Manifolds – Problem Solutions

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These notes are slightly modified from those written by Neil Lambert and Alice Rogers

1. Manifolds

Problem 1.1. *Show that the induced topology indeed satisfies the definition of a topology.*

Solution:

Let $\mathcal{U} = \{U_i\}$ be the topology of Y and $X \subset Y$. The induced topology is $\mathcal{V} = \{U_i \cap X | U_i \in \mathcal{U}\}$.

i) $\emptyset = \emptyset \cap X$ so since $\emptyset \in \mathcal{U}$ we have $\emptyset \in \mathcal{V}$.

ii) $X = X \cap Y$ so since $Y \in \mathcal{U}$ we have $X \in \mathcal{V}$

iii) Take $V_i = U_i \cap X \in \mathcal{V}$ then

$$\begin{aligned} V_1 \cap V_2 \cap ... \cap V_n &= (U_1 \cap X) \cap (U_2 \cap X) \cap ... \cap (U_n \cap X) \\ &= (U_1 \cap U_2 \cap ... \cap U_n) \cap X \\ &\in \mathcal{V} \end{aligned} \tag{1.1}$$

iv) For an arbitrary number of V_i's:

$$\bigcup_i V_i = \bigcup_i (U_i \cap X) = \left(\bigcup_i U_i \right) \cap X \in \mathcal{V} \tag{1.2}$$

Problem 1.2. *Why aren't closed subsets of $\mathbb{R}^n$, e.g. a disk with boundary or a line in $\mathbb{R}^2$, along with the identity map charts (note that in its own induced topology any subset of $\mathbb{R}^n$ is an open set)?*

Solution: If $C \subset \mathbb{R}^n$ is closed we may still view it as an open set in its own induced topology. The identity map $id : C \to \mathbb{R}^n$ is certainly continuous (in inverse image of an open set is open by definition of the induced topology). It is also clearly 1-1 and hence is a bijection onto its image. However consider $id^{-1} : id(C) \subset \mathbb{R}^n \to C$ this has

$$(id^{-1})^{-1}(C) = C \tag{1.3}$$

but since C is open in its own induced topology and closed in the topology of $\mathbb{R}^n$ we see that id^{-1} is not continuous (as $(id^{-1})^{-1}$ of an open set is closed). Thus id is not a homeomorphism.

Problem 1.3. *What is $\mathbb{R}P^1$?*

Solution:

By definition $\mathbb{R}P^1 = \{(x,y) \in \mathbb{R}^2 - (0,0) | (x,y) \sim (\lambda x, \lambda y), \lambda \in \mathbb{R} - 0\}$. A point $(x,y) \in \mathbb{R}^2 - (0,0)$ defines a line through the origin. The point $(\lambda x, \lambda y)$ with $\lambda \neq 0$ will define the same line as (x,y). Thus $\mathbb{R}P^1$ is the space of lines through the origin.

On the other and a line through the origin is specified by the angle (roughly $\theta = \arctan(y/x)$) it makes with the postive x-axis. Since (x,y) and $(-x,-y)$ define the same line this angle is identified modulo π rather than 2π. Thus $\mathbb{R}P^1$ can be identified with a circle.

To make this more precise one should construct a diffeomorhpism from $\mathbb{R}P^1$ to S^1. Try this.

Problem 1.4. *Show that the following:*

$$\begin{aligned} U_1 &= \{(x,y) \in S^1 | y > 0\} \ , \qquad & \phi_1(x,y) = x \\ U_2 &= \{(x,y) \in S^1 | y < 0\} \ , \qquad & \phi_2(x,y) = x \\ U_3 &= \{(x,y) \in S^1 | x > 0\} \ , \qquad & \phi_3(x,y) = y \\ U_4 &= \{(x,y) \in S^1 | x < 0\} \ , \qquad & \phi_4(x,y) = y \end{aligned} \tag{1.4}$$

are a set of charts which cover S^1.

Solution:

It should be clear that all the U_i are open and cover S^1 and that the ϕ_i continuous with $\phi_i(U_i) = (-1,1)$. Their inverses are

$$\begin{aligned} \phi_1^{-1}(\theta) &= (\theta, \sqrt{1-\theta^2}) \\ \phi_2^{-1}(\theta) &= (\theta, -\sqrt{1-\theta^2}) \\ \phi_3^{-1}(\theta) &= (\sqrt{1-\theta^2}, \theta) \\ \phi_4^{-1}(\theta) &= (-\sqrt{1-\theta^2}, \theta) \end{aligned} \tag{1.5}$$

which are continuous for $\theta \in (-1,1)$ hence they are homeomorphisms (onto their image).

Next we must check that $\phi_i \circ \phi_j^{-1} : (-1,1) \to (-1,1)$ are C^∞ for all non-intersecting pairs. Thus we must check that

$$\begin{aligned} \phi_1 \circ \phi_3^{-1}(\theta) &= \sqrt{1-\theta^2} \ , \qquad & \phi_3 \circ \phi_1^{-1}(\theta) = \sqrt{1-\theta^2} \\ \phi_1 \circ \phi_4^{-1}(\theta) &= -\sqrt{1-\theta^2} \ , \qquad & \phi_4 \circ \phi_1^{-1}(\theta) = \sqrt{1-\theta^2} \\ \phi_2 \circ \phi_3^{-1}(\theta) &= \sqrt{1-\theta^2} \ , \qquad & \phi_3 \circ \phi_2^{-1}(\theta) = -\sqrt{1-\theta^2} \\ \phi_2 \circ \phi_4^{-1}(\theta) &= -\sqrt{1-\theta^2} \ , \qquad & \phi_4 \circ \phi_2^{-1}(\theta) = -\sqrt{1-\theta^2} \end{aligned} \tag{1.6}$$

These are all C^∞ since $\theta \in (-1,1)$.

Problem 1.5. *Show that the 2-sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2+y^2+z^2 = 1\}$ is a 2-dimensional manifold.*

Solution:

The hint was to consider stereographic projection. This requires using two charts

$$U_S = \{(x,y,z) \in S^2 | z < 1\} \qquad \text{and} \qquad U_N = \{(x,y,z) \in S^2 | z > -1\} \tag{1.7}$$

these are clearly open and cover S^2. In each chart one constructs $\phi_{N/S} : U_{N/S} \to \mathbb{R}^2$ by taking a straight line through either the south pole $(0,0,-1)$ or north pole $(0,0,1)$ and then through the point $p \in U_{N/S}$. These lines are defined by the equation

$$X(\lambda) = \begin{pmatrix} 0 \\ 0 \\ \pm 1 \end{pmatrix} + \lambda \begin{pmatrix} x \\ y \\ z \mp 1 \end{pmatrix} \tag{1.8}$$

so that $X(0)$ is either the north or south pole and $X(1)$ is a point on S^2. We define $\phi_{N/S}(p)$ to be the point in the (x, y)-plane where the line intersects $z = 0$.

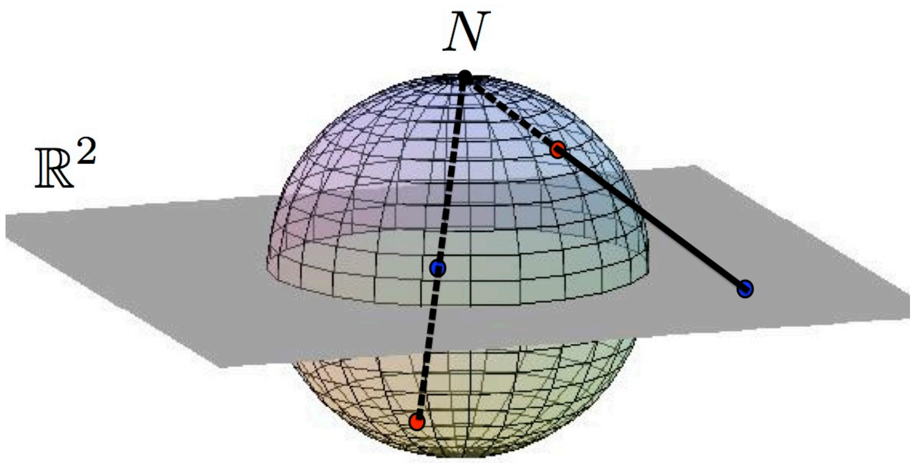

Figure 1: Stereographic projection from $U_S = S^2 - \{(0, 0, 1)\} \to \mathbb{R}^2$

Note that this construction works on all of U_N and on all of U_S respectively but not on all of S^2. Explicitly one has

$$\begin{aligned} \phi_S(x, y, z) &= \left(\frac{x}{1-z}, \frac{y}{1-z} \right) \\ \phi_N(x, y, z) &= \left(\frac{x}{1+z}, \frac{y}{1+z} \right) \end{aligned} \tag{1.9}$$

By construction these maps are injective as they define a unique line and this will intersect the $z = 0$ plane at a unique point. They are continuous and their inverses (to construct them consider the line through $(u, v, 0)$ and $(0, 0, \pm 1)$ and see where it intersects S^2):

$$\begin{aligned} \phi_S^{-1}(u, v) &= \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2} \right) \\ \phi_N^{-1}(u, v) &= \left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{1-u^2-v^2}{1+u^2+v^2} \right) \end{aligned} \tag{1.10}$$

These are also continuous. Finally we must simply observe that

$$\begin{aligned} \phi_S \circ \phi_N^{-1}(u, v) &= \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right) \\ \phi_N \circ \phi_S^{-1}(u, v) &= \left(\frac{u}{u^2+v^2}, \frac{v}{u^2+v^2} \right) \end{aligned} \tag{1.11}$$

are C^∞ on $\phi_N(U_N \cap U_S) = \phi_S(U_N \cap U_S) = \mathbb{R} - (0,0)$.

2. The Tangent Space

Problem 2.1. *Consider the circle S^1 as above. Show that $f : S^1 \to \mathbb{R}$ with $f(x,y) = x^2 + y$ is C^∞.*

Solution:

We can take the coordinates above. We need to consider $f \circ \phi_i^{-1}(\theta) : \phi_i(U_i) \to \mathbb{R}$:

$$\begin{aligned}
f \circ \phi_1^{-1}(\theta) &= \theta^2 + \sqrt{1-\theta^2} \\
f \circ \phi_2^{-1}(\theta) &= \theta^2 - \sqrt{1-\theta^2} \\
f \circ \phi_3^{-1}(\theta) &= 1 - \theta^2 + \theta \\
f \circ \phi_4^{-1}(\theta) &= 1 - \theta^2 + \theta
\end{aligned} \tag{2.1}$$

clearly all these functions are C^∞ on $\phi_i(U_i) = (-1, 1)$.

3. Maps Between Manifolds

Problem 3.1. *Show that $f : S^1 \to S^1$ defined by $f(e^{2\pi i\theta}) = e^{2\pi i n\theta}$ is C^∞ for any n.*

Solution:

Again we choose the same charts and note that $\theta \in [0, 1]$. First observe that

$$\begin{aligned}
f \circ \phi_1^{-1}(\theta) &= e^{in\arctan(\sqrt{1-\theta^2}/\theta)} \\
f \circ \phi_2^{-1}(\theta) &= e^{-in\arctan(\sqrt{1-\theta^2}/\theta)} \\
f \circ \phi_3^{-1}(\theta) &= e^{in\arctan(\theta/\sqrt{1-\theta^2})} \\
f \circ \phi_4^{-1}(\theta) &= e^{-in\arctan(\theta/\sqrt{1-\theta^2})}
\end{aligned} \tag{3.1}$$

Note that this is well defined since $\theta \neq 0$ on $\phi_1(U_1), \phi_2(U_2)$ and $\theta \neq \pm 1$ on $\phi_3(U_3), \phi_4(U_4)$. For any function χ we also have that

$$\begin{aligned}
\phi_1(e^{in\chi}) &= \cos(n\chi) \\
\phi_2(e^{in\chi}) &= \cos(n\chi) \\
\phi_3(e^{in\chi}) &= \sin(n\chi) \\
\phi_4(e^{in\chi}) &= \sin(n\chi)
\end{aligned} \tag{3.2}$$

thus we see that $\phi_i \circ f \circ \phi_j^{-1}(\theta)$ has the form:

$$\begin{aligned}
&\cos(n \arctan(\sqrt{1-\theta^2}/\theta) \\
&\pm \sin(n \arctan(\sqrt{1-\theta^2}/\theta) \\
&\cos(n \arctan(\theta/\sqrt{1-\theta^2}) \\
&\pm \sin(n \arctan(\theta/\sqrt{1-\theta^2})
\end{aligned} \tag{3.3}$$

And these are all C^∞ on the appropriate range of θ.

Problem 3.2. *Show that the charts of two diffeomorphic manifolds are in a one to one correspondence.*

Solution:

Let $\{U_i, \phi_i\}$ and $\{V_a, \psi_a\}$ be differential structures for $\mathcal{M}$ and $\mathcal{N}$ respectively and $f : \mathcal{M} \to \mathcal{N}$ a diffeomorphism.

First we show that $\{f(U_i), \phi_i \circ f^{-1}\}$ is a set of charts that cover $\mathcal{N}$. We note that these cover $\mathcal{N}$:

$$\cup_i f(U_i) = f(\cup_i U_i) = f(\mathcal{M}) = \mathcal{N} \tag{3.4}$$

Furthermore $\phi_i \circ f^{-1}$ are clearly homeomorphisms (bijective, continuous with the inverse continuous).

Similarly $\{f^{-1}(V_a), \psi_a \circ f\}$ is a set of charts that cover $\mathcal{M}$.

Now on $V_a \cap f(U_i)$ we have that

$$\psi_a \circ f^{-1} \circ \phi_i^{-1} : \phi_i(V_a \cap f(U_i)) \to \psi_a(V_a \cap f(U_i)) \tag{3.5}$$

and this is C^∞ as f is C^∞ (recall the definition). Thus the charts $\{f(U_i), \phi_i \circ f^{-1}\}$ are compatible with the charts $\{V_a, \psi_a\}$. Since we take the differential structure to be maximal we find that the charts $\{f(U_i), \phi_i \circ f^{-1}\}$ must be included in the differential struture $\{V_a, \psi_a\}$ of $\mathcal{N}$.

Similarly $\{f^{-1}(V_a), \psi_a \circ f\}$ are compatible with the charts $\{U_i, \phi_i\}$ and since we assume the differential structure to be maximal it follows that the $\{f^{-1}(V_a), \psi_a \circ f\}$ are included in $\{U_i, \phi_i\}$.

Thus it follows that $\{V_a, \psi_a\}$ and $\{U_i, \phi_i\}$ are in a one-to-one correspondence with each other.

Problem 3.3. *Show that the set of diffeomorphisms from a manifold to itself forms a group under composition.*

Solution:

Suppose $f, g : \mathcal{M} \to \mathcal{M}$ are diffeomorphims. Then

$$\phi_i \circ (f \circ g) \circ \phi_j^{-1} = \phi_i \circ (f \circ \phi_k^{-1} \circ \phi_k \circ g) \circ \phi_j^{-1} = (\phi_i \circ f \circ \phi_k^{-1}) \circ (\phi_k \circ g \circ \phi_j^{-1}) \tag{3.6}$$

is C^∞ for all i, j, k since f and g are C^∞. Similarly for $g^{-1} \circ f^{-1}$. Furthermore $f \circ g$ is a bijection if both f and g are. Thus $f \circ g$ is a diffeomorphism.

Clearly $id : \mathcal{M} \to \mathcal{M}$ is a diffeomorhism and by definition (the properties are symmetric between f and f^{-1}) if f is a diffeomorphism then so is f^{-1}.

4. Vector Fields

Problem 4.1. *What goes wrong if try to define $(X \cdot Y)(f) = X(f) \cdot Y(f)$?*

Solution:

With this definition we find that

$$\begin{aligned} X \cdot Y(f+g) &= (X(f)+X(g))(Y(f)+Y(g)) \\ &= X(f)Y(f) + X(f)Y(g) + X(g)Y(f) + X(f)Y(g) \\ &= X \cdot Y(f) + X \cdot Y(g) + X(f)Y(g) + X(g)Y(f) \end{aligned} \tag{4.1}$$

and the last two terms are unwanted.

If we try the same trick that we used for the commutator and define $[X,Y](f) = X(f)Y(f) - Y(f)X(f)$ then this clearly vanishes identically.

Problem 4.2. *Show that, if in a particular coordinate system,*

$$X = \sum_\mu X^\mu(x) \frac{\partial}{\partial x^\mu}\Big|_p , \qquad Y = \sum_\mu Y^\mu(x) \frac{\partial}{\partial x^\mu}\Big|_p \tag{4.2}$$

then

$$[X,Y] = \sum_\mu \sum_\nu (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) \frac{\partial}{\partial x^\nu}\Big|_p \tag{4.3}$$

Solution:

We simply calculate:

$$\begin{aligned} X(Y)(f) &= X\left(\sum_\nu Y^\nu \frac{\partial}{\partial x^\nu}(f \circ \phi_i^{-1}) \circ \phi_i\right) \\ &= \sum_\mu \left(X^\mu \frac{\partial}{\partial x^\mu}\left(\sum_\nu Y^\nu \frac{\partial}{\partial x^\nu}(f \circ \phi_i^{-1}) \circ \phi_i \circ \phi_i^{-1}\right) \circ \phi_i\right) \\ &= \sum_\mu \sum_\nu \left(X^\mu \partial_\mu Y^\nu \frac{\partial}{\partial x^\nu}(f \circ \phi_i^{-1}) + X^\mu Y^\nu \frac{\partial^2}{\partial x^\nu \partial x^\mu}(f \circ \phi_i^{-1})\right) \circ \phi_i \end{aligned} \tag{4.4}$$

Since the second term is symmetric in X^μ and Y^ν we find that

$$X(Y)(f) - Y(X)(f) = \sum_\mu \sum_\nu (X^\mu \partial_\mu Y^\nu - Y^\mu \partial_\mu X^\nu) \frac{\partial}{\partial x^\nu}(f \circ \phi_i^{-1}) \circ \phi_i \tag{4.5}$$

and we prove the theorem.

Problem 4.3. *Show that for three vector fields X, Y, Z on $\mathcal{M}$ the Jacobi identity holds:*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{4.6}$$

Solution:

Here we simply expand things out; suppose $f \in C^\infty(\mathcal{M})$, then

$$\begin{aligned}\Big([X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]\Big)(f) &= X([Y,Z]f)-[Y,Z](Xf)+Y([Z,X]f)\\ &\quad -[Z,X](Yf)+Z([X,Y]f)-[X,Y](Zf)\\ &= X(Y(Zf))-X(Z(Yf))-Y(Z(Xf))+Z(Y(Xf))\\ &\quad +Y(Z(Xf))-Y(X(Zf))-Z(X(Yf))+X(Z(Yf))\\ &\quad +Z(X(Yf))-Z(Y(Xf))-X(Y(Zf))+Y(X(Zf))\\ &= 0 \end{aligned} \tag{4.7}$$

where we used the linearity properties of vectors, i.e. $X(Y+Z)f = X(Yf)+X(Zf)$.

Problem 4.4. *Consider a manifold with a local coordinate system* $\phi_i = (x^1, ..., x^n)$.

i) Show that $\left[\frac{\partial}{\partial x^\mu}\Big|_p, \frac{\partial}{\partial x^\nu}\Big|_p\right] = 0$

ii) Evaluate $\left[\frac{\partial}{\partial x^1}\Big|_p, \varphi(x^1,x^2)\frac{\partial}{\partial x^2}\Big|_p\right]$ *where* $\varphi(x^1,x^2)$ *is a* C^∞ *function of* x^1, x^2.

Solution:

In the first case we find

$$\begin{aligned}\left[\frac{\partial}{\partial x^\mu},\frac{\partial}{\partial x^\nu}\right](f) &= \frac{\partial}{\partial x^\mu}\Big|_p\left(\frac{\partial}{\partial x^\nu}(f\circ\phi_i^{-1})\circ\phi_i\right)-\frac{\partial}{\partial x^\nu}\Big|_p\left(\frac{\partial}{\partial x^\mu}(f\circ\phi_i^{-1})\circ\phi_i\right)\\ &= \frac{\partial^2}{\partial x^\nu\partial x^\mu}(f\circ\phi_i^{-1})\circ\phi_i-\frac{\partial^2}{\partial x^\mu\partial x^\nu}(f\circ\phi_i^{-1})\circ\phi_i\\ &= 0\end{aligned} \tag{4.8}$$

And in the second case:

$$\begin{aligned}\left[\frac{\partial}{\partial x^\mu},\varphi\frac{\partial}{\partial x^\nu}\right](f) &= \frac{\partial}{\partial x^\mu}\Big|_p\left(\varphi\frac{\partial}{\partial x^\nu}(f\circ\phi_i^{-1})\circ\phi_i\right)-\varphi\frac{\partial}{\partial x^\nu}\Big|_p\left(\frac{\partial}{\partial x^\mu}(f\circ\phi_i^{-1})\circ\phi_i\right)\\ &= \varphi\frac{\partial^2}{\partial x^\nu\partial x^\mu}(f\circ\phi_i^{-1})\circ\phi_i+\left(\frac{\partial}{\partial x^\mu}\Big|_p\varphi\right)\frac{\partial}{\partial x^\nu}\Big|_p(f)\\ &\quad -\varphi\frac{\partial^2}{\partial x^\mu\partial x^\nu}(f\circ\phi_i^{-1})\circ\phi_i\\ &= \left(\frac{\partial}{\partial x^\mu}\Big|_p\varphi\right)\frac{\partial}{\partial x^\nu}\Big|_p(f)\end{aligned} \tag{4.9}$$

so

$$\left[\frac{\partial}{\partial x^\mu},\varphi\frac{\partial}{\partial x^\nu}\right] = \left(\frac{\partial}{\partial x^\mu}\Big|_p\varphi\right)\frac{\partial}{\partial x^\nu}\Big|_p \tag{4.10}$$