx {} & -2.76 \cdot SC2 + 7.44 \times 10^{-1} \cdot C223 + 0.60 \cdot SC23 \\
& - 2.13 \times 10^{-2} \cdot (1 - 2 \cdot SS23) .
\end{aligned}
$$

$$
\begin{aligned}
b_{113} = {} & 2 \cdot \{ I_5 \cdot C2 \cdot C23 + I_7 \cdot SC23 - I_{12} \cdot C2 \cdot S23 \\
& + I_{15} \cdot (2 \cdot SC23 \cdot C5 + (1 - 2 \cdot SS23) \cdot C4 \cdot S5) \\
& + I_{16} \cdot C2 \cdot (C23 \cdot C5 - S23 \cdot C4 \cdot S5) + I_{21} \cdot SC23 \cdot CC4 \\
& + I_{20} \cdot ((1 + CC4) \cdot SC23 \cdot SS5 - (1 - 2 \cdot SS23) \cdot C4 \cdot SC5) \\
& + I_{22} \cdot ((1 - 2 \cdot SS23) \cdot C5 - 2 \cdot SC23 \cdot C4 \cdot S5) \} \\
& + I_{10} \cdot (1 - 2 \cdot SS23) ; \\
\approx {} & 7.44 \times 10^{-1} \cdot C2 \cdot C23 + 0.60 \cdot SC23 \\
& + 2.20 \times 10^{-2} \cdot C2 \cdot S23 - 2.13 \times 10^{-2} \cdot (1 - 2 \cdot SS23) .
\end{aligned}
$$

$$
\begin{aligned}
b_{114} = {} & 2 \cdot \{ -I_{15} \cdot SC23 \cdot S4 \cdot S5 - I_{16} \cdot C2 \cdot C23 \cdot S4 \cdot S5 \\
& + I_{18} \cdot C4 \cdot S5 - I_{20} \cdot (SS23 \cdot SS5 \cdot SC4 - SC23 \cdot S4 \cdot SC5) \\
& - I_{22} \cdot CC23 \cdot S4 \cdot S5 - I_{21} \cdot SS23 \cdot SC4 \} ; \\
\approx {} & -2.50 \times 10^{-3} \cdot SC23 \cdot S4 \cdot S5 + 8.60 \times 10^{-4} \cdot C4 \cdot S5 \\
& - 2.48 \times 10^{-3} \cdot C2 \cdot C23 \cdot S4 \cdot S5 .
\end{aligned}
$$

$$
\begin{aligned}
b_{115} = {} & 2 \cdot \{ I_{20} \cdot (SC5 \cdot (CC4 \cdot (1 - CC23) - CC23) \\
& - SC23 \cdot C4 \cdot (1 - 2 \cdot SS5)) - I_{15} \cdot (SS23 \cdot S5 - SC23 \cdot C4 \cdot C5) \\
& - I_{16} \cdot C2 \cdot (S23 \cdot S5 - C23 \cdot C4 \cdot C5) + I_{18} \cdot S4 \cdot C5 \\
& + I_{22} \cdot (CC23 \cdot C4 \cdot C5 - SC23 \cdot S5) \} ; \\
\approx {} & -2.50 \times 10^{-3} \cdot (SS23 \cdot S5 - SC23 \cdot C4 \cdot C5) \\
& - 2.48 \times 10^{-3} \cdot C2 \cdot (S23 \cdot S5 - C23 \cdot C4 \cdot C5) \\
& + 8.60 \times 10^{-4} \cdot S4 \cdot C5.
\end{aligned}
$$

$$b_{116} = 0 .$$

$$
\begin{aligned}
b_{123} = {} & 2 \cdot \{ -I_5 \cdot S23 + I_{13} \cdot C23 + I_{15} \cdot S23 \cdot S4 \cdot S5 \\
& + I_{18} \cdot (C23 \cdot C4 \cdot S5 + S23 \cdot C5) + I_{19} \cdot C23 \cdot SC4 \\
& + I_{20} \cdot S4 \cdot (C23 \cdot C4 \cdot CC5 - S23 \cdot SC5) \\
& + I_{22} \cdot C23 \cdot S4 \cdot S5 \} ; \\
\approx {} & 2.67 \times 10^{-1} \cdot S23 - 7.58 \times 10^{-3} \cdot C23 .
\end{aligned}
$$

$$
\begin{aligned}
b_{124} = {} & -I_{18} \cdot 2 \cdot S23 \cdot S4 \cdot S5 + I_{19} \cdot S23 \cdot (1 - (2 \cdot SS4)) \\
& + I_{20} \cdot S23 \cdot (1 - 2 \cdot SS4 \cdot CC5) - I_{14} \cdot S23 ; \qquad \approx 0 .
\end{aligned}
$$

$$
\begin{aligned}
b_{125} = {} & I_{17} \cdot C23 \cdot S4 + I_{18} \cdot 2 \cdot (S23 \cdot C4 \cdot C5 + C23 \cdot S5) \\
& + I_{20} \cdot S4 \cdot (C23 \cdot (1 - 2 \cdot SS5) - S23 \cdot C4 \cdot 2 \cdot SC5) ; \\
\approx {} & 0 .
\end{aligned}
$$

$$b_{126} = -I_{23} \cdot (S23 \cdot C5 + C23 \cdot C4 \cdot S5) ; \qquad \approx 0 .$$

$$b_{134} = b_{124} . \qquad b_{135} = b_{125} . \qquad b_{136} = b_{126} .$$

$$
\begin{aligned}
b_{145} = {} & 2 \cdot \{ I_{15} \cdot S23 \cdot C4 \cdot C5 + I_{16} \cdot C2 \cdot C4 \cdot C5 \\
& + I_{18} \cdot C23 \cdot S4 \cdot C5 + I_{22} \cdot C23 \cdot C4 \cdot C5 \} + I_{17} \cdot S23 \cdot C4 \\
& - I_{20} \cdot (S23 \cdot C4 \cdot (1 - 2 \cdot SS5) + 2 \cdot C23 \cdot SC5) ; \\
\approx {} & 0 .
\end{aligned}
$$

$$b_{146} = I_{23} \cdot S23 \cdot S4 \cdot S5 ; \qquad \approx 0 .$$

$$b_{156} = -I_{23} \cdot (C23 \cdot S5 + S23 \cdot C4 \cdot C5) ; \qquad \approx 0 .$$

$$b_{212} = 0 . \qquad b_{213} = 0 .$$

$$
\begin{aligned}
b_{214} = {} & I_{14} \cdot S23 + I_{19} \cdot S23 \cdot (1 - (2 \cdot SS4)) \\
& + 2 \cdot \{ -I_{15} \cdot C23 \cdot C4 \cdot S5 + I_{16} \cdot S2 \cdot C4 \cdot S5 \\
& + I_{20} \cdot (S23 \cdot (CC5 \cdot CC4 - 0.5) + C23 \cdot C4 \cdot SC5) \\
& + I_{22} \cdot S23 \cdot C4 \cdot S5 \} ; \\
\approx {} & 1.64 \times 10^{-3} \cdot S23 - 2.50 \times 10^{-3} \cdot C23 \cdot C4 \cdot S5 + \\
& 2.48 \times 10^{-3} \cdot S2 \cdot C4 \cdot S5 + 0.30 \times 10^{-3} \cdot S23 \cdot (1 - (2 \cdot SS4)) .
\end{aligned}
$$

$$
\begin{aligned}
b_{215} = {} & 2 \cdot \{ -I_{15} \cdot C23 \cdot S4 \cdot C5 + I_{22} \cdot S23 \cdot S4 \cdot C5 \\
& + I_{16} \cdot S2 \cdot S4 \cdot C5 \} - I_{17} \cdot C23 \cdot S4 \\
& + I_{20} \cdot (C23 \cdot S4 \cdot (1 - 2 \cdot SS5) - 2 \cdot S23 \cdot SC4 \cdot SC5) ; \\
\approx {} & -2.50 \times 10^{-3} \cdot C23 \cdot S4 \cdot C5 + 2.48 \times 10^{-3} \cdot S2 \cdot S4 \cdot C5 \\
& - 6.42 \times 10^{-4} \cdot C23 \cdot S4 .
\end{aligned}
$$

$$b_{216} = -b_{126} .$$

$$
\begin{aligned}
b_{223} = {} & 2 \cdot \{ -I_{12} \cdot S3 + I_5 \cdot C3 + I_{16} \cdot (C3 \cdot C5 - S3 \cdot C4 \cdot S5) \} ; \\
\approx {} & 2.20 \times 10^{-2} \cdot S3 + 7.44 \times 10^{-1} \cdot C3 .
\end{aligned}
$$

PUMA 560 dynamics equations (page 3)

$$\begin{aligned} b_{224} &= 2 \bullet \{-I_{16} \bullet C3 \bullet S4 \bullet S5 + I_{20} \bullet SC4 \bullet SS5 \\ &\quad + I_{21} \bullet SC4 - I_{22} \bullet S4 \bullet S5\}\ ; \\ &\approx -2.48\times10^{-3} \bullet C3 \bullet S4 \bullet S5\ . \end{aligned}$$

$$\begin{aligned} b_{225} &= 2 \bullet \{-I_{15} \bullet S5 + I_{16} \bullet (C3 \bullet C4 \bullet C5 - S3 \bullet S5) \\ &\quad + I_{20} \bullet SS4 \bullet SC5 + I_{22} \bullet C4 \bullet C5\}\ ; \\ &\approx -2.50\times10^{-3} \bullet S5 \ + \ 2.48\times10^{-3} \bullet (C3 \bullet C4 \bullet C5 - S3 \bullet S5)\ . \end{aligned}$$

$b_{226} = 0\ .$ $\qquad b_{234} = b_{224}\ .$

$b_{235} = b_{225}\ .$ $\qquad b_{236} = 0\ .$

$$\begin{aligned} b_{245} &= 2 \bullet \{-I_{15} \bullet S4 \bullet C5 - I_{16} \bullet S3 \bullet S4 \bullet C5\} \\ &\quad - I_{17} \bullet S4 + I_{20} \bullet S4 \bullet (1 - 2 \bullet SS5)\ ; \qquad \approx 0\ . \end{aligned}$$

$b_{246} = I_{23} \bullet C4 \bullet S5\ ; \qquad \approx 0\ .$

$b_{256} = I_{23} \bullet S4 \bullet C5\ ; \qquad \approx 0\ .$

$b_{312} = 0\ .$ $\qquad b_{313} = 0\ .$

$$\begin{aligned} b_{314} &= 2 \bullet \{-I_{15} \bullet C23 \bullet C4 \bullet S5 + I_{22} \bullet S23 \bullet C4 \bullet S5 \\ &\quad + I_{20} \bullet (S23 \bullet (CC5 \bullet CC4 - 0.5) + C23 \bullet C4 \bullet SC5)\} \\ &\quad + I_{14} \bullet S23 + I_{19} \bullet S23 \bullet (1 - (2 \bullet SS4))\ ; \\ &\approx -2.50\times10^{-3} \bullet C23 \bullet C4 \bullet S5 \ + \ 1.64\times10^{-3} \bullet S23 \\ &\quad + 0.30\times10^{-3} \bullet S23 \bullet (1 - 2 \bullet SS4)\ . \end{aligned}$$

$$\begin{aligned} b_{315} &= 2 \bullet (-I_{15} \bullet C23 \bullet S4 \bullet C5 + I_{22} \bullet S23 \bullet S4 \bullet C5) \\ &\quad - I_{17} \bullet C23 \bullet S4 \\ &\quad + I_{20} \bullet S4 \bullet (C23 \bullet (1 - 2 \bullet SS5) - 2 \bullet S23 \bullet C4 \bullet SC5)\ ; \\ &\approx -2.50\times10^{-3} \bullet C23 \bullet S4 \bullet C5 \ - \ 6.42\times10^{-4} \bullet C23 \bullet S4\ . \end{aligned}$$

$b_{316} = -b_{136}\ .$ $\qquad b_{323} = 0\ .$

$$\begin{aligned} b_{324} &= 2 \bullet \{I_{20} \bullet SC4 \bullet SS5 + I_{21} \bullet SC4 - I_{22} \bullet S4 \bullet S5\}\ ; \\ &\approx 0\ . \end{aligned}$$

$$\begin{aligned} b_{325} &= 2 \bullet \{-I_{15} \bullet S5 + I_{20} \bullet SS4 \bullet SC5 + I_{22} \bullet C4 \bullet C5\}\ ; \\ &\approx -2.50\times10^{-3} \bullet S5\ . \end{aligned}$$

$b_{326} = 0\ .$ $\qquad b_{334} = b_{324}\ .$

$b_{335} = b_{325}\ .$ $\qquad b_{336} = 0\ .$

$$\begin{aligned} b_{345} &= -I_{15} \bullet 2 \bullet S4 \bullet C5 - I_{17} \bullet S4 + I_{20} \bullet S4 \bullet (1 - 2 \bullet SS5)\ ; \\ &\approx -2.50\times10^{-3} \bullet S4 \bullet C5\ . \end{aligned}$$

$b_{346} = b_{246}\ .$ $\qquad b_{356} = b_{256}\ .$ $\qquad b_{412} = -b_{214}\ .$

$b_{413} = -b_{314}\ .$ $\qquad b_{414} = 0\ .$

$$\begin{aligned} b_{415} &= -I_{20} \bullet (S23 \bullet C4 \bullet (1 - 2 \bullet SS5) + 2 \bullet C23 \bullet SC5) \\ &\quad - I_{17} \bullet S23 \bullet C4\ ; \\ &\approx -6.42\times10^{-4} \bullet S23 \bullet C4\ . \end{aligned}$$

$b_{416} = -b_{146}\ .$ $\qquad b_{423} = -b_{324}\ .$ $\qquad b_{424} = 0\ .$

$$\begin{aligned} b_{425} &= I_{17} \bullet S4 + I_{20} \bullet S4 \bullet (1 - 2 \bullet SS5)\ ; \\ &\approx 6.42\times10^{-4} \bullet S4\ . \end{aligned}$$

$b_{426} = -b_{246}\ .$ $\qquad b_{434} = 0\ .$

$b_{435} = b_{425}\ .$ $\qquad b_{436} = -b_{346}\ .$

$b_{445} = -I_{20} \bullet 2 \bullet SC5\ ; \qquad \approx 0\ .$

$b_{446} = 0\ ;$

$b_{456} = -I_{23} \bullet S5\ ; \qquad \approx 0\ .$

$b_{512} = -b_{215}\ .$ $\qquad b_{513} = -b_{315}\ .$ $\qquad b_{514} = -b_{415}\ .$

$b_{515} = 0\ .$ $\qquad b_{516} = -b_{156}\ .$ $\qquad b_{523} = -b_{325}\ .$

$b_{524} = -b_{425}\ .$ $\qquad b_{525} = 0\ .$ $\qquad b_{526} = -b_{256}\ .$

$b_{534} = b_{524}\ .$ $\qquad b_{535} = 0\ .$ $\qquad b_{536} = -b_{356}\ .$

$b_{545} = 0\ .$ $\qquad b_{546} = -b_{456}\ .$ $\qquad b_{556} = 0\ .$

$b_{612} = b_{126}\ .$ $\qquad b_{613} = b_{136}\ .$ $\qquad b_{614} = b_{146}\ .$

$b_{615} = b_{156}\ .$ $\qquad b_{616} = 0\ .$ $\qquad b_{623} = 0\ .$

$b_{624} = b_{246}\ .$ $\qquad b_{625} = b_{256}\ .$ $\qquad b_{626} = 0\ .$

$b_{634} = b_{624}\ .$ $\qquad b_{635} = b_{625}\ .$ $\qquad b_{636} = 0\ .$

$b_{645} = b_{456}\ .$ $\qquad b_{646} = 0\ .$ $\qquad b_{656} = 0\ .$

Table A6. The expressions for the terms of the centrifugal matrix.
(The Abbreviated Expressions have units of kg-m^2.)

$c_{11} = 0\ .$

$$\begin{aligned} c_{12} &= +I4 \bullet C2 - I8 \bullet S23 - I9 \bullet S2 + I13 \bullet C23 \\ &\quad + I_{15} \bullet S23 \bullet S4 \bullet S5 + I_{16} \bullet C2 \bullet S4 \bullet S5 \\ &\quad + I_{18} \bullet (C23 \bullet C4 \bullet S5 + S23 \bullet C5) + I_{19} \bullet C23 \bullet SC4 \\ &\quad + I_{20} \bullet S4 \bullet (C23 \bullet C4 \bullet cC5 - S23 \bullet SC5) \\ &\quad + I_{22} \bullet C23 \bullet S4 \bullet S5\ ; \\ &\approx 6.90\times10^{-3} \bullet C2 \ + \ 1.34\times10^{-1} \bullet S23 \ - \ 2.38\times10^{-2} \bullet S2\ . \end{aligned}$$

$c_{13} = 0.5 \bullet b_{123}\ .$

$$\begin{aligned} c_{14} &= -I_{15} \bullet S23 \bullet S4 \bullet S5 - I_{16} \bullet C2 \bullet S4 \bullet S5 \\ &\quad + I_{18} \bullet C23 \bullet C4 \bullet S5 + I_{20} \bullet S23 \bullet S4 \bullet SC5 \\ &\quad - I_{22} \bullet C23 \bullet S4 \bullet S5\ ; \qquad \approx 0\ . \end{aligned}$$

$$\begin{aligned} c_{15} &= -I_{15} \bullet S23 \bullet S4 \bullet S5 - I_{16} \bullet C2 \bullet S4 \bullet S5\ ; \\ &\quad + I_{18} \bullet (S23 \bullet C5 + C23 \bullet C4 \bullet S5) - I_{22} \bullet C23 \bullet S4 \bullet S5 \\ &\approx 0\ . \end{aligned}$$

$c_{16} = 0\ .$ $\qquad c_{21} = -0.5 \bullet b_{112}\ .$

$c_{22} = 0\ .$ $\qquad c_{23} = 0.5 \bullet b_{223}\ .$

$$\begin{aligned} c_{24} &= -I_{15} \bullet C4 \bullet S5 - I_{16} \bullet S3 \bullet C4 \bullet S5 + I_{20} \bullet C4 \bullet SC5\ ; \\ &\approx 0\ . \end{aligned}$$

$$\begin{aligned} c_{25} &= -I_{15} \bullet C4 \bullet S5 + I_{16} \bullet (C3 \bullet C5 - S3 \bullet C4 \bullet S5) \\ &\quad + I_{22} \bullet C5\ ; \qquad \approx 0\ . \end{aligned}$$

$c_{26} = 0\ .$ $\qquad c_{31} = -0.5 \bullet b_{113}\ .$

$c_{32} = -c_{23}\ .$ $\qquad c_{33} = 0\ .$

$$\begin{aligned} c_{34} &= -I_{15} \bullet C4 \bullet S5 + I_{20} \bullet C4 \bullet SC5\ ; \\ &\approx -1.25\times10^{-3} \bullet C4 \bullet S5\ . \end{aligned}$$

$c_{35} = -I_{15} \bullet C4 \bullet S5 + I_{22} \bullet C5\ ; \qquad \approx c_{34}\ .$

$c_{36} = 0\ .$ $\qquad c_{41} = -0.5 \bullet b_{114}\ .$ $\qquad c_{42} = -0.5 \bullet b_{224}\ .$

$c_{43} = 0.5 \bullet b_{423}\ .$ $\qquad c_{44} = 0\ .$ $\qquad c_{45} = 0\ .$

$c_{46} = 0\ .$ $\qquad c_{51} = -0.5 \bullet b_{115}\ .$ $\qquad c_{52} = -0.5 \bullet b_{225}\ .$

$c_{53} = 0.5 \bullet b_{523}\ .$ $\qquad c_{54} = -0.5 \bullet b_{445}\ .$ $\qquad c_{55} = 0\ .$

$c_{56} = 0\ .$ $\qquad c_{61} = 0\ .$ $\qquad c_{62} = 0\ .$

$c_{63} = 0\ .$ $\qquad c_{64} = 0\ .$ $\qquad c_{65} = 0\ .$

$c_{66} = 0\ .$

Table A7. Gravity Terms.
(The Abbreviated Expressions have units of newton-meters.)

$g_1 = 0\ .$

$$\begin{aligned} g_2 &= g1 \bullet C2 + g2 \bullet S23 + g3 \bullet S2 + g4 \bullet C23\ ; \\ &\quad + g5 \bullet (S23 \bullet C5 + C23 \bullet C4 \bullet S5) \\ &\approx -37.2 \bullet C2 \ - \ \varepsilon\,1 \bullet S23 \ + \ 1.02 \bullet S2\ . \end{aligned}$$

$$\begin{aligned} g_3 &= g2 \bullet S23 + g4 \bullet C23 + g5 \bullet (S23 \bullet C5 + C23 \bullet C4 \bullet S5)\ ; \\ &\approx -8.4 \bullet S23 \ + \ 0.25 \bullet C23\ . \end{aligned}$$

$$\begin{aligned} g_4 &= -g5 \bullet S23 \bullet S4 \bullet S5\ ; \\ &\approx 2.8\times10^{-2} \bullet S23 \bullet S4 \bullet S5\ . \end{aligned}$$

$$\begin{aligned} g_5 &= g5 \bullet (C23 \bullet S5 + S23 \bullet C4 \bullet C5)\ ; \\ &\approx -2.8\times10^{-2} \bullet (C23 \bullet S5 \ + \ S23 \bullet C4 \bullet C5)\ . \end{aligned}$$

$g_6 = 0\ .$

Recursive dynamics to the rescue

- Recursive formulations of Newton-Euler dynamics already derived in early 1980s.
- Recursive formulations based on screw theory (Featherstone), spatial operator algebra (Rodriguez and Jain), Lie group concepts.

  - **Initialization**: $V_0 = \dot{V}_0 = F_{n+1} = 0$
  - **For** $i = 1$ to n **do**:
    - $T_{i-1,i} = M_i e^{S_i q_i}$
    - $V_i = \mathrm{Ad}_{T_{i,i-1}}(V_{i-1}) + S_i \dot{q}_i$
    - $\dot{V}_i = S_i \ddot{q}_i + Ad_{T_{i,i-1}}(\dot{V}_{i-1}) + [\mathrm{Ad}_{T_{i,i-1}}(V_{i-1}), S_i \dot{q}_i]$
  - **For** $i = n$ to 1 **do**:
    - $F_i = \mathrm{Ad}^*_{T_{i,i-1}}(F_{i+1}) + G_i \dot{V}_i \mathrm{ad}^*_{V_i}(G_i V_i)$
    - $\tau_i = S_i^T F_i$

The importance of analytic gradients

- Finite difference approximations of gradients (and Hessians) often lead to poor convergence and numerical instabilities.
- Derivation of recursive algorithms for analytic gradients and Hessians using Lie group operators and transformations:

Initialization

$$\frac{\partial V_0}{\partial q_i}, \frac{\partial \dot{V}_0}{\partial q_i}, \frac{\partial F_{n+1}}{\partial q_i} = 0_{6\times n}$$

$$\frac{\partial V_0}{\partial \dot{q}_i}, \frac{\partial \dot{V}_0}{\partial \dot{q}_i}, \frac{\partial F_{n+1}}{\partial \dot{q}_i} = 0_{6\times n}$$

$$\frac{\partial V_0}{\partial \ddot{q}_i}, \frac{\partial \dot{V}_0}{\partial \ddot{q}_i}, \frac{\partial F_{n+1}}{\partial \ddot{q}_i} = 0_{6\times n}$$

$$\frac{\partial V_0}{\partial V_0}, \frac{\partial \dot{V}_0}{\partial \dot{V}_0} = I_{6\times 6}$$

$$\frac{\partial V_0}{\partial \dot{V}_0}, \frac{\partial \dot{V}_0}{\partial V_0}, \frac{\partial F_{n+1}}{\partial V_0}, \frac{\partial F_{n+1}}{\partial \dot{V}_0} = 0_{6\times 6}$$

Forward recursion: for $k = 1$ to n

$$\frac{\partial V_k}{\partial q_i} = Ad_{f^{-1}_{k-1,k}} \frac{\partial V_{k-1}}{\partial q_i} - \delta_{i,k} ad_{S_k} V_k$$

$$\frac{\partial \dot{V}_k}{\partial q_i} = Ad_{f^{-1}_{k-1,k}} \frac{\partial \dot{V}_{k-1}}{\partial q_i} - ad_{S_k}\{\delta_{i,k} Ad_{f^{-1}_{k-1,k}} \dot{V}_{k-1} + \frac{\partial V_k}{\partial q_i}\dot{q}_k\}$$

$$\frac{\partial V_k}{\partial \dot{q}_i} = Ad_{f^{-1}_{k-1,k}} \frac{\partial V_{k-1}}{\partial \dot{q}_i} - \delta_{i,k} S_k$$

$$\frac{\partial \dot{V}_k}{\partial \dot{q}_i} = Ad_{f^{-1}_{k-1,k}} \frac{\partial \dot{V}_{k-1}}{\partial \dot{q}_i} - ad_{S_k}\{\delta_{i,k} V_k + \frac{\partial V_k}{\partial \dot{q}_i}\dot{q}_k\}$$

$$\frac{\partial V_k}{\partial \ddot{q}_i} = Ad_{f^{-1}_{k-1,k}} \frac{\partial V_{k-1}}{\partial \ddot{q}_i}$$

$$\frac{\partial \dot{V}_k}{\partial \ddot{q}_i} = Ad_{f^{-1}_{k-1,k}} \frac{\partial \dot{V}_{k-1}}{\partial \ddot{q}_i} + \delta_{i,k} S_k - ad_{S_k}\{\frac{\partial V_k}{\partial \ddot{q}_i}\dot{q}_k\}$$

$$\frac{\partial V_k}{\partial V_{0,i}} = Ad_{f^{-1}_{k-1,k}} \frac{\partial V_{k-1}}{\partial V_{0,i}}$$

$$\frac{\partial \dot{V}_k}{\partial V_{0,i}} = Ad_{f^{-1}_{k-1,k}} \frac{\partial \dot{V}_{k-1}}{\partial V_{0,i}} - ad_{S_k} \frac{\partial V_k}{\partial V_{0,i}}\dot{q}_k$$

$$\frac{\partial V_k}{\partial \dot{V}_{0,i}} = Ad_{f^{-1}_{k-1,k}} \frac{\partial V_{k-1}}{\partial \dot{V}_{0,i}}$$

$$\frac{\partial \dot{V}_k}{\partial \dot{V}_{0,i}} = Ad_{f^{-1}_{k-1,k}} \frac{\partial \dot{V}_{k-1}}{\partial \dot{V}_{0,i}} - ad_{S_k} \frac{\partial V_k}{\partial \dot{V}_{0,i}}\dot{q}_k$$

Backward recursion: for $k = n$ to 0

$$\frac{\partial F_k}{\partial q_i} = Ad^*_{f^{-1}_{k,k+1}}\{\delta_{i,k+1} ad^*_{-S_{k+1}} F_{k+1} + \frac{\partial F_{k+1}}{\partial q_i}\} + J_k \frac{\partial \dot{V}_k}{\partial q_i} - ad^*_{\frac{\partial V_k}{\partial q_i}} J_k V_k - ad^*_{V_k} J_k \frac{\partial V_k}{\partial q_i}$$

$$\frac{\partial F_k}{\partial \dot{q}_i} = Ad^*_{f^{-1}_{k,k+1}} \frac{\partial F_{k+1}}{\partial \dot{q}_i} + J_k \frac{\partial \dot{V}_k}{\partial \dot{q}_i} - ad^*_{\frac{\partial V_k}{\partial \dot{q}_i}} J_k V_k - ad^*_{V_k} J_k \frac{\partial V_k}{\partial \dot{q}_i}$$

$$\frac{\partial F_k}{\partial \ddot{q}_i} = Ad^*_{f^{-1}_{k,k+1}} \frac{\partial F_{k+1}}{\partial \ddot{q}_i} + J_k \frac{\partial \dot{V}_k}{\partial \ddot{q}_i} - ad^*_{\frac{\partial V_k}{\partial \ddot{q}_i}} J_k V_k - ad^*_{V_k} J_k \frac{\partial V_k}{\partial \ddot{q}_i}$$

$$\frac{\partial F_k}{\partial V_{0,i}} = Ad^*_{f^{-1}_{k,k+1}} \frac{\partial F_{k+1}}{\partial V_{0,i}} + J_k \frac{\partial \dot{V}_k}{\partial V_{0,i}} - ad^*_{\frac{\partial V_k}{\partial V_{0,i}}} J_k V_k - ad^*_{V_k} J_k \frac{\partial V_k}{\partial V_{0,i}}$$

$$\frac{\partial F_k}{\partial \dot{V}_{0,i}} = Ad^*_{f^{-1}_{k,k+1}} \frac{\partial F_{k+1}}{\partial \dot{V}_{0,i}} + J_k \frac{\partial \dot{V}_k}{\partial \dot{V}_{0,i}} - ad^*_{\frac{\partial V_k}{\partial \dot{V}_{0,i}}} J_k V_k - ad^*_{V_k} J_k \frac{\partial V_k}{\partial \dot{V}_{0,i}}$$

Maximum payload lifting

J. Bobrow et al, *ICRA Video Proceedings*, 1999.

Things a robot must do (in parallel)

- Vision processing, object recognition, classification
- Sensing (joint, force, tactile, laser, sonar, etc.)
- Localization/SLAM
- Manipulation planning
- Control (arms, legs, torso, hands, wheels)
- Communication
- Task scheduling and planning

Robots are being asked to simultaneously do more and more with only limited resources available for computation, communication, memory, etc.

Control laws and trajectories need to be designed in a way that minimizes the use of such resources.


생활의 달인

Measuring the cost of control

- Control depends on both time and state
- Simplest control is a constant one
- Cost of control implementation (“attention”) is proportional to the rate at which the control changes with respect to state and time.
- A control that requires little attention is one that is robust to discretization of time and state

Brockett's attention functional

Given system $\dot{x} = f(x, u, t)$, consider the following controller cost:

$$\iint \left\|\frac{\partial u}{\partial x}\right\|^2 + \left\|\frac{\partial u}{\partial t}\right\|^2 \mathrm{dx}\,\mathrm{dt}$$

- A multi-dimensional calculus of variations problem (integral over both space and time)
- Existence of solutions not always guaranteed

An approximate solution

Assuming control of the form $u = K(t)x + v(t)$ and state space integration is bounded, a minimum attention LQR control law* can be formulated as a finite-dimensional optimization problem:

$$\min_{P_f, Q>0} \quad J_{att} = \int_{t_0}^{t_f} \|K(t)\|^2 + \|\dot{u}^* - K(t)\,\dot{x}^*\|^2 dt$$

$$u(x,t) = K(t)\big(x - x^*(t)\big) + u^*(t)$$

$$K(t) = -R^{-1}B^T(t)P(t)$$

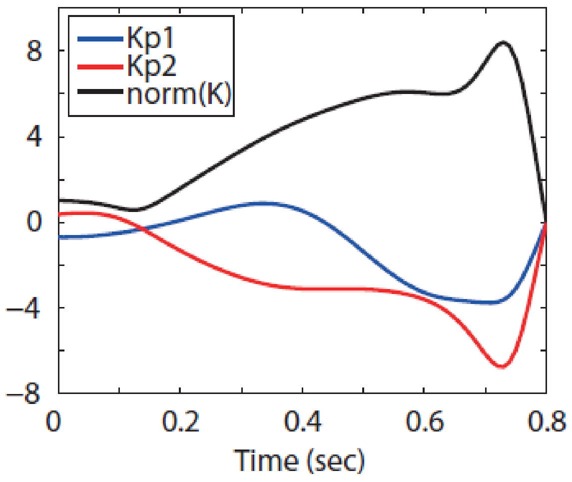
$$-\dot{P} = PA(t) + A^T(t) - PB(t)R^{-1}B^T(t)P + Q, P\big(t_f\big) = P_f$$

x,u* are given, e.g., as the outcome of some offline optimization procedure or supplied by the user.*

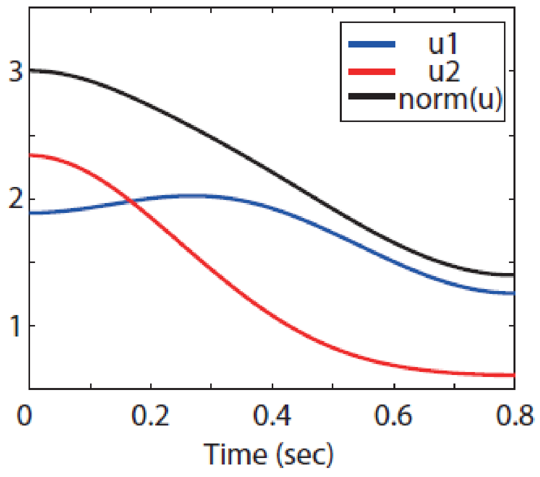
Example: Robot ball catching

Numerical experiments for a 2-dof arm catching a ball while tracking a minimum torque change trajectory:

Feedback gain



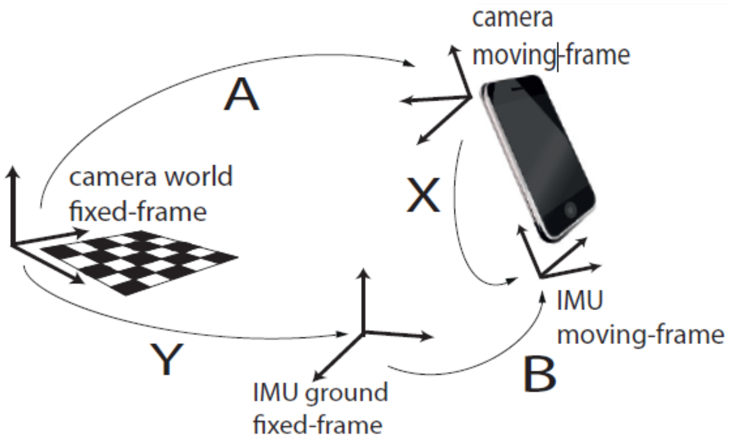
Feedforward input

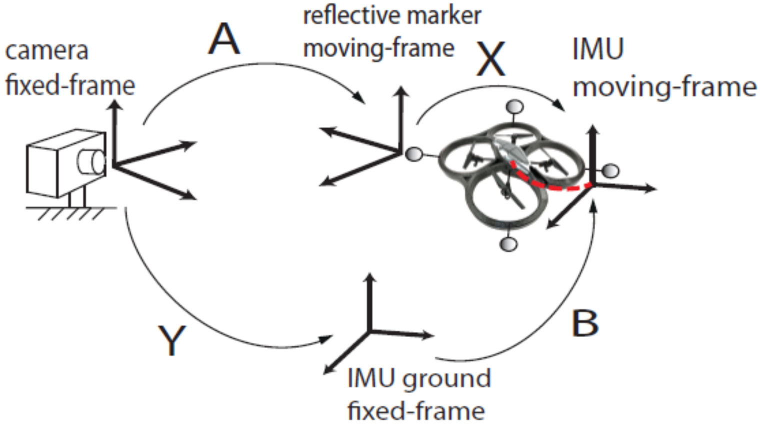


Feedback gains increase with time
Feedforward inputs decrease with time

Vision and Image Analysis

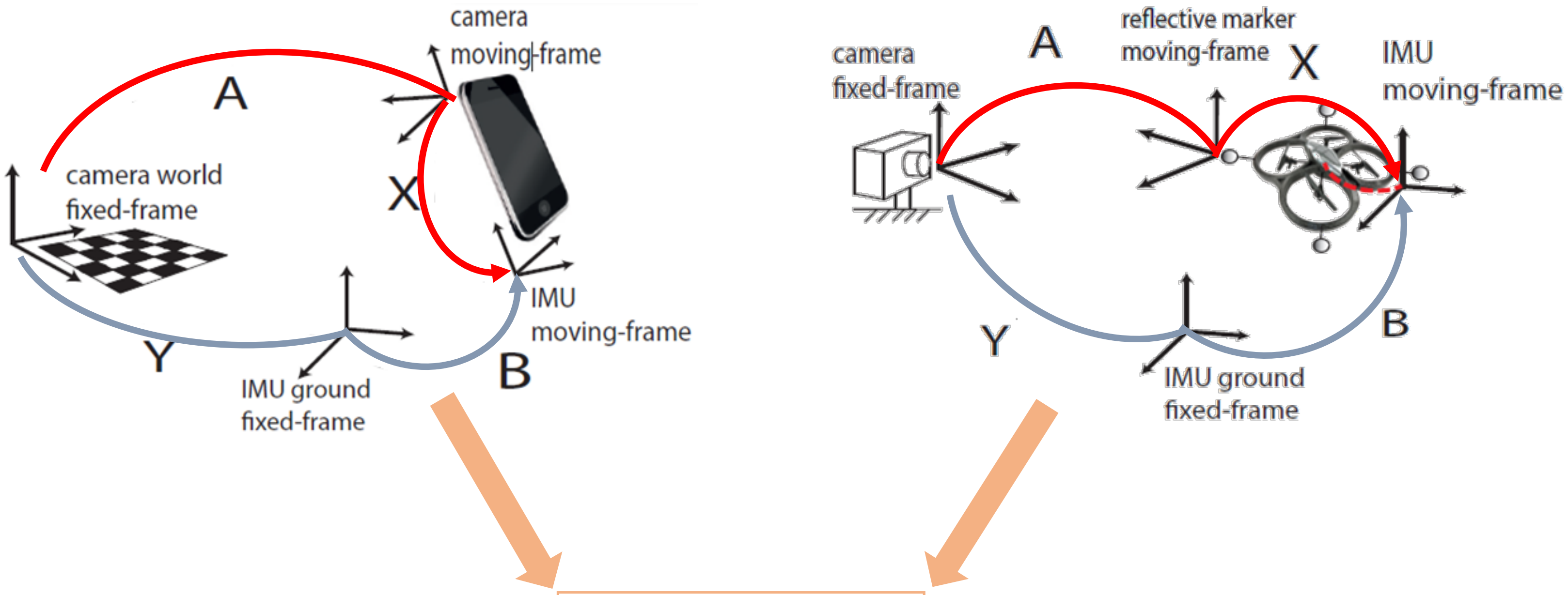
Two-frame sensor calibration





- A, B, X, Y can be elements of $SO(3)$ or $SE(3)$
- A, B are obtained from sensor measurements
- X, Y are unknowns to be determined.

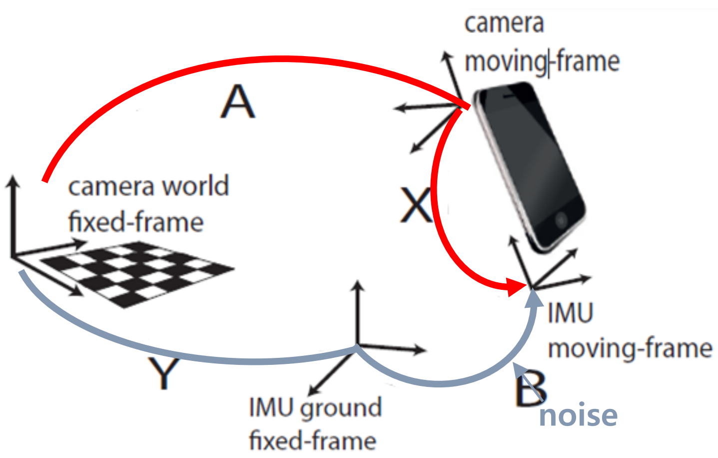
Two-frame sensor calibration

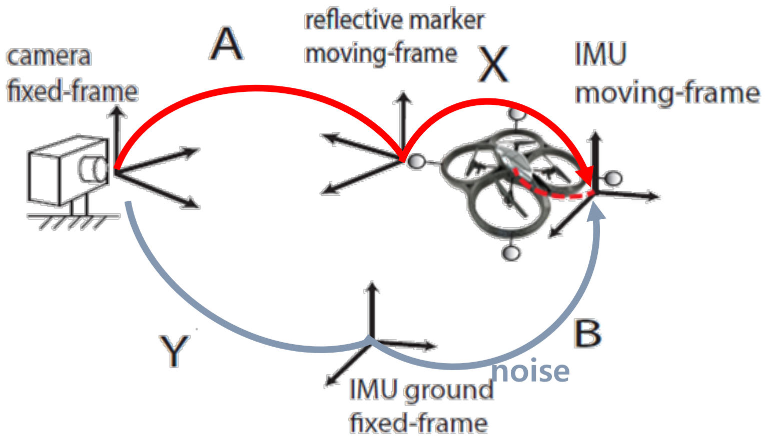


$$A_i X = Y B_i$$

- Given N measurement pairs $(A_i, B_i)_{i=1,\ldots,N}$
- Find the optimal (X, Y) pair that minimizes the fitting criterion.

Two-frame sensor calibration



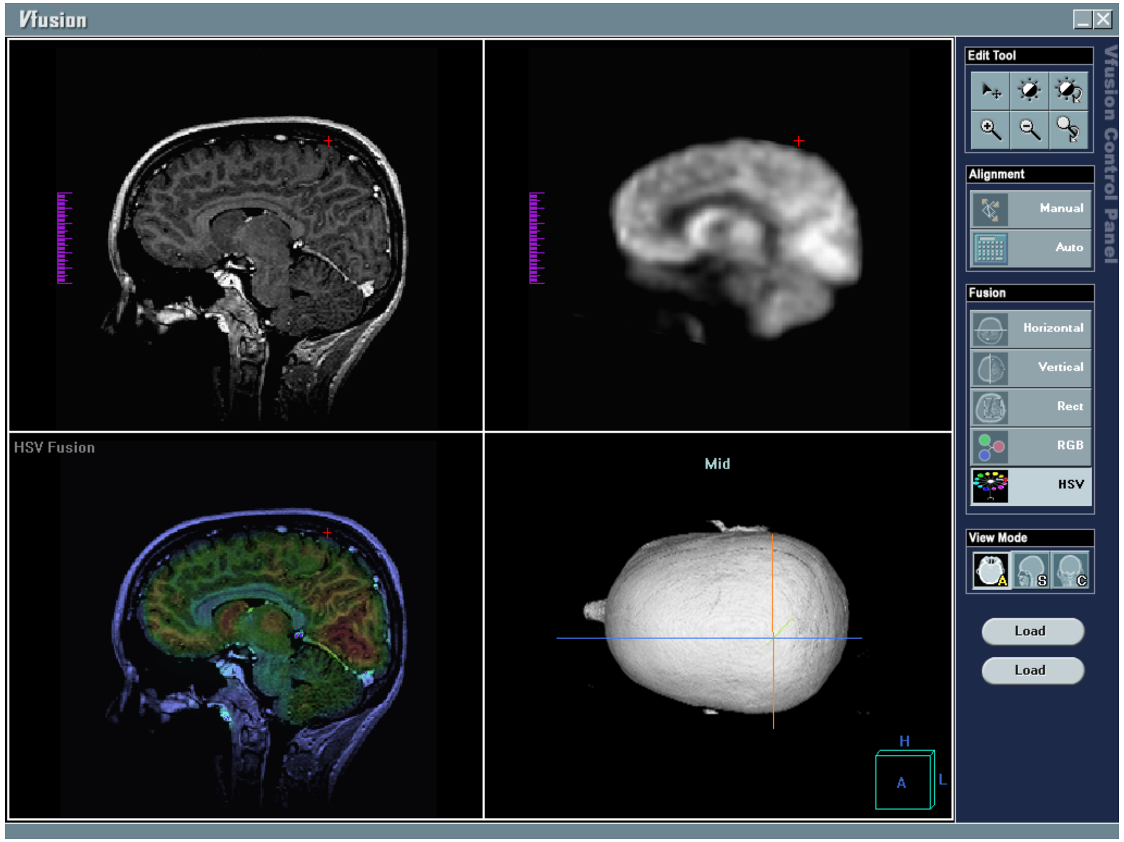


- $(A_i, B_i)_{i=1,\ldots,N}$ are noisy; there does not exist any (X, Y) that perfectly satisfies $A_i X = Y B_i$
- Determine (X, Y) that minimizes

$$\min_{X,Y \in SE(3)} \frac{1}{2} \sum_{i=1}^{N} \|A_i X - Y B_i\|^2$$

Multimodal image registration

Multimodal image volume registration: Find optimal transformation that maximizes the mutual information between two image volumes.



Detailed tissue structure provided by MRI (upper left) is combined with abnormal regions detected by PET (upper right). The red regions in the fused image represent the anomalous regions.

Multimodal image registration

Problem definition: $T^* = \arg\max_T I(A(Tx), B(x))$ where

- T is an element of some transformation group (SO(3), SE(3), SL(3) are widely used).
- $x \in \Re^3$ are the volume coordinates,
- *A*,*B* are volume data,
- $I(\cdot,\cdot)$ is the mutual information criterion,

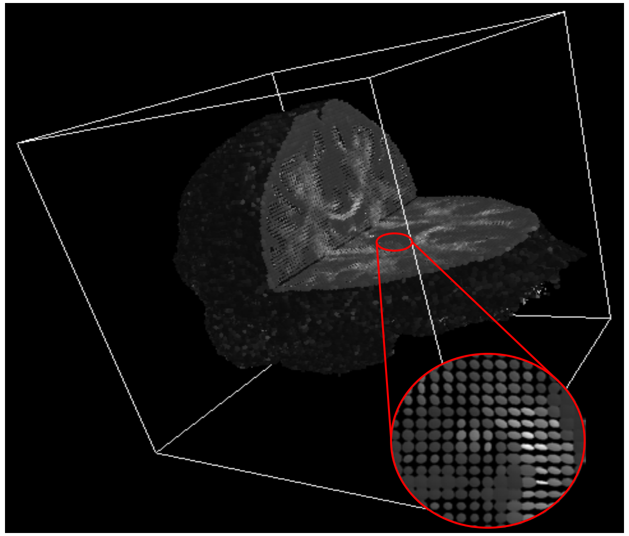
Evaluating the objective function is numerically expensive and analytic gradients are not available. Instead, it is common to resort to **direct search methods** like the Nelder-Mead algorithm.

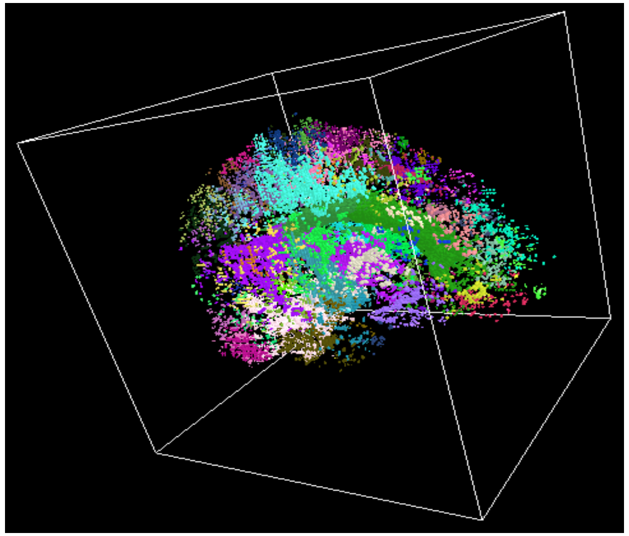
Optimization on matrix Lie groups

The above reduce to an optimization problem on matrix Lie groups:

- For the n-frame sensor calibration problem, the objective function reduces to the form $\sum_i \; Tr(XA_iX^TB_i \; - XC_i)$. Analytic gradients and Hessians are available, and steepest descent along minimal geodesics seems to work quite well.
- In the multimodal image registration problem, Nelder-Mead can be generalized to the group by using minimal geodesics as the edges of the simplex.
- There is a well-developed literature on optimization on Lie groups, including generalizations of common vector space algorithms to matrix Lie groups and manifolds.

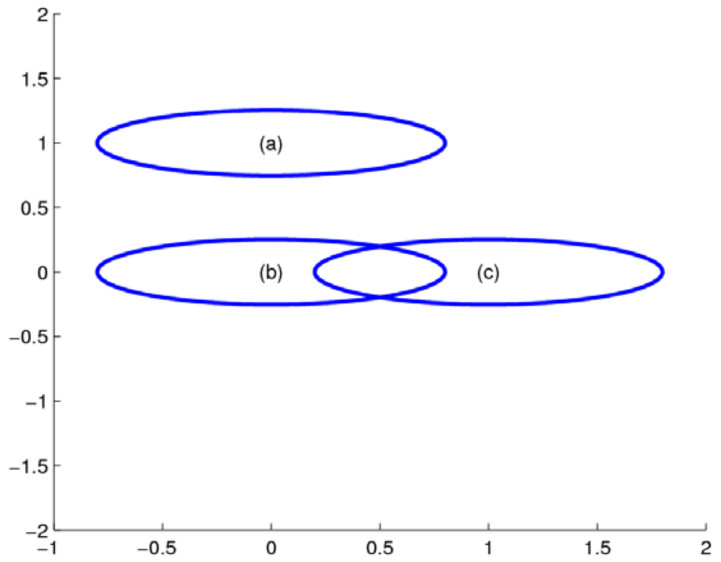
Diffusion tensor image segmentation





Each voxel is a 3D multivariate normal distribution. The mean indicates the position, while the covariance indicates the direction of diffusion of water molecules. Segmentation of a DTI image requires a metric on the manifold of multivariate Gaussian distributions.

Geometry of DTI segmentation

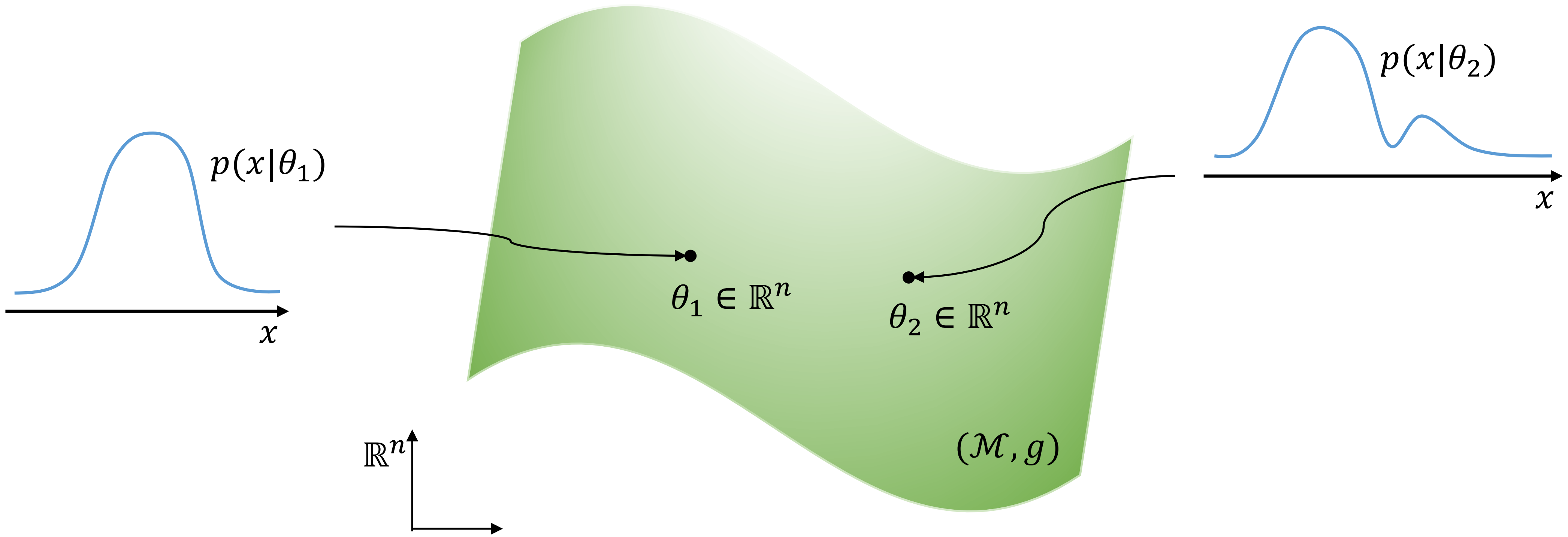


In this example, water molecules are able to move more easily in the *x*-axis direction. Therefore, diffusion tensors (b) and (c) are closer than (a) and (b)

Using the standard approach of calculating distances on the means and covariances separately, and summing the two for the total distance, results in dist(a,b) = dist(b,c), which is unsatisfactory.

Geometry of statistical manifolds

An n-dimensional statistical manifold $\mathcal{M}$ is a set of probability distributions parametrized by some smooth, continuously-varying parameter $\theta \in \mathbb{R}^n$.



Geometry of statistical manifolds

- The Fisher information defines a Riemannian metric g on a statistical manifold $\mathcal{M}$:

$$g_{ij}(\theta) = \mathbb{E}_{x\sim p(.|\theta)}\left[\frac{\partial \log p(x|\theta)}{\partial \theta_i}\frac{\partial \log p(x|\theta)}{\partial \theta_j}\right]$$

- Connection to KL divergence:

$$D_{KL}\big(p(.|\theta)||p(.|\theta + d\theta)\big) = \frac{1}{2}d\theta^T g(\theta)d\theta + o(\|d\theta\|^2)$$

Geometry of Gaussian distributions

- The manifold of Gaussian distributions $\mathcal{N}(n)$

$$\mathcal{N}(n) = \{\theta = (\mu, \Sigma) |\, \mu \in \mathbb{R}^n, \Sigma \in \mathcal{P}(n)\},$$

$$\text{where } \mathcal{P}(n) = \{P \in \mathbb{R}^{n\times n} | P = P^T, P \succ 0\}$$

- Fisher information metric on $\mathcal{N}(n)$

$$ds^2 = d\theta^T g(\theta) d\theta = d\mu^T \Sigma^{-1} d\mu + \frac{1}{2} tr\left((\Sigma^{-1} d\Sigma)^2\right)$$

- Euler-Lagrange equations for geodesics on $\mathcal{N}(n)$

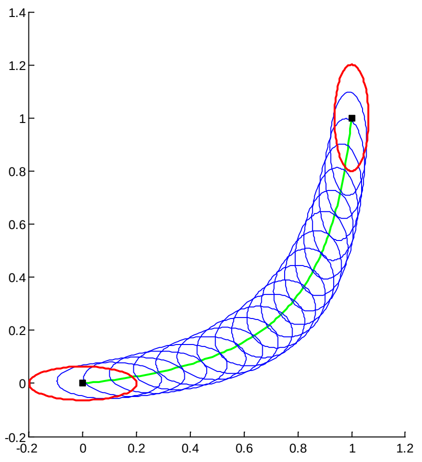
$$\frac{d^2\mu}{dt^2} - \frac{d\Sigma}{dt}\Sigma^{-1}\frac{d\mu}{dt} = 0$$

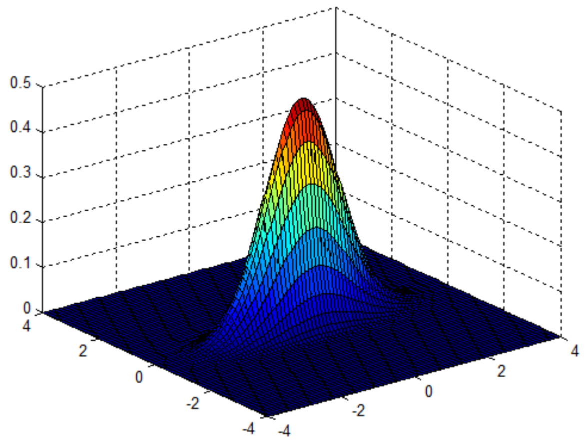
$$\frac{d^2\Sigma}{dt^2} + \frac{d\mu}{dt}\frac{d\mu^T}{dt} - \frac{d\Sigma}{dt}\Sigma^{-1}\frac{d\Sigma}{dt} = 0$$

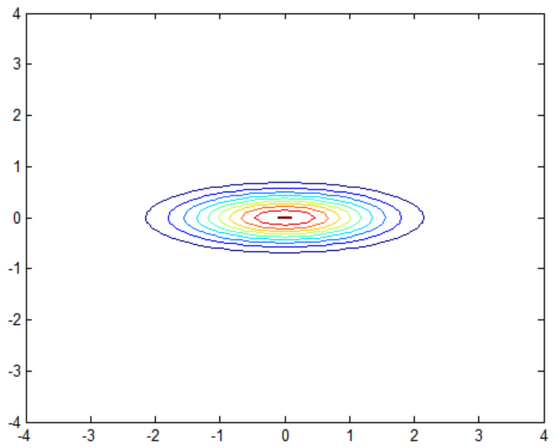
Geometry of Gaussian distributions

- Geodesic Path on $\mathcal{N}(2)$

$$\mu_0 = \begin{bmatrix}0\\0\end{bmatrix}, \Sigma_0 = \begin{bmatrix}1 & 0\\0 & 0.1\end{bmatrix}, \qquad \mu_1 = \begin{bmatrix}1\\1\end{bmatrix}, \Sigma_1 = \begin{bmatrix}0.1 & 0\\0 & 1\end{bmatrix}$$







Restriction to covariances

- Fisher information metric on $\mathcal{N}(n)$ with fixed mean $\bar{\mu}$

$$ds^2 = \frac{1}{2} tr\left((\Sigma^{-1} d\Sigma)^2\right)$$

Affine-invariant metric on $\mathcal{P}(n)$

- Invariant under general linear group $GL(n)$ action

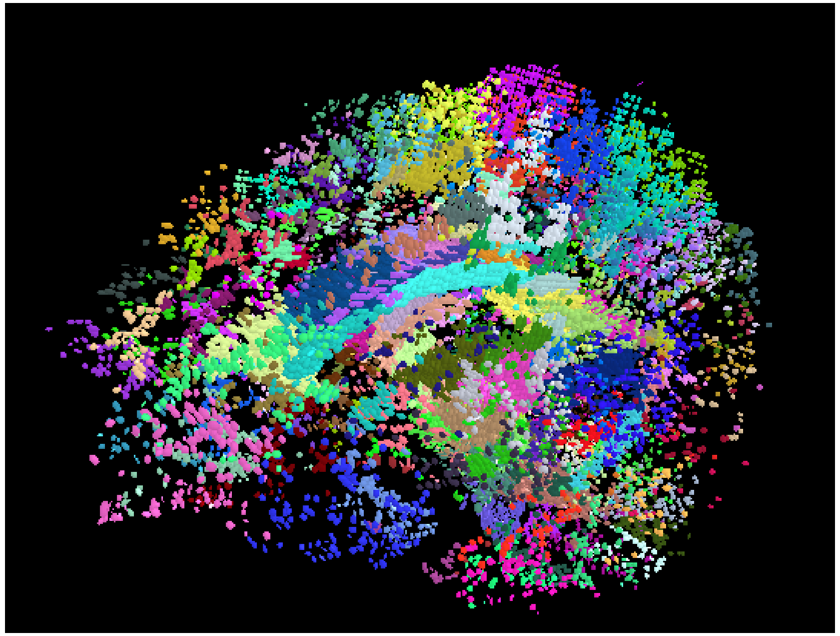
$$\Sigma \rightarrow S^T \Sigma S, S \in GL(n)$$

which implies **coordinate invariance**.

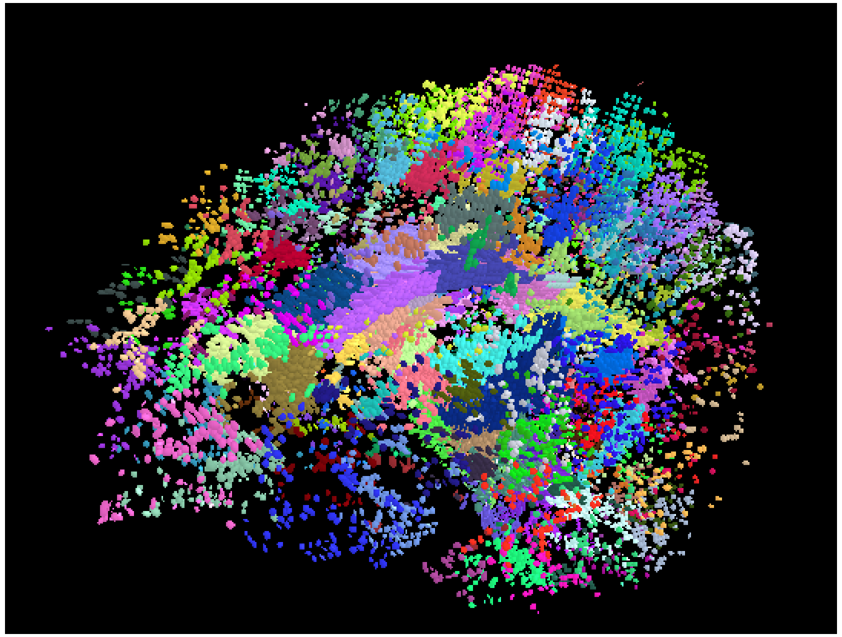
- **Closed-form** geodesic distance

$$d_{\mathcal{P}(n)}(\Sigma_1, \Sigma_2) = \left[\sum_{i=1}^{n} (\log \lambda_i(\Sigma_1^{-1}\Sigma_2))^2\right]^{1/2}$$

Results of segmentation for brain DTI



Using covarianceand Euclidean distance



Using MND distance

Human and Robot Model Identification

Inertial parameter identification

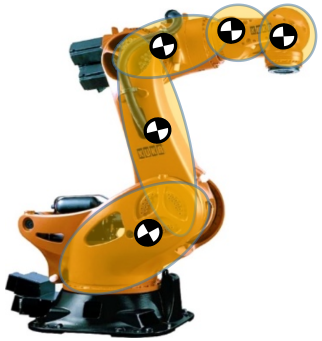
Dynamics:

$$\hat{\tau} = \tau - J^T(q)F_{ext}$$

$$= M(q,\Phi)\ddot{q} + b(q,\dot{q},\Phi)$$

$$= \Gamma(q,\dot{q},\ddot{q}) \cdot \Phi$$

- Need to identify mass-inertial parameters Φ.
- Φ is **linear** with respect to the dynamics.

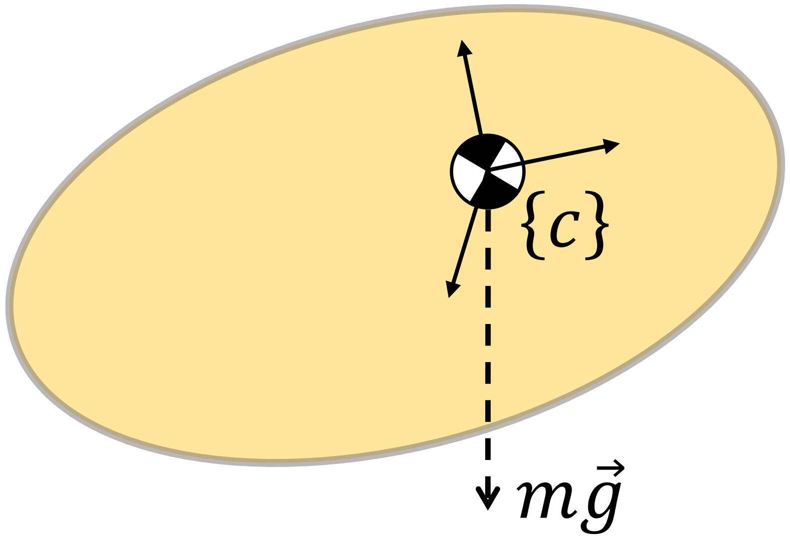


Rigid body mass-inertial properties

- Clearly $m > 0$ and $I_c > 0$
- The fact that $I_c > 0$ is necessary but not sufficient: Because **mass density is non-negative everywhere**, the following must also hold:

$$\boldsymbol{\lambda_1 + \lambda_2 > \lambda_3}$$
$$\boldsymbol{\lambda_2 + \lambda_3 > \lambda_1}$$
$$\boldsymbol{\lambda_3 + \lambda_1 > \lambda_2}$$

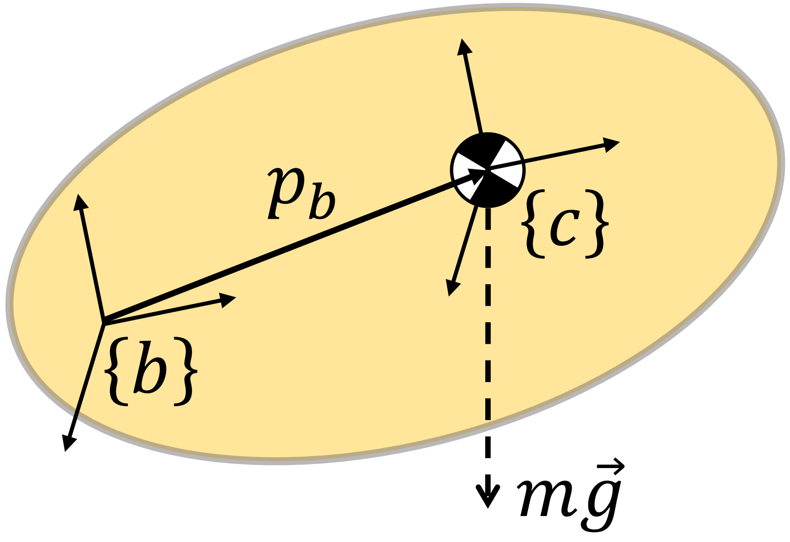
where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of I_c (so-called **triangle inequality relation** for rigid body inertias)



$$I_c = \begin{bmatrix} I_c^{xx} & I_c^{xy} & I_c^{xz} \\ I_c^{xy} & I_c^{yy} & I_c^{yz} \\ I_c^{xz} & I_c^{yz} & I_c^{zz} \end{bmatrix}$$

Rigid body mass-inertial properties

For the purposes of dynamic calibration it is more convenient to identify the inertia parameters with respect to a body frame {b}, i.e., the six parameters associated with I_b, the three parameters associated with h_b, and the mass m, resulting in a total of 10 parameters, denoted $\phi \in \mathbb{R}^{10}$, per rigid body.



$$I_b = \begin{bmatrix} I_b^{xx} & I_b^{xy} & I_b^{xz} \\ I_b^{xy} & I_b^{yy} & I_b^{yz} \\ I_b^{xz} & I_b^{yz} & I_b^{zz} \end{bmatrix}$$

$$h_b = m \cdot p_b$$

Rigid body mass-inertial properties

Wensing et al (2017) showed that Traversaro's sufficiency conditions are equivalent to the following:

$$P(\phi) = \begin{bmatrix} S_b & h_b \\ h_b^T & m \end{bmatrix} \in \mathbb{R}^{4\times 4}$$

is **positive definite** (i.e. $P(\phi) \in \mathcal{P}(4)$) , where

$$S_b = \int xx^T \rho(x) dV = \frac{1}{2} tr(I_b) \cdot \mathbb{1} - I_b$$

with $\rho(x)$ the mass density function.

Inertial parameter identification

- Let $\phi_i \in \mathbb{R}^{10}$ be the inertial parameters for link i, and

$$\Phi = [\phi_1, \cdots, \phi_N] \in \mathbb{R}^{10N}.$$

- Sampling the dynamics at T time instances, the identification problem reduces to a least-squares problem:

$$\boldsymbol{A} \cdot \boldsymbol{\Phi} = \boldsymbol{b} \in \mathbb{R}^{mT}$$

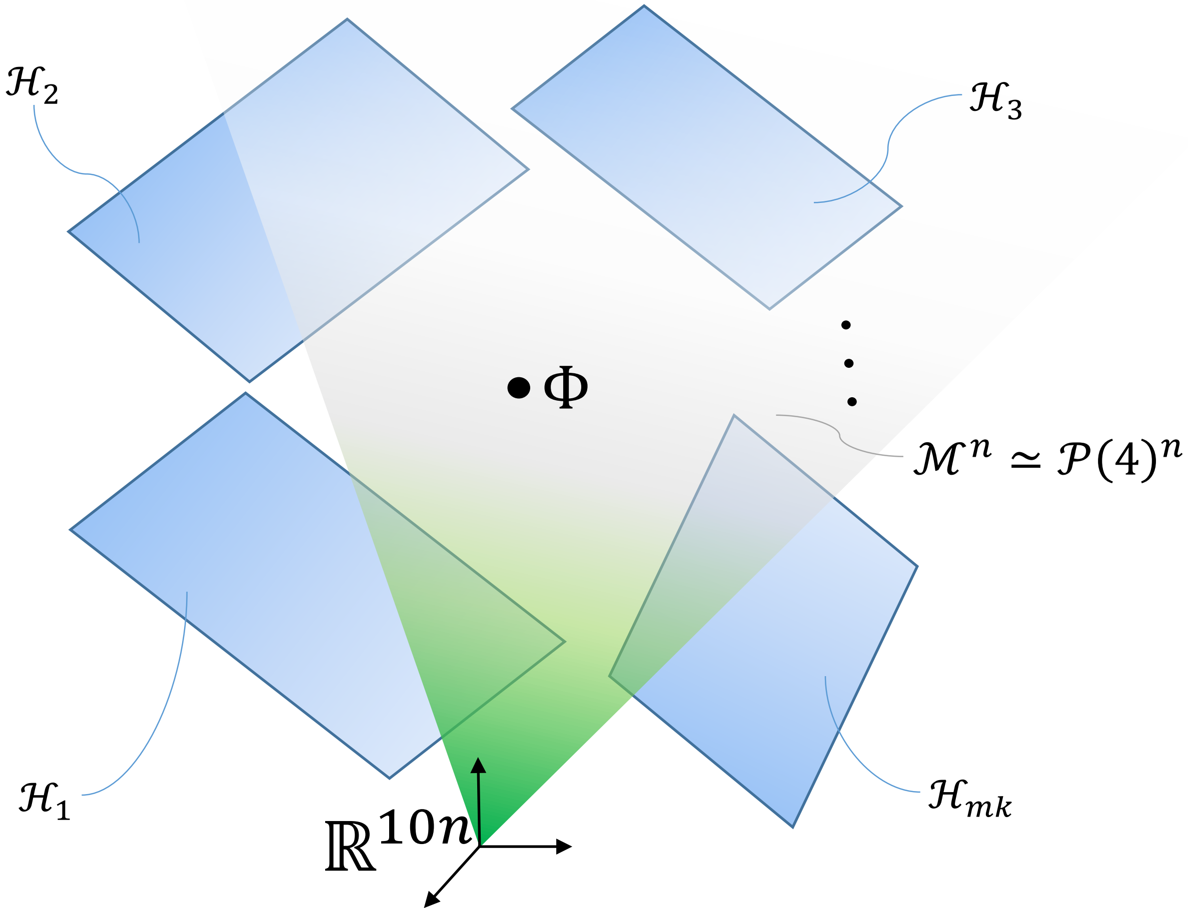
where $A = \begin{bmatrix} \Gamma(q(t_1), \dot{q}(t_1), \ddot{q}(t_1)) \in \mathbb{R}^{m\times 10N} \\ \vdots \\ \Gamma(q(t_T), \dot{q}(t_T), \ddot{q}(t_T)) \in \mathbb{R}^{m\times 10N} \end{bmatrix} = \begin{bmatrix} a_1^T \\ \vdots \\ a_{mT}^T \end{bmatrix}$ and $b = \begin{bmatrix} \hat{\tau}(t_1) \in \mathbb{R}^m \\ \vdots \\ \hat{\tau}(t_T) \in \mathbb{R}^m \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_{mT} \end{bmatrix}$

- Φ should also satisfy

$$\boldsymbol{P(\phi_i) > 0 \, , i = 1, \cdots, N}.$$

Geometry of $A \cdot \Phi = b$

Find Φ in $\mathcal{M}^n \simeq \mathcal{P}(4)^n$ **closest** to each of the hyperplanes $\mathcal{H}_i = \{x : a_i^T x = b_i\}$. Implies the need for a **distance metric** $d(\cdot,\cdot)$ on Φ.



$$\min_{\Phi, \{\hat{\Phi}_i\}_{i=1}^{mk}} \sum_{i=1}^{mk} w_i \cdot d(\Phi, \hat{\Phi}_i)^2 + \underbrace{\gamma \cdot d(\Phi, \hat{\Phi}_0)^2}_{\text{regularization term}}$$

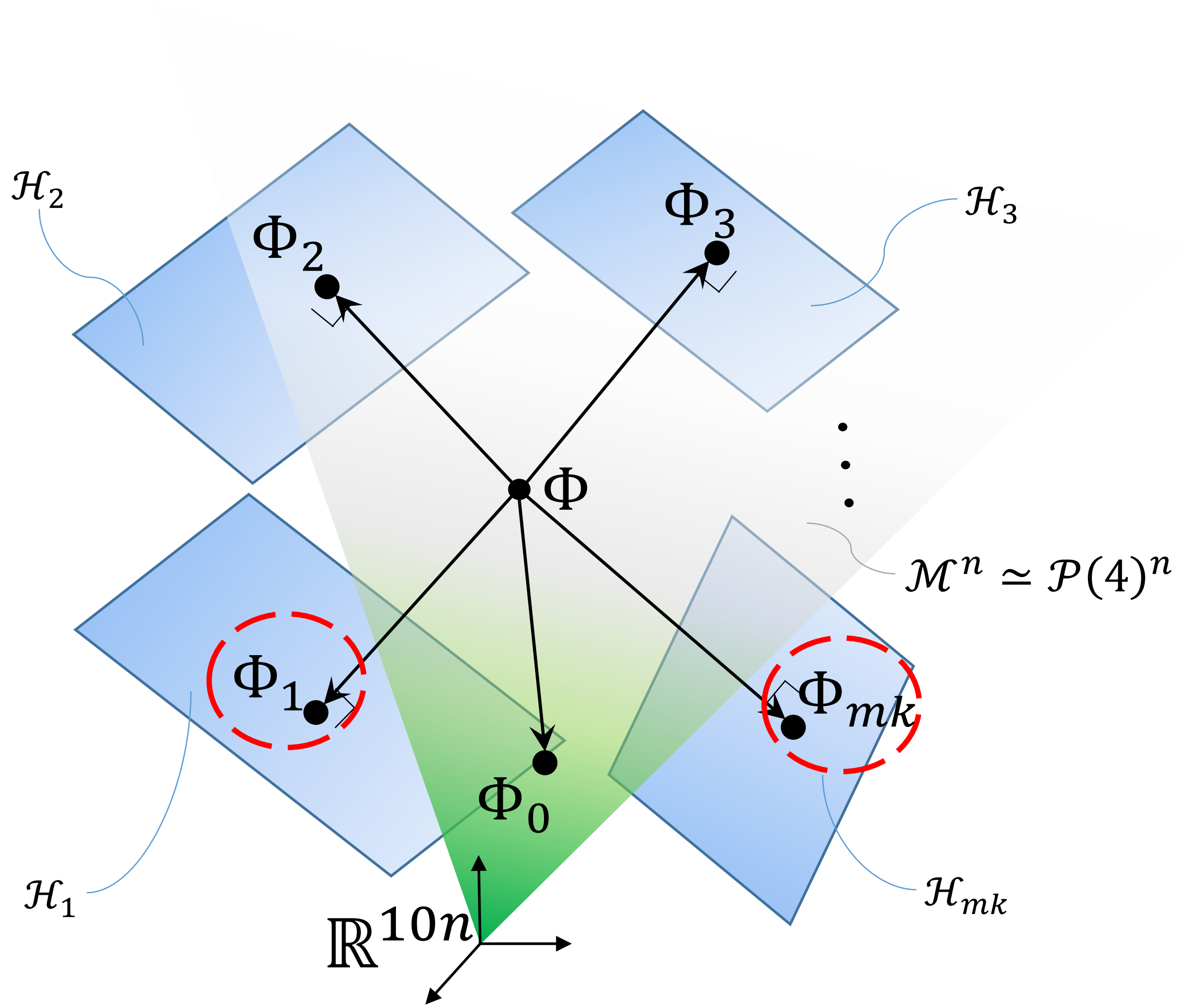
$$\text{s.t.}\ \hat{\Phi}_i \in \mathcal{H}_i\ ,\ i = 1, \cdots, mk$$

$\hat{\Phi}_i$: **Projection** of Φ onto $\mathcal{H}_i$ $(i = 1, \cdots, mT)$

$\hat{\Phi}_0$: Nominal values (from, e.g., CAD orstatistical data)

Geometry of ordinary least squares

- For **Standard Euclidean metric** on Φ : $d(\Phi, \widehat{\Phi}) = \|\Phi - \widehat{\Phi}\|$



Ordinary least squares

$$\min_{\Phi} \|A\Phi - b\|^2 + \underbrace{\gamma \left\|\Phi - \widehat{\Phi}_0\right\|^2}_{regularization-term}$$

with closed-form solution $\Phi^{LS} = (A^T A + \gamma I)^{-1} (A^T b + \gamma \Phi_0)$ is equivalent to the following :

$$\min_{\Phi, \{\widehat{\Phi}_i\}_{i=1}^{mk}} \sum_{i=1}^{mk} \|a_i\|^2 \left\|\Phi - \widehat{\Phi}_i\right\|^2 + \underbrace{\gamma \left\|\Phi - \widehat{\Phi}_0\right\|^2}_{\text{regularization term}}$$

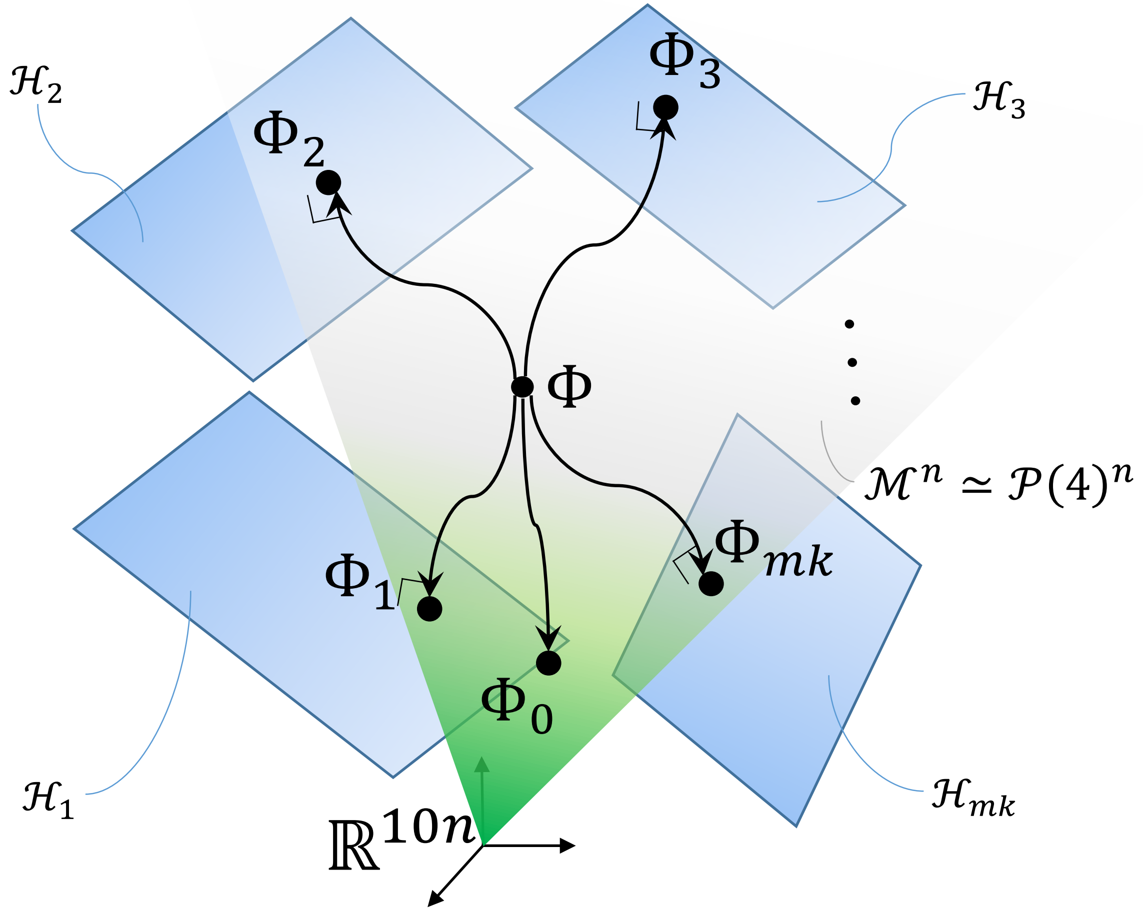
$$\text{s.t.} \widehat{\Phi}_i \in \mathcal{H}_i \ , \ i = 1, \cdots, mk$$

$\widehat{\Phi}_i$: **Euclidean Projection** of Φ onto $\mathcal{H}_i$

$\widehat{\Phi}_0$: Prior value

Using geodesic distance on P(n)

- **Geodesic distance** based least-squares solution:
- $d_{\mathcal{M}^n}(\Phi, \widehat{\Phi}) = \sum_{j=1}^{n} d_{\mathcal{P}(4)}\left(P(\phi_j), P(\hat{\phi}_j)\right)$



$$\min_{\Phi, \{\widehat{\Phi}_i\}_{i=1}^{mk}} \sum_{i=1}^{mk} w_i \cdot d_{\mathcal{M}^n}(\Phi, \widehat{\Phi}_i)^2 + \underbrace{\gamma d_{\mathcal{M}^n}(\Phi, \widehat{\Phi}_0)^2}_{regularization-term}$$

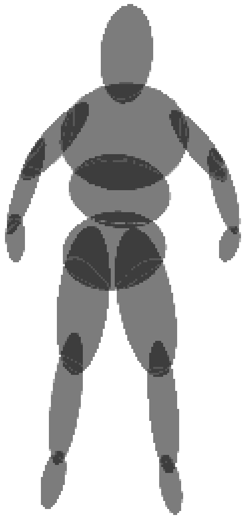
$$\text{s.t.} \widehat{\Phi}_i \in \mathcal{H}_i,\ i = 1, \cdots, mk$$

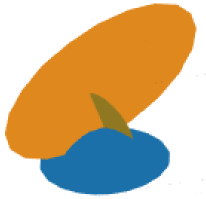
$\widehat{\Phi}_i$: **Geodesic Projection** of Φ onto $\mathcal{H}_i$

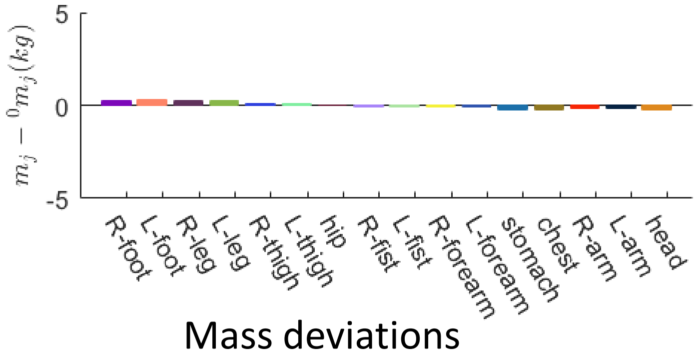
$\widehat{\Phi}_0$: Prior value

Example

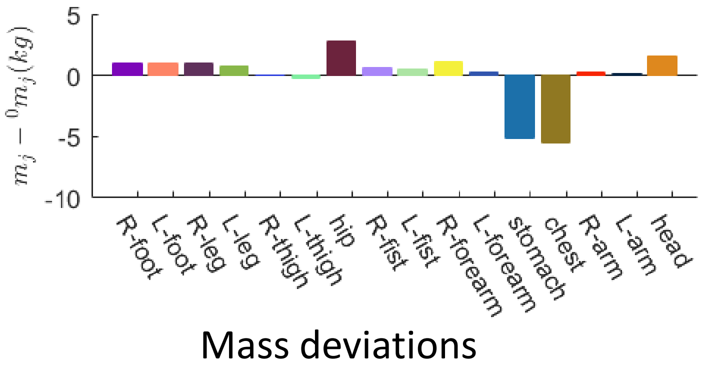
Prior







Ordinary least-squares with LMI

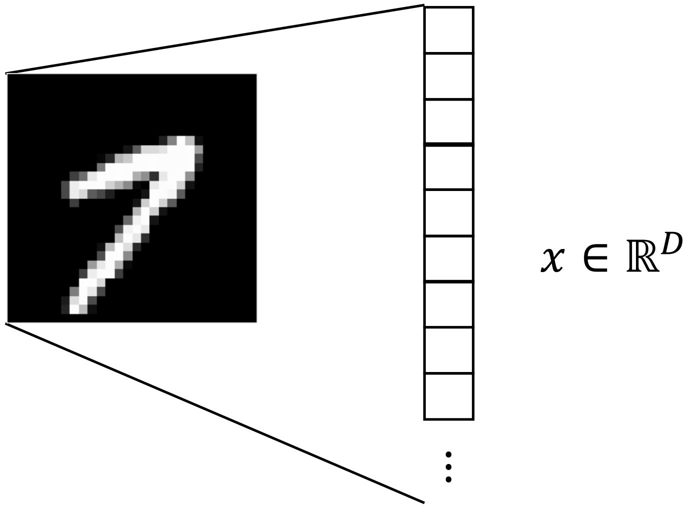


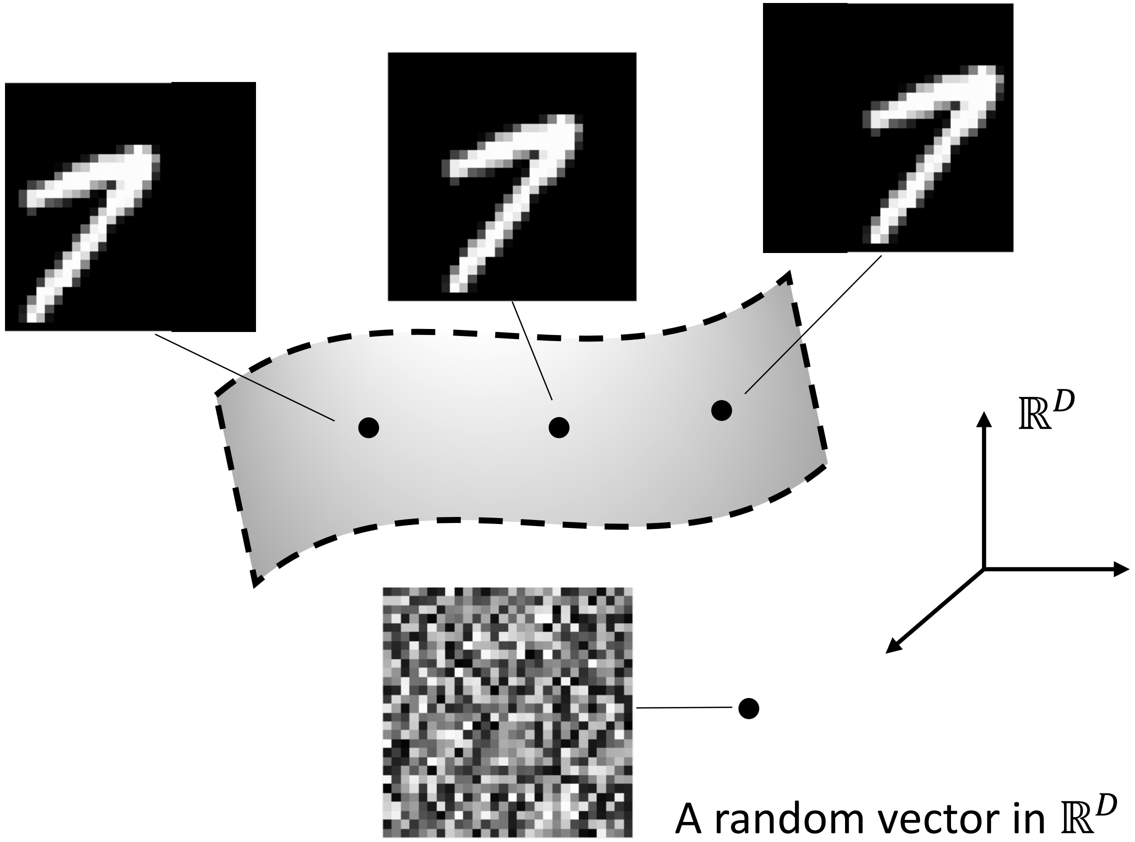
Geodesic distance on P(n)

Machine Learning

Manifold hypothesis

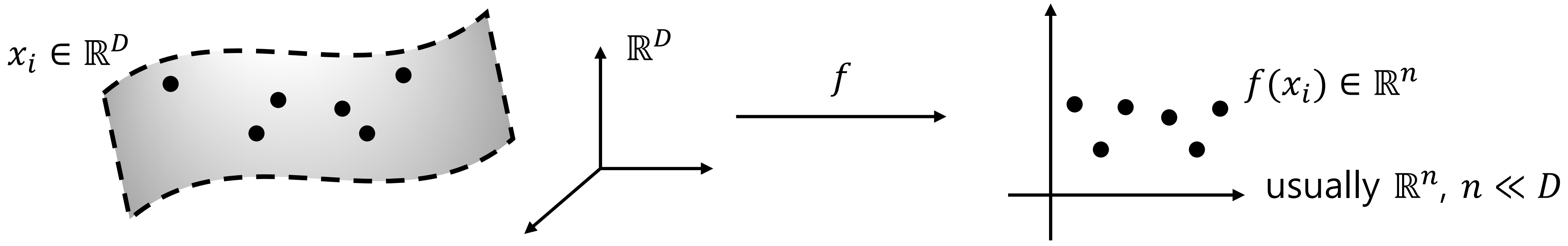
- Consider vector representation of data, e.g., an image, as $x \in \mathbb{R}^D$
- Meaningful data lie on a d-dim. manifold in $\mathbb{R}^D$, $d \ll D$
  - Ex) image of number '7'





Manifold learning and dimension reduction

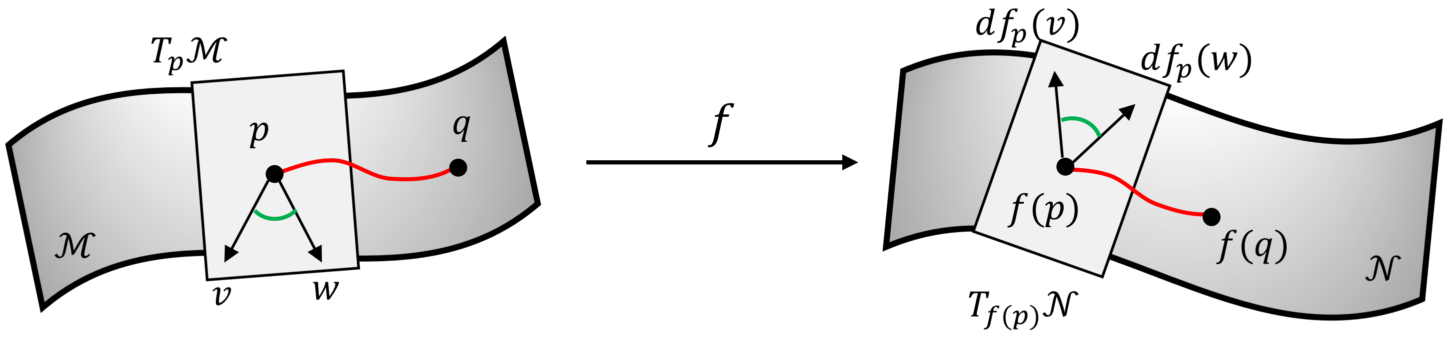
- Given data $\{x_i\}_{i=1,\cdots,N}$, $x_i \in \mathbb{R}^D$, find a map f from data space to lower dimensional space while minimizing a global measure of **distortion**:



- Existing methods
  - Linear methods: PCA (Principal Component Analysis), MDS (Multi-Dimensional Scaling), …
  - Nonlinear methods (manifold learning): LLE (Locally Linear Embedding), Isomap, LE (Laplacian Eigenmap), DM (Diffusion Map), …

Coordinate-Invariant Distortion Measures

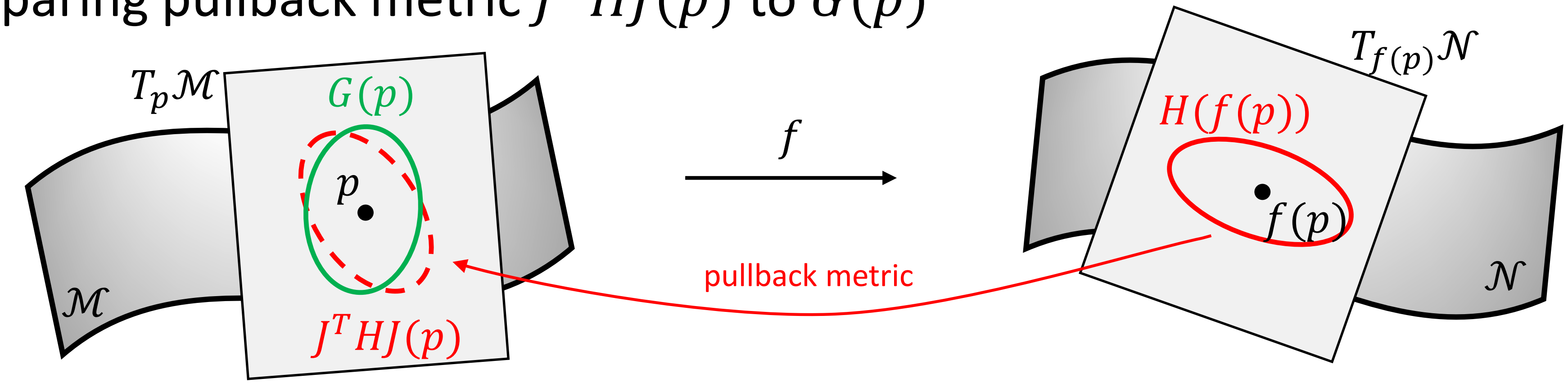
- Consider a smooth map $f: \mathcal{M} \rightarrow \mathcal{N}$ between two compact Riemannian manifolds
  - $(\mathcal{M}, g)$: local coord. $x = (x_1, \cdots, x_m)$, Riemannain metric $G = (g_{ij})$
  - $(\mathcal{N}, h)$: local coord. $y = (y_1, \cdots, y_n)$, Riemannian metric $H = (h_{\alpha\beta})$
- Isometry
  - Map preserving length, angle, and volume - the ideal case of no distortion
  - If $\dim(\mathcal{M}) \leq \dim(\mathcal{N})$, f is an **isometry** when $\boldsymbol{J(x)^\top H(f(x))J(x) = G(x)}$ for all $x \in \mathcal{M}$, where $J = \left(\frac{\partial f^\alpha}{\partial x^i}\right) \in \mathbb{R}^{n \times m}$



$df_x: T_x\mathcal{M} \rightarrow T_{f(x)}\mathcal{N}$ is the differential of $f: \mathcal{M} \rightarrow \mathcal{N}$

Coordinate-Invariant Distortion Measures

- Comparing pullback metric $J^\top HJ(p)$ to $G(p)$



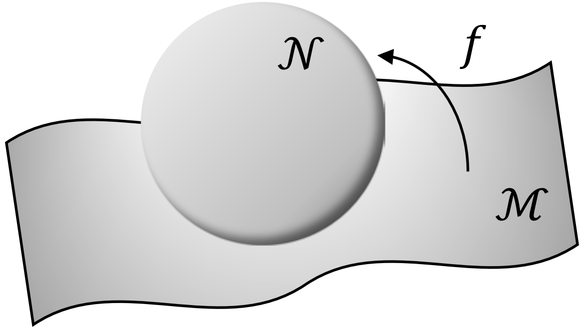
- Let $\lambda_1, \cdots, \lambda_m$ be roots of $\det(J^\top HJ - \lambda G) = 0$
  ($\lambda_1, \cdots, \lambda_m = 1$ in the case of isometry)
- $\sigma(\lambda_1, \cdots, \lambda_m)$ denote any symmetric function, e.g., $\sigma(\lambda_1, \cdots, \lambda_m) = \sum_{i=1}^{m} \frac{1}{2} \|\lambda_i - 1\|^2$
- Global distortion measure

$$\int_{\mathcal{M}} \sigma(\lambda_1, \cdots, \lambda_m)\sqrt{\det G}\, dx^1 \cdots dx^m$$

Harmonic Maps

- Assume $\sigma(\lambda_1, \cdots, \lambda_m) = \sum_{i=1}^{m} \lambda_i$, and boundary conditions $\partial\mathcal{N} = f(\partial\mathcal{M})$ are given
- Define the global distortion measure as

$$D(f) = \int_{\mathcal{M}} Tr(J^\top H J G^{-1})\sqrt{\det G}\, dx^1 \cdots dx^m$$



- Extremals of $D(f)$ are known as **harmonic maps**
- Variational equation (for $\alpha = 1, \cdots, n$)

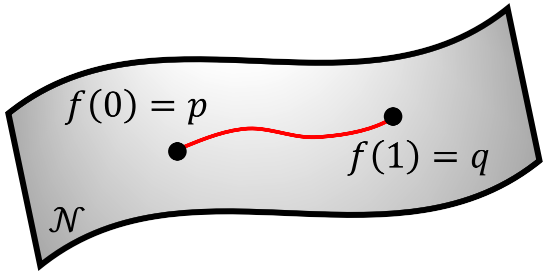
$$\sum_{i=1}^{m}\sum_{j=1}^{m} \frac{1}{\sqrt{\det G}} \frac{\partial}{\partial x^i}\left(\frac{\partial f^\alpha}{\partial x^j} g^{ij}\sqrt{\det G}\right) + \sum_{\beta=1}^{n}\sum_{\gamma=1}^{n} g^{ij}\Gamma_{\beta\gamma}^{\alpha} \frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j} = 0$$

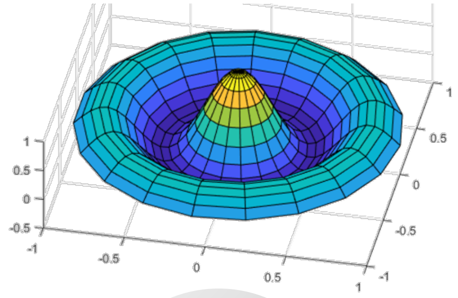
where g^{ij} is (i,j) entry of G^{-1}, $\Gamma_{\beta\gamma}^{\alpha}$ denote the Christoffel symbols of the second kind

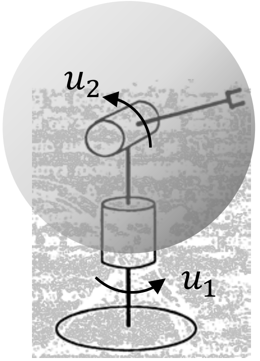
- **Physical interpretation: Wrap marble ($\mathcal{N}$) with rubber ($\mathcal{M}$) (Harmonic maps correspond to elastic equilibria)**

Examples of Harmonic Maps

- Lines ($f\colon [0,1] \to [0,1]$)
  - $D(f) = \int_0^1 \dot{f}^2 dt$
  - $\ddot{f} = 0$
- Geodesics ($f\colon [0,1] \to \mathcal{N}$)
  - $D(f) = \int_0^1 \dot{f}^\top H \dot{f} dt$
  - $\frac{d^2 f^\alpha}{dt^2} + \sum_{\beta=1}^n \sum_{\gamma=1}^n \Gamma_{\beta\gamma}^\alpha \frac{df^\beta}{dt} \frac{df^\gamma}{dt} = 0$
- Laplace's equation on $\mathbb{R}^2$ ($f\colon \mathbb{R}^2 \to \mathbb{R}$)
  - $\nabla^2 f = 0$
  - Harmonic functions
- 2R Spherical mechanism ($f\colon T^2 \to S^2$)
  - $ds^2 = \epsilon_1 du_1^2 + \epsilon_2 du_2^2, \epsilon_1 \epsilon_2 = 1$
  - $D(f) = \pi^2(\epsilon_1 + 2\epsilon_2), \epsilon_1^* = \sqrt{2}, \epsilon_2^* = 1/\sqrt{2}$

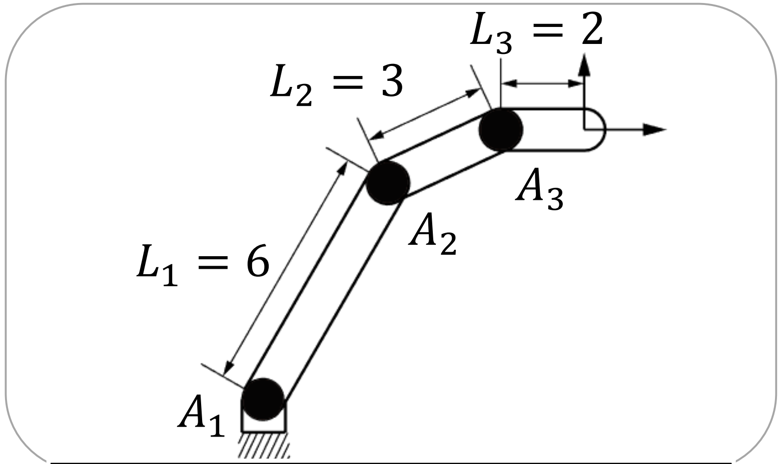






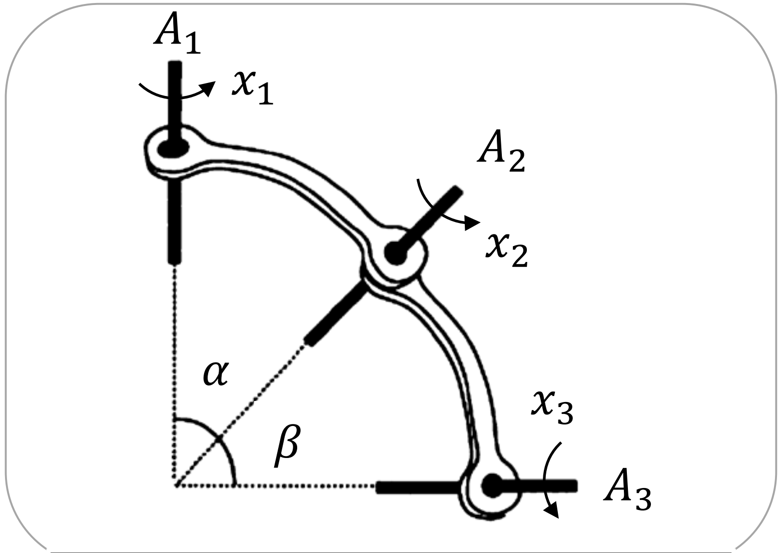
Harmonic Maps and Robot Kinematics

- Open chain planar mechanism ($f: T^n \rightarrow \mathrm{SE}(2)$)
  - $f(x_1, \cdots, x_n) = Me^{A_1 x_1} \ldots e^{A_n x_n}$
  - $D(f) = (L_1^2 + 2L_2^2 + \cdots + nL_n^2)d + nc$
  - Optimal link length: $(L_1^*, \cdots, L_n^*) = (1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n})$



3-link finger example

- 3R Spherical mechanism ($f: T^3 \rightarrow \mathrm{SO}(3)$)
  - $f(x_1, x_2, x_3) = e^{A_1 x_1} e^{A_2 x_2} e^{A_3 x_3}$
  - $D(f)$ is invariant to α, β
  - Workspace volume $W(f) = \sin\alpha \sin\beta$ Volume(SO(3))
  - $\alpha = 90°, \beta = 90°$ maximize the workspace volume



3-link spherical wrist mechanism

Taxonomy of Manifold Learning Algorithms

	$\mathcal{M}$	$\mathcal{N}$	G^{-1} (inverse pseudo-metric)	H	$\sigma(\lambda)$	Volume form	Constraint
LLE (Locally Linear Embedding)	Bounded region in $\mathbb{R}^D$ containing data points	$\mathbb{R}^d$	$\Delta x \Delta x^\top$ (Δx is local reconstruction error obtained when running the algorithm)	I	$\sum_{i=1}^{m} \lambda_i$	$\rho(x)dx$	$\int_{\mathcal{M}} f(x)f(x)^\top \rho(x)dx = I$
LE (Laplacian Eigenmap)	Same as above	$\mathbb{R}^d$	$\int_{\mathcal{M}\cap B_\epsilon(x)} k(x,z)(x-z)(x-z)^\top \rho(z)dz$	I	$\sum_{i=1}^{m} \lambda_i$	$\rho(x)dx$	$\int_{\mathcal{M}} d(x)f(x)f(x)^\top \rho(x)dx = I$ ($d(x) = \int_{\mathcal{M}\cap B_\epsilon(x)} k(x,z)\rho(z)dz$)
DM (Diffusion Map)	Same as above	$\mathbb{R}^d$	$\int_{\mathcal{M}\cap B_\epsilon(x)} k(x,z)(x-z)(x-z)^\top dz$	I	$\sum_{i=1}^{m} \lambda_i$	$1dx$	$\int_{\mathcal{M}} f(x)f(x)^\top dx = I$

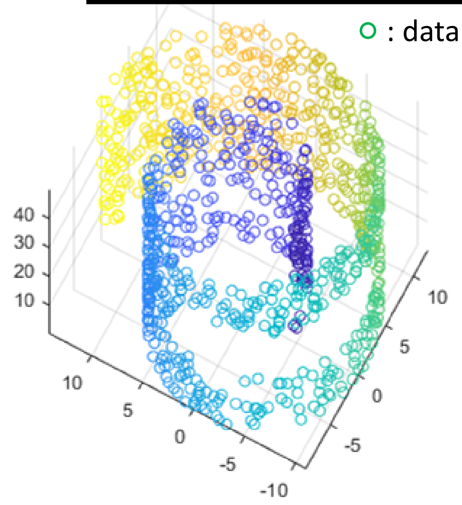
Manifold learning algorithms such as LLE, LE, DM share a similar objective as **harmonic maps** while having equality constraint to avoid trivial solution $f = const. \in \mathbb{R}^d$

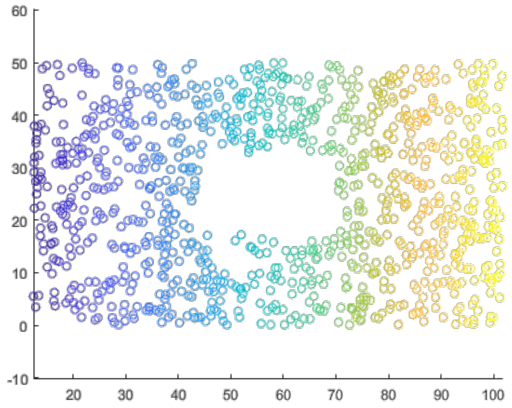
- $k(x,y)$ is a kernel function usually chosen by the user
- Using heat kernel $k(x,y) = (4\pi h)^{-d/2} \exp(-\frac{\|x-z\|^2}{4h})$ gives a way to estimate the Laplace-Beltrami operator

Example: Swiss roll (diffusion map)

Swiss roll data
(2-dim manifold in 3-dim space)

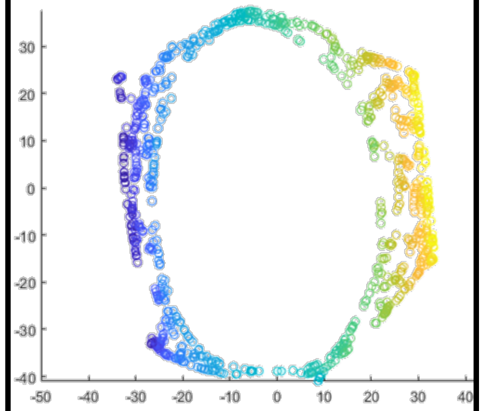
○ : data points





Flattened swiss roll

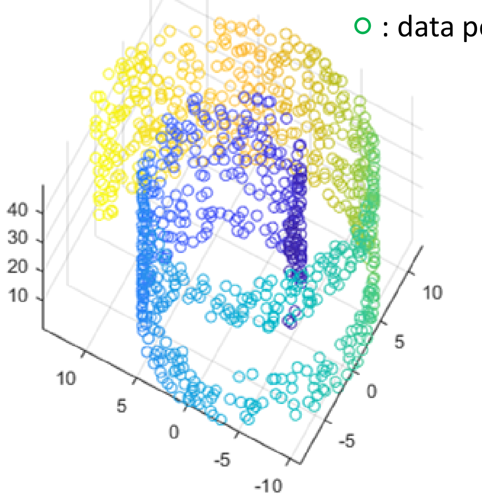
Diffusion map embedding

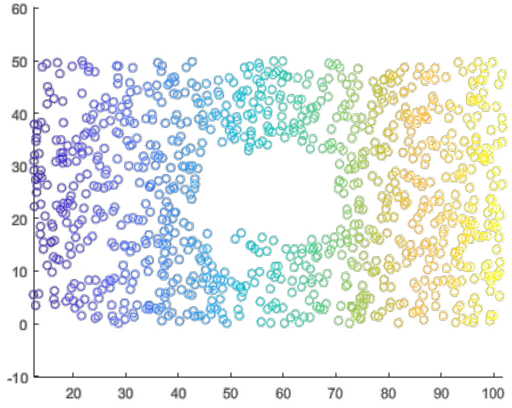


Example: Swiss roll (geometric distortion)

Swiss roll data
(2-dim manifold in 3-dim space)

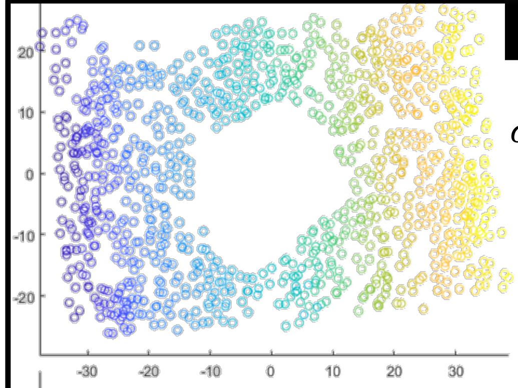
○ : data points



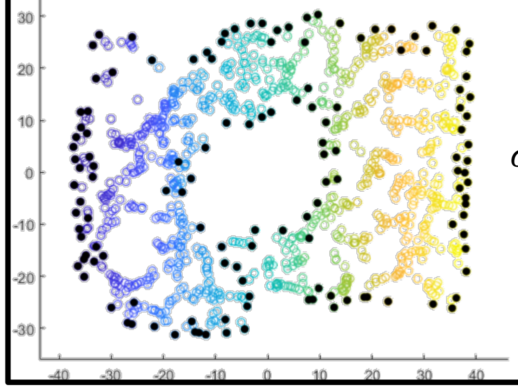


Flattened swiss roll

Minimum distortion results



$$\sigma(\boldsymbol{\lambda}) = \sum_{i=1}^{m} \frac{1}{2} \|\lambda_i - 1\|^2$$



Harmonic mapping +

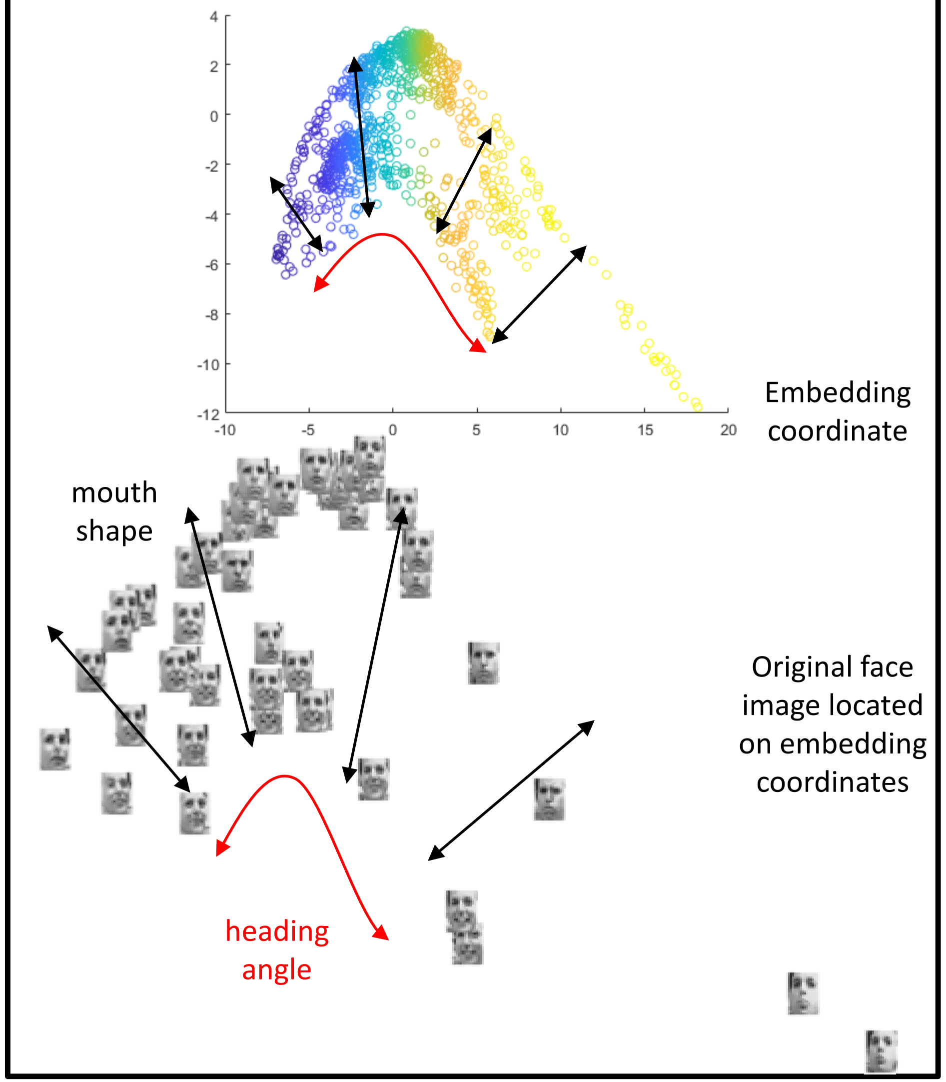
$$\sigma(\boldsymbol{\lambda}) = \sum_{i=1}^{m} \frac{1}{2} \|\lambda_i - 1\|^2$$

for boundary points

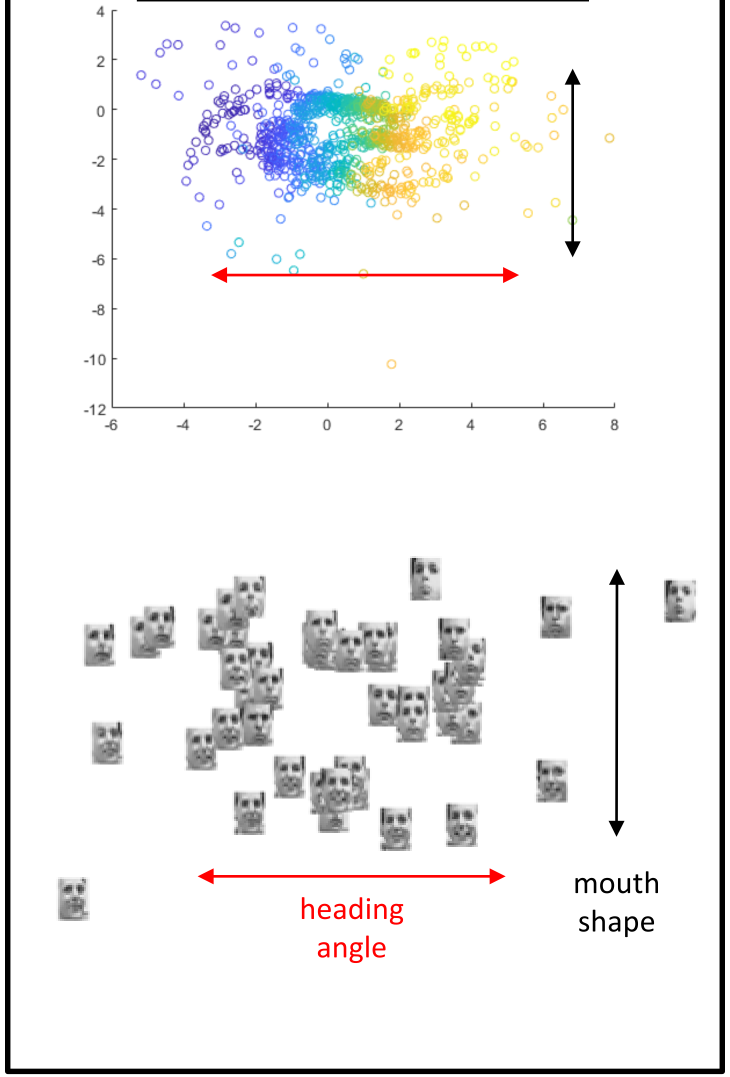
● : boundary points

Example: Faces

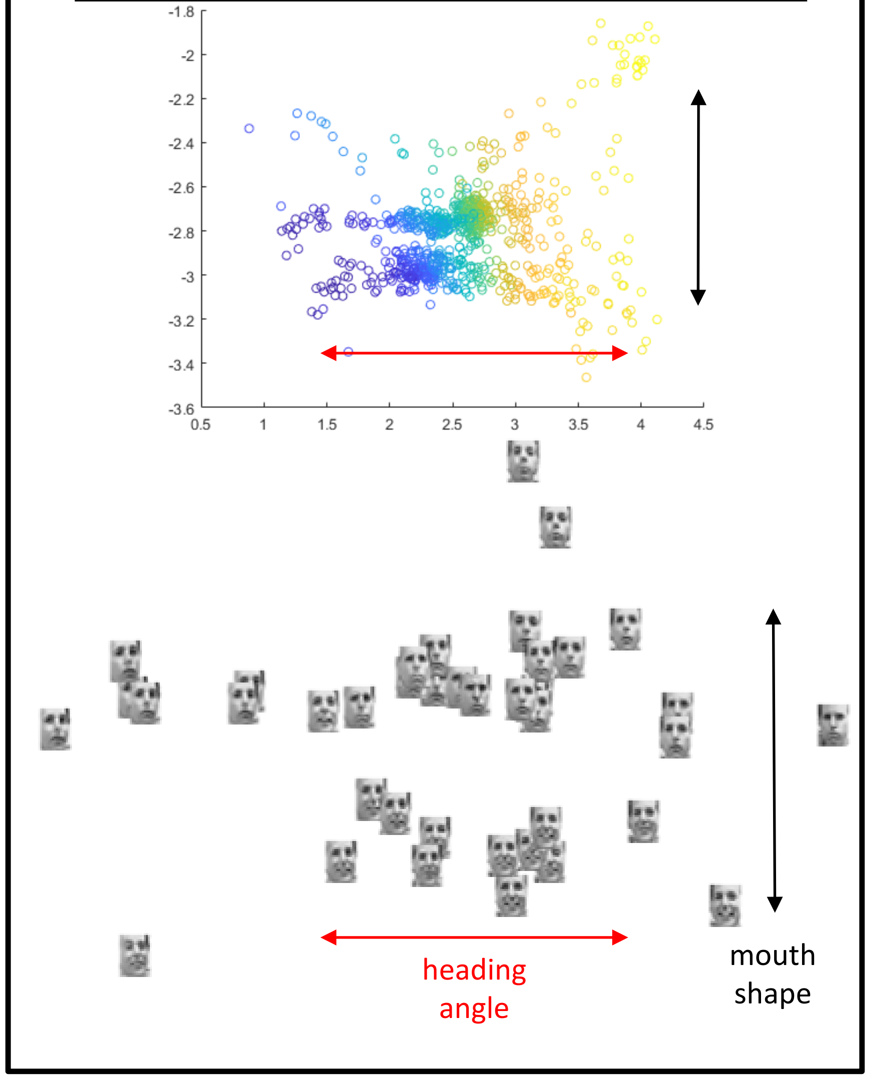
Laplacian eigenmap embedding



Min distortion
$(\sigma(\boldsymbol{\lambda}) = \sum_{i=1}^{m} \frac{1}{2} \|\lambda_i - 1\|^2)$



Min distortion (Harmonic mapping + $\sigma(\boldsymbol{\lambda}) = \sum_{i=1}^{m} \frac{1}{2} \|\lambda_i - 1\|^2$ for boundary points)



Concluding Remarks

Concluding Remarks

- Geometric methods have unquestionably been effective in solving a wide range of problems in robotics
- Many problems in perception, planning, control, learning, and other aspects of robotics can be reduced to a geometric problem, and geometric methods offer a powerful set of coordinate-invariant tools for their solution.

References

References

(http://robotics.snu.ac.kr/fcp)

Robot Mechanics and Control

- "Modern Robotics: Mechanics, Planning, and Control," KM Lynch, FC Park, Cambridge University Press, 2017 (http://modernrobotics.org)
- "A Mathematical Introduction to Robotic Manpulation," RM Murray, ZX Li, S Sastry, CRC Press, 1994
- A Lie group formulation of robot dynamics, FC Park, J Bobrow, S Ploen, *Int. J. Robotics Research*, 1995
- Distance metrics on the rigid body motions with applications to kinematic design, FC Park, *ASME J. Mech. Des.*, 1995
- Singularity analysis of closed kinematic chains, Jinwook Kim, FC Park, *ASME J. Mech. Des*, 1999
- Kinematic dexterity of robotic mechanisms, FC Park, R. Brockett, *Int. J. Robotics Research*, 1994
- Least squares tracking on the Euclidean group, *IEEE Trans. Autom.* Contr., Y. Han and F.C. Park, 2001
- Newton-type algorithms for dynamics-based robot motion optimization, SH Lee, JG Kim, FC Park et al, *IEEE Trans. Robotics*, 2005

References

Trajectory Optimization and Motion Planning

- Newton-type algorithms for dynamics-based robot motion optimization, SH Lee, JG Kim, FC Park et al, *IEEE Trans. Robotics*, 2005
- Smooth invariant interpolation of rotations, FC Park and B. Ravani, *ACM Trans. Graphic*s, 1997
- Progress on the algorithmic optimization of robot motion, A Sideris, J Bobrow, FC Park, *in Lecture Notes in Control and Information Sciences*, vol. 340, Springer-Verlag, October 2006
- Convex optimization algorithms for active balancing of humanoid robots, Juyong Park, J Haan, FC Park, *IEEE Trans. Robotics*, 2007
- Tangent bundle RRT: a randomized algorithm for constrained motion planning, B Kim, T Um, C Suh, FC Park, *Robotica*, 2016
- Randomized path planning for tasks requiring the release and regrasp of objects, J Kim, FC Park, *Advanced Robotics*, 2016
- Randomized path planning on vector fields, I Ko, FC Park, *Int. J. Robotics Res.,* 2014
- Motion optimization using Gaussian processs dynamical models, H Kang, FC Park, *Multibody Sys. Dyn.* 2014

References

Vision and Image Analysis

- Numerical optimization on the Euclidean group with applications to camera calibration, S Gwak, JG Kim, FC Park*, IEEE Trans. Robotics Autom*., 2003.
- Geometric direct search algorithms for image registration, S Lee, M Choi, H. Kim, FC Park, *IEEE Trans. Image Proc.*, 2007.
- Particle filtering on the Euclidean group: framework and applications, JH Kwon, M Choi, FC Park, CM Chun, *Robotica*, 2007.
- Visual tracking via particle filtering on the affine group, JH Kwon, FC Park, Int. J. Robotics Res., 2010.
- A geometric particle filter for template-based visual tracking, JH Kwon, HS Lee, FC Park, KM Lee, *IEEE Trans. PAMI*, 2014
- DTI segmentation and fiber tracking using metrics on multivariate normal distributions, M Han, FC Park, *J. Math. Imaging and Vision*, 2014
- A stochastic global optimization algorithm for the two-frame sensor calibration problem, JH Ha, DH Kang, FC Park, *IEEE Trans. Ind. Elec* 2016

References

Attention, Learning, Geometry

- A minimum attention control law for ball catching, CJ Jang, JE Lee, SH Lee, FC Park, *Bioinsp. Biomim*. 2015
- Kinematic feedback control laws for generating natural arm movements, DH Kim, CJ Jang, FC Park, *Bioinsp. Biomim.* 2013
- Kinematic dexterity of robotic mechanisms, FC Park, RW Brockett, I*nt. J. Robotics Res.* 1994