Grade 12 Introduction to Calculus and Grade 12 Advanced Mathematics

Manitoba Curriculum
Framework of Outcomes

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Contents

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Introduction

Purpose of Document

This document provides specific learning outcomes and achievement indicators for planning of scope, sequence, and depth for both Introduction to Calculus (45S) and Advanced Mathematics (45S or 40S). This document is intended to communicate high expectations for students' mathematical learning to all education partners.

A student can earn up to three different optional half-credit mathematics courses (or equivalent full-credit courses). These optional half-credit courses are designed to meet the needs of students who demonstrate particular aptitude or strong interest in mathematics and who desire to study more advanced topics. These courses are intended to assist students in their transition from secondary to post-secondary mathematics courses.

These optional half-credit courses provide an introduction to areas of mathematics that are covered in post-secondary programs within Manitoba and out of province. These courses will be most helpful for those students planning to study engineering, mathematics, science, computer science, or other mathematics-oriented programs. When choosing the optional half courses and topics within them, students should consider their interests, both future and current. Students, parents, and educators are encouraged to research the admission requirements for post-secondary programs of study, as they vary by institution and by year.

Beliefs about Students and Mathematics Learning

Students are curious, active learners with individual interests, abilities, needs, and career goals. They come to school with varying knowledge, life experiences, expectations, and backgrounds. A key component in developing students' mathematical literacy is making connections to these backgrounds, experiences, goals, and aspirations. Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences.

This meaning is best developed when learners encounter mathematical experiences that proceed from the simple to the complex, and that connect concrete and abstract representations. The use of manipulatives, visuals, and a variety of pedagogical and assessment approaches can address the diversity of learning styles. At all levels of understanding, students benefit from working with a variety of materials, tools, and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial, and symbolic representations of mathematics.

Students need frequent opportunities to develop and reinforce their conceptual understanding, procedural thinking, and problem-solving abilities. By addressing these three interrelated components, students will strengthen their ability to apply mathematical learning to their daily lives.

The learning environment should value, respect, and address all students' experiences and ways of thinking so that students are comfortable taking intellectual risks, asking questions, and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to realize that it is acceptable to solve problems in different ways and that solutions may vary, depending upon how the problem is understood.

Assessment *for* learning, assessment *as* learning, and assessment *of* learning are all critical to helping students learn mathematics. A variety of evidence and a variety of assessment approaches should be used in the mathematics classroom.

First Nations, Métis, and Inuit Perspectives

First Nations, Métis, and Inuit students in Manitoba come from diverse geographic areas and have varied cultural and linguistic backgrounds. Students attend schools in a variety of settings, including urban, rural, and isolated communities. Teachers need to recognize and understand the diversity of cultures within schools and the diverse experiences of students.

First Nations, Métis, and Inuit students often have a whole-world view of the environment; as a result, many of these students live and learn best in a holistic way. This means that students look for connections in learning and learn mathematics best when it is contextualized and not taught as discrete content.

Many First Nations, Métis, and Inuit students come from cultural environments where learning takes place through active hands-on participation. Traditionally, little or no emphasis was placed upon the written word. Oral communication, along with practical applications and experiences, is important to student learning and understanding.

A variety of teaching and assessment strategies is required to build upon the diverse knowledge, cultures, communication styles, skills, attitudes, experiences, and learning styles of students.

The strategies used must go beyond the incidental inclusion of topics and objects unique to a culture or region and strive to achieve higher levels of multicultural education (Banks and Banks).

Affective Domain

A positive attitude is an important aspect of the affective domain, which has a profound effect on learning. Environments that create a sense of belonging, support risk taking, and provide opportunities for success help students to develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, to participate willingly in classroom activities, to persist in challenging situations, and to engage in reflective practices.

Teachers, students, and parents need to recognize the relationship between the affective and cognitive domains and to nurture those aspects of the affective domain that contribute to positive attitudes. To experience success, students must be taught to set achievable goals and assess themselves as they work toward these goals.

Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Goals for Students

The main goals of mathematics education are to prepare students to

- communicate and reason mathematically
- use mathematics confidently, accurately, and efficiently to solve problems
- appreciate and value mathematics
- make connections between mathematical knowledge and skills and their applications
- commit themselves to lifelong learning
- become mathematically literate citizens, using mathematics to contribute to society and to think critically about the world

Students who have met these goals

- gain an understanding and appreciation of the role of mathematics in society
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical problem solving
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity about mathematics and situations involving mathematics

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding through

- taking risks
- thinking and reflecting independently
- sharing and communicating mathematical understanding
- solving problems in individual and group projects
- pursuing greater understanding of mathematics
- appreciating the value of mathematics throughout history

Mathematical Processes

The seven mathematical processes are critical aspects of learning, doing, and understanding mathematics. Students must encounter these processes regularly in a mathematics program in order to achieve the goals of mathematics education. The following interrelated mathematical processes are intended to permeate the teaching and learning of mathematics. Students are expected to

- use communication in order to learn and express their understanding
- make connections among mathematical ideas, other concepts in mathematics, everyday experiences, and other disciplines
- demonstrate fluency with mental mathematics and estimation
- develop and apply new mathematical knowledge through problem solving
- develop mathematical reasoning
- select and use technology as a tool for learning and solving problems
- develop visualization skills to assist in processing information, making connections, and solving problems

All seven processes should be used in the teaching and learning of mathematics.

Communication

Students need opportunities to read about, represent, view, write about, listen to, and discuss mathematical ideas. These opportunities allow students to create links among their own language and ideas, the language and ideas of others, and the formal language and symbols of mathematics. Communication is important in clarifying, reinforcing, and modifying ideas, attitudes, and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics. Students also need to communicate their learning using mathematical terminology. Communication can play a significant role in helping students make connections among concrete, pictorial, graphical, symbolic, verbal, written, and mental representations of mathematical ideas. Explanations of ideas should include the various representations as appropriate. Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

Connections

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to each other or to real-world phenomena, students begin to view mathematics as useful, relevant, and integrated. Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections. “Because the learner is constantly searching for connections on many levels, educators need to orchestrate the experiences from which learners extract understanding.... Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching” (Caine and Caine 5).

Mental Mathematics and Estimation

Mental mathematics and estimation is a combination of cognitive strategies that enhance flexible thinking and number sense. It involves using strategies to perform mental calculations.

Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy, and flexibility in reasoning and calculating. “Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math” (NCTM, May 2005).

Students proficient with mental mathematics “become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers and are more able to use multiple approaches to problem solving” (Rubenstein 442).

Mental mathematics “provides a cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers” (Hope v).

Estimation is used for determining approximate values or quantities, usually by referring to benchmarks or referents, or for determining the reasonableness of calculated values. Estimation is also used to make mathematical judgments and to develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it. Developing number sense is a lifelong process. Students benefit from opportunities to practise and solidify previously learned skills and procedures in new contexts.

Problem Solving

”Problem solving is an integral part of all mathematics learning” (NCTM, 2000). Learning through problem solving should be the focus of mathematics at all grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts. Problem solving is to be employed throughout all of mathematics and should be embedded throughout all the topics.

When students encounter new situations and respond to questions of the type “How would you...?” or “How could you...?”, the problem-solving approach is being modelled. Students develop their own problem-solving strategies by listening to, discussing, and trying different strategies.

In order for an activity to be based on problem solving, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately.

A true problem requires students to use prior knowledge in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families, or current events.

Both conceptual understanding and student engagement are fundamental in moulding students' willingness to persevere in future problem-solving tasks.

Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill, or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive and deductive reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem, they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple creative and innovative solutions. Creating an environment where students openly look for and engage in the process of finding a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

Reasoning

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. Questions that challenge students to think, analyze, and synthesize help them to develop an understanding of mathematics. All students need to be challenged to answer questions such as "Why do you believe that's true/correct?" or "What would happen if...?".

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns, and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

When explaining ideas, students should be encouraged to use concrete, pictorial, symbolic, graphical, verbal, and written representations of their mathematical ideas.

Technology

Technology can be used effectively to contribute to and support the learning of a wide range of mathematical learning outcomes. Technology enables students to explore and create patterns, examine relationships, test conjectures, and solve problems. Some post-secondary institutions expect fluency with technology.

Technology has the potential to enhance the teaching and learning of mathematics. It can be used to

- explore and demonstrate mathematical relationships and patterns
- organize and display data
- generate and test inductive conjectures
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
- reinforce the learning of basic facts
- develop personal procedures for mathematical operations
- model situations
- develop number and spatial sense
- create geometric figures

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. Students need to know when it is appropriate to use technology such as a calculator and when to apply their mental computation, reasoning, and estimation skills to predict and check answers. The use of technology can enhance, although it should not replace, conceptual understanding, procedural thinking, and problem solving.

Visualization

Visualization "involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world" (Armstrong 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them.

Visual images and visual reasoning are important components of number, spatial, and measurement sense. Number visualization occurs when students create mental representations of numbers.

Being able to create, interpret, and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure and when to estimate, and it involves knowledge of several estimation strategies (Shaw and Cliatt 150).

Visualization is fostered through the use of concrete materials, technology, and a variety of visual representations. It is through visualization that abstract concepts can be understood concretely by the student. Visualization is a foundation to the development of abstract understanding, confidence, and fluency.

Nature of Mathematics

Mathematics is one way of understanding, interpreting, and describing our world. There are a number of characteristics that define the nature of mathematics, including change, constancy, number sense, patterns, relationships, spatial sense, and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12... can be described as

- skip-counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain (Steen 184)

Constancy

Many important properties in mathematics do not change when conditions change. Examples of constancy include

- the conservation of equality in solving equations
- the sum of the interior angles of any triangle
- theoretical probability of an event

Some problems in mathematics require students to focus on properties that remain constant. The recognition of constancy enables students to solve problems such as those involving constant rates of change, lines with constant slope, or direct variation situations.

Number Sense

Number sense, which can be thought of as deep understanding and flexibility with numbers, is the most important foundation of numeracy (British Columbia Ministry of Education 146). Continuing to foster number sense is fundamental to growth of mathematical understanding.

Number sense is an awareness and understanding of what numbers are, their relationships, their magnitude, and the relative effect of operating on numbers, including the use of mental mathematics and estimation (Fennell and Landis 187).

Patterns

Mathematics is about recognizing, describing, and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics. Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment.

Patterns may be represented in concrete, visual, auditory, or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create, and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

Relationships

Mathematics is used to describe and explain relationships. Within the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables, and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns, and describing possible relationships visually, symbolically, orally, or in written form. Technology should be used to aid in the search for relationships.

Spatial Sense

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. Some of these experiences should involve the use of technology. These experiences offer a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations. Graphing calculators or graphing software can aid students in developing this understanding.

Uncertainty

In mathematics, interpretations of data and the predictions made from data inherently lack certainty.

Events and experiments generate statistical data that can be used to make predictions. It is important that students recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of an interpretation or conclusion is directly related to the quality of the data it is based upon. An awareness of uncertainty provides students with an understanding of why and how to assess the reliability of data and data interpretation.

Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

Use of Mathematics

It is important to study mathematics because of its many applications to everyday life and to specialized areas of work. A knowledge of mathematics is necessary if students are to have an awareness and an understanding of their environment and to enjoy a fuller life. With this in mind, it is suggested that mathematics be considered in the following ways:

- **As a Mode of Communication**

  Mathematics is a language. For students to communicate successfully in mathematics, they must learn the basic concepts without having to meet demands for logical rigour. At first, the language acquired should be used to describe patterns and relationships observed in the real world, and it should allow students to share their mathematical thinking. At a later stage, symbols should be introduced to record these relationships, expand mathematical vocabulary, and communicate understanding.

- **As a Tool**

  Mathematics is a tool used to answer practical questions, thus demonstrating to students that to function in society they need a good understanding of arithmetic, algebra, and statistics. Having a deeper understanding of mathematics allows students to solve problems in different ways, see mathematics concepts from a variety of perspectives, develop a broad range of mathematical tools, and apply the tools appropriately and efficiently. Students should become capable of employing an organized and sometimes creative process leading to the solution of a problem. This may involve applying fundamental operations and information-processing skills—such as collecting, organizing, and interpreting data—to familiar and unfamiliar contexts. Realistic problems relating to the students' environment can be engaging for some students as they make connections to contexts they may encounter in the future.

- **As a Source of Aesthetic Enjoyment**

  Mathematics is an aesthetic discipline like music. Many students can develop an appreciation of mathematics through the discovery of patterns in numbers and in created or naturally occurring designs. An elegant solution to a mathematics problem can be aesthetically pleasing. Methodologies and techniques conducive to developing positive inquiring attitudes toward mathematics should be employed.

Students should be given opportunities to recognize and appreciate these three ways in which the subject can be used. Classroom emphasis on these uses cannot always be equal, however, and it must vary depending on the nature of the topic and the interest and abilities of the students.

Curriculum Framework Details

The curriculum framework for Introduction to Calculus and for Advanced Mathematics is stated in terms of big ideas and specific learning outcomes, with details provided as achievement indicators.

Big Ideas describe concepts that broadly encompass a topic and give an overall sense of purpose and meaning to the specific learning outcomes. The big ideas are statements providing direction to guide teachers and students as they navigate and work to connect the details of a mathematics topic. Teachers should share and discuss the big ideas with their students to help them see and make connections to concepts within a topic and to other mathematics concepts that are part of students' prior learning.

Specific Learning Outcomes are statements that identify the specific knowledge, skills, and understandings that students are required to attain by the end of a given course. Each outcome is to be achieved through a variety of learning strategies and experiences.

Achievement Indicators specify the depth and breadth of learning expected for each outcome. They may be used to determine whether students have met the corresponding specific learning outcome. Additional notes are sometimes given as elaborations to help teachers in their planning of the scope and sequence of instruction. The development of the concepts of the outcome do not need to be presented in the order given. Furthermore, a teacher may choose to develop the concepts of the outcome using indicators other than those listed.

Course Options Overview

These are optional courses and they do not qualify for the compulsory Grade 12 Mathematics credit. They are intended for students who have completed or are completing their compulsory Grade 12 Mathematics credit (usually Pre-Calculus Mathematics 40S). They are designed for students who show an aptitude for, or a strong interest in, mathematics and plan to study further mathematics at the post-secondary level.

Consistent with the scheduling of other high school courses, 55 hours of instruction is required for each half credit and 110 hours is required for each full credit. Each half-credit course is composed of four topics, described in this document.

- ½ credit—Introduction to Calculus
- ½ credit—Advanced Mathematics I (1st half credit—4 topics)
- ½ credit—Advanced Mathematics II (2nd half credit—4 topics different from Advanced Mathematics I)
- 1 credit—Introduction to Calculus and Advanced Mathematics I (4 topics)
- 1 credit—Advanced Mathematics I and II (8 topics)

To earn the Introduction to Calculus half credit, a student must successfully complete all four topics with outcomes as described in this document. To earn an Advanced Mathematics half credit, a student must successfully complete four topics from the list of core or additional topics. Alternatively, to earn a full credit, a student must successfully complete eight topics from the list of core or additional topics of Advanced Mathematics. The Advanced Mathematics topics may be chosen by the teacher with the students' interests in mind or they may be chosen by the students with the teacher's approval.

Introduction to Calculus

Students should have taken Pre-Calculus Mathematics 40S prior to the Introduction to Calculus course. In special circumstances, Introduction to Calculus could be taken concurrently with Pre-Calculus Mathematics 40S if the sequence of topics is chosen carefully. The Introduction to Calculus is a half credit consisting of the following four topics:

- Limits
- Derivatives
- Applications of Derivatives
- Integration

Advanced Mathematics

It is recommended that students will have taken Pre-Calculus Mathematics 40S prior to the Advanced Mathematics course. Alternatively, students with an Applied Mathematics 40S credit can take this course and should intentionally choose topics with minimal algebra.

Grade 12 Advanced Mathematics consists of four chosen topics that are different for each half credit. The flexibility of the course allows teachers to choose the topics and encourages the input of students. For a full credit in Advanced Mathematics, eight different topics must be chosen.

The seven core Advanced Mathematics topics have specific learning outcomes (SLOs) and achievement indicators given later in this document. The additional Advanced Mathematics topics' outcomes are to be determined by the teacher or by a student in consultation with the teacher. A suggested (but not comprehensive) list of additional advanced mathematics topics follows. Although the order of the topics is not prescribed, there are some connections among topics. In addition, there is an overview statement for each topic to help teachers and students see connections among prior, current, and future learning with an end goal of choosing appropriate topics.

Core Advanced Mathematics Topics (outcome details provided in this document):

- Complex Numbers and Polar Coordinates
- Statistics
- Number Theory
- Matrices and Systems of Equations
- 3-Dimensional Geometry
- Vectors
- Conic Sections

Additional Advanced Mathematics Topics (outcome details determined by the teacher):

- Fractal geometry
- Calculus topics (to extend beyond Introduction to Calculus content)
- History of mathematics
- Applications of mathematics to computer science (e.g., cryptography)
- Combinatorics extending beyond permutations and combinations (e.g., pigeonhole principle)
- Interdisciplinary project

Specific Learning Outcomes by Course

The following pages present the specific learning outcomes and achievement indicators for all four topics of Introduction to Calculus and for the core topics of Advanced Mathematics.

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Topic: Limits

Big Ideas:

- Limits can describe function values as the input values approach a number or infinity.
- Limits are especially useful when the input value is not part of the domain of a function.

Specific Learning Outcomes: *It is expected the students will…*		Achievement Indicators:
IC.1.1	Demonstrate an understanding of the concept of the limit.	■ Explore the concept of limits by analyzing a function's graph and table of values. ■ Utilize the definition and proper notation of limits to express a function's limit at a specific point. ■ Verify limit theorems: ▫ identity function ▫ constant function ▫ constant times a function ▫ sum/difference of functions ▫ product/quotient of functions ▫ power of a function ■ Utilize limit theorems to determine the limit of functions by direct substitution.
IC.1.2	Evaluate limits to analyze functions.	■ Explain why $\frac{0}{0}$ is called an indeterminate form. ■ Solve limits of indeterminate form by algebraic manipulation. ■ Define one-sided limits. ■ Evaluate one-sided limits of functions (including piecewise) graphically and algebraically. ■ Explain the behaviour of limits in the form $\lim_{x \to 0}\left(\frac{\text{number}}{x}\right)$. ■ Determine limits at infinity. ■ Apply limits to determine the equations of horizontal and vertical asymptotes.
IC.1.3	Apply the concept of limit to the continuity of a function.	■ Determine graphically whether a function is continuous. ■ Determine algebraically whether a function is continuous.

Topic: Derivatives

Big Ideas:

- The derivative extends the concept of slope to the slope of a curve at a point.
- A derivative function can help someone describe the "shape" of the curve with that derivative.

Specific Learning Outcomes: *It is expected the students will…*		**Achievement Indicators:**
IC.2.1	Develop the definition of the derivative as the slope of a curve at a point.	*Note: Prerequisite knowledge includes determining the slope and the equation of a line.* ■ Explain how slopes of secant lines can approximate the slope of a tangent line. ■ Define the derivative as the limit of the difference quotient, which is the slope of the tangent line at a point. ■ Describe the derivative function, $f'(x)$, as a function that determines the slope at any point of the function, $f(x)$. *Note: Students should be exposed to different notations for derivatives* $\left(f',\ y',\ \text{and } \frac{dy}{dx}\right)$.

IC.2.2	Develop and apply differentiation rules.	■ Develop and apply differentiation rules:
		  ▪ a constant times $f(x)$
		  ▪ the power rule with rational exponents
		  ▪ sum and difference
		  ▪ product
		  ▪ quotient
		  ▪ chain rule
		■ Apply the derivative rules to determine the equation of a tangent line at a point, given a function equation and a point on the function.
		■ Define and determine higher-order derivatives of a function.
		Note: The functions explored in this introductory course do not include trigonometric, exponential, and logarithmic functions.

IC.2.3	Demonstrate an understanding of implicit differentiation.	■ Determine the derivative of a relation implicitly. ■ Determine the equation of a tangent line to a relation, given a point. ■ Determine higher-order derivatives of a relation using implicit differentiation.

Topic: Applications of Derivatives

Big Idea:

- Applying derivatives can help someone solve problems based on many other function models as accurately and efficiently as those with linear or quadratic models.

Specific Learning Outcomes: *It is expected the students will…*		**Achievement Indicators:**
IC.3.1	Apply derivatives to solve problems involving the motion of particles.	■ Describe the meaning of a displacement function. ■ Determine average and instantaneous velocity given a displacement function. ■ Determine average and instantaneous acceleration given a displacement function. ■ Solve particle motion problems.
IC.3.2	Determine features of a function using derivatives to sketch the function accurately.	*Note: Prerequisite skills for this topic include describing domain using interval notation, set notation, and number line graphs. Teachers may want to review solving linear and non-linear inequalities using a sign diagram.* ■ Determine the critical values of a function. ■ Determine the intervals where a function is increasing and decreasing. ■ Determine relative extremes and absolute extremes graphically and algebraically. ■ Determine intervals where the graph of a function is concave up and concave down. ■ Determine the points of inflection. ■ Sketch a polynomial function accurately using its characteristics, including intercepts, domain, range, maxima, minima, points of inflection, and concavity. *Note: The functions explored in this introductory course do not include trigonometric, exponential, and logarithmic functions.*
IC.3.3	Apply derivatives to solve optimization and related rates problems.	■ Solve optimization problems. ■ Apply the chain rule and implicit differentiation to determine rates of change. ■ Solve problems involving related rates.

Topic: Integrals

Big Ideas:

- Integration extends the area of geometric shapes to the area under a function curve where the height of a region is changing.
- Derivatives and integrals are inversely related.

Specific Learning Outcomes: *It is expected the students will…*		**Achievement Indicators:**
IC.4.1	Demonstrate an understanding of the relationship between anti-differentiation and integration of functions.	■ Describe anti-differentiation as the inverse operation of differentiation. ■ Determine the general antiderivative (family of functions), given the derivative of a function. ■ Define integration in terms of the area bounded by a function curve and the x-axis. ■ Relate anti-differentiation and integration as the fundamental theorem of calculus (first part). ■ Define the indefinite integral.
IC.4.2	Apply integration to solve problems.	■ Determine a specific antiderivative, given the derivative function and the coordinates of a point. ■ Apply integration in a context such as particle motion.
IC.4.3	Demonstrate and apply an understanding of the definite integral.	■ Define the definite integral. ■ Evaluate definite integrals geometrically by calculating area. ■ Evaluate definite integrals using antiderivatives and the fundamental theorem of calculus (second part). ■ Evaluate the definite integral of functions algebraically and geometrically where parts of the function may be below the x-axis. ■ Relate the total area bounded by a function curve, $f(x)$, and the x-axis on interval $[a, b]$ to the definite integral of the absolute value of the function, $\int_a^b \lvert f(x) \rvert dx$. ■ Determine the area between any two functions on a given interval. ■ Determine the area between two functions where intersecting points determine the interval.

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Topic: Complex Numbers and Polar Coordinates

Big Ideas:

- All other number systems are a subset of the complex number system.
- The operations and properties that apply to other number systems also apply to the complex number system.
- Complex numbers can be represented on a two-dimensional plane in rectangular or polar form.

Overview: The set of complex numbers were developed to describe all the roots of polynomial functions including the real number roots. Unlike the real number system, which can be represented on a one-dimensional number line, the complex number system is represented on a two-dimensional plane. Often "i" is referred to as an imaginary unit that is a symbol with the property that $i = \sqrt{-1}$ or $i^2 = -1$. Complex numbers apply to electrical engineering, aircraft design, medicine, and graphic design. Polar coordinates allow us to sketch some relations that are not functions with much more ease. Some applications of polar coordinates include guiding vessels and guiding industrial robots.

Specific Learning Outcomes: *It is expected the students will...*		**Achievement Indicators:**
AM.1.1	Define and perform operations on complex numbers.	■ Define the complex number system and describe its history. ■ Determine the absolute value of a complex number. ■ Determine the geometrical representation of a complex number. ■ Compare complex numbers. ■ Perform operations on complex numbers. ■ Define complex conjugates and apply them to the division of complex numbers.
AM.1.2	Make connections between complex numbers and quadratic equation solutions.	■ Solve quadratic equations with complex roots. ■ Solve quadratic equations with complex coefficients. ■ Given the roots, real or complex, determine the corresponding quadratic equation.
AM.1.3	Demonstrate an understanding of polar coordinates and their graphs.	■ Demonstrate how to read polar coordinates. ■ Sketch a point expressed in polar coordinates. ■ Convert rectangular coordinates to polar coordinates and vice versa. ■ Convert equations in polar form to rectangular form and vice versa. ■ Sketch a variety of polar equations. ■ Apply symmetry to sketch a polar graph.
AM.1.4	Make connections between complex numbers and polar coordinates.	■ Determine the argument of a complex number. ■ Represent a complex number in polar form. ■ Convert a complex number in polar form to rectangular form and vice versa.

Topic: Statistics

Big Ideas:

- Using a data sample, statistics can be used to describe a set of data or allow us to predict, based on probability, things about a set of data.
- Statistics allow someone to explore, describe, model, and explain data.
- Statistics can describe the central tendency of a set of data and how the data is spread out.

Overview: Statistics is the study of data and how to represent it. By analyzing the data, statistics can help us understand data and make inferences about it. Data can be described in terms of measures of central tendency and measures of spread. Statistics connects to the concept of probability when applied to probability distributions such as the binomial distribution and the normal distribution. Binomial distributions build on previous knowledge students gained about the binomial theorem. Data analysis is prevalent and integral to the work in a broad array of fields including social sciences, sports teams, business, scientific research, and data analytics.

Specific Learning Outcomes: *It is expected the students will...*		**Achievement Indicators:**
AM.2.1	Demonstrate an understanding of the concepts of measures of central tendency and spread.	■ Define vocabulary used in statistics (including measures of central tendency and spread). ■ Represent data in histograms, frequency tables, and grouped frequency tables. ■ Calculate quartiles, the interquartile range, and the range of data. ■ Determine whether a data point is an outlier. ■ Represent data with a box and whisker plot. ■ Demonstrate an understanding of the properties of variance and standard deviation. ■ Calculate the variance and standard deviation of data.
AM.2.2	Demonstrate an understanding of probability distributions including the binomial distribution.	■ Define discrete random variables and probability distributions. ■ Recognize that the probabilities in a probability distribution always add to 1. ■ Represent data with tree diagrams and probability distribution charts. ■ Use probability distributions to solve problems. ■ Define binomial distribution. ■ Use the formula for calculating probabilities within a binomial distribution. ■ Find the mean and standard deviation of a binomial distribution. *Note: Teachers are encouraged to connect the binomial expansion formula from Pre-Calculus Mathematics 40S to the formula for calculating binomial distributions.*
AM.2.3	Develop and apply the properties of a normal distribution.	■ Describe the properties of the normal distribution and the standard normal distribution. ■ Analyze the normal distribution to show probabilities can be estimated with the 68-95-99.7 rule. ■ Apply the formula for calculating z-scores and use z-scores to compare data. ■ Calculate probabilities in normal distributions, given scores. ■ Calculate scores in normal distributions, given probabilities. ■ Use a normal distribution to estimate a binomial distribution when appropriate. ■ Determine confidence intervals. ■ Calculate the margin of error of a confidence interval.

Topic: Number Theory

Big Ideas:

- Many ideas are true about all integers, some about subsets of integers, and some apply to only one integer.
- It is necessary to provide some type of proof that a conjecture is true for all integers, since we cannot test every possibility.

Overview: Number theory is a branch of mathematics and is, in large part, devoted to the study of integers. Important number theory topics include primes, prime factorization, and properties of numbers made out of integers such as rational numbers. Proofs are examples of deductive or inductive reasoning, and they demonstrate that a statement is always true under the given conditions. Many unanswered questions in mathematics originate in number theory. Applications of number theory are modular arithmetic, cryptography, and topics of computer science; however, number theory is often studied just for fun.

Specific Learning Outcomes: *It is expected the students will…*		**Achievement Indicators:**

AM.3.1	Apply proof techniques to prove mathematical theorems or statements.	■ Demonstrate an understanding of proof techniques:
		  ■ direct proof (proof by construction)
		  ■ proof by contradiction
		  ■ proof by induction
		Note: Other proof techniques may be introduced.

AM.3.2	Explore, develop, and apply the properties of integers.	■ Illustrate and explain the divisibility of integers and the Archimedean property. ■ Demonstrate an understanding of modular arithmetic. ■ Define, develop, and apply the greatest common divisor (GCD). ■ Apply the Euclidean algorithm to find the GCD. ■ Express the GCD as a linear equation. ■ Define prime and composite numbers. ■ Demonstrate an understanding of the sieve of Eratosthenes. ■ Prove there are an infinite number of primes. ■ Demonstrate an understanding of the fundamental theorem of arithmetic. ■ Define the least common multiple (LCM). ■ Apply prime factorization to determine the LCM and GCD. *Note: Teachers are encouraged to demonstrate the formal proof of the Fundamental Theorem of Arithmetic*
AM.3.3	Represent numbers in different bases.	■ Define decimal representation (base 10). ■ Represent an integer in bases other than 10. ■ Convert a number from one base to another. ■ Demonstrate an understanding of binary and hexadecimal notation.

Topic: Matrices and Systems of Equations

Big Ideas:

- The intrinsic relationship of the four operations when applied to numbers is the same when they are applied to matrices.
- Writing the coefficients and constants in a linear system in the form of a matrix is a way to represent the system and can assist in determining solutions to that system.

Overview: A matrix is a rectangular array of numbers with algebraic properties and geometric connections to linear systems and vectors in 2-space, 3-space, and n-space (with any number of dimensions). These topics are studied in depth in post-secondary courses involving linear algebra. There are many matrix applications in computer science.

Specific Learning Outcomes: *It is expected the students will…*		**Achievement Indicators:**
AM.4.1	Demonstrate an understanding of matrices.	■ Define the following terms: □ matrix □ dimension or order of a matrix □ entry of a matrix □ equal matrices □ square matrix □ zero matrix □ row and column matrix □ transpose of a matrix ■ Give examples of where matrices are used.
AM.4.2	Perform operations on matrices.	■ Perform addition and subtraction on matrices where possible. ■ Perform scalar multiplication of a matrix. ■ Apply matrix operations to solve simple matrix equations. ■ Multiply matrices where possible. ■ Explain why matrix multiplication is not commutative. ■ Determine the inverse of a 2×2 matrix using a formula. ■ Explain the three operations used when row-reducing a matrix. ■ Explain the difference between a row-echelon matrix and a reduced row-echelon matrix. ■ Apply matrix row reduction to find the inverse of a square matrix.
AM.4.3	Solve systems of equations using matrices.	■ Define a determinant. ■ Determine the determinant of a matrix using: □ row reduction □ cofactor expansion □ the arrow method for a 3×3 matrix ■ Solve a system of equations using: □ row reduction □ inverse □ Cramer's rule *Note: Solving a system of equations using either the inverse or Cramer's rule is not expected beyond a 2×2 system.*

Topic: 3-Dimensional Geometry

Big Ideas:

- It takes three pieces of information to describe a point in 3-space.
- Algebraic concepts developed for geometry in two dimensions can be extended to three dimensions.

Overview: Geometry in three dimensions is an extension of two-dimensional Euclidean geometry studied in Middle and Senior Years. Applications of three-dimensional geometry include computer graphics, computer assisted design, architecture, interior design, and 3-D modelling. The topic of three-dimensional geometry has connections with the topics of matrices (3×3), vectors, and calculus.

Specific Learning Outcomes: *It is expected the students will…*		**Achievement Indicators:**

AM.5.1	Demonstrate an understanding of 3-space.	■ Define the following terms in 3-space:
		  ■ coordinate planes
		  ■ coordinates of a point
		  ■ octant
		■ Sketch the coordinate axes and points given the coordinates.
		■ Sketch a rectangular prism or a plane in 3-space.
		■ Determine the distance between two points in 3-space.

AM.5.2	Represent and analyze lines, planes, and surfaces algebraically and graphically in 3-space.	■ Determine the point-normal equation of a plane. ■ Prove two planes are parallel. ■ Determine the general equation of a plane. ■ Determine the equation of a sphere. ■ Sketch the curve of intersection of surfaces. ■ Determine the points of intersection of non-parallel planes. ■ Determine the equation of a sphere. ■ Determine the equation of a solid of revolution.

Topic: Vectors

Big Ideas:

- Quantities that have both a magnitude and a direction can be efficiently represented by a vector.
- Vectors can be represented both geometrically and algebraically and some ideas about vectors are easier to see with one representation than the other.

Overview: A vector is a quantity that can be represented with a directed line segment. The study of vectors in mathematics helps us understand how vectors interact with each other, and it can be applied to the study of kinematics and forces in physics. Vector mathematics connects to analytic geometry in two dimensions and helps extend the concept of an equation of a line to three dimensions. This topic has connections to matrices and three-dimensional geometry. Vectors are commonly used in physics and have applications in branches of engineering.

Specific Learning Outcomes: *It is expected the students will…*		**Achievement Indicators:**
AM.6.1	Develop an understanding of vectors and perform basic vector operations.	■ Define a vector and a scalar. ■ Determine the magnitude of a vector. ■ Determine the unit vector. ■ Add and subtract vectors. ■ Apply vector addition and subtraction to geometrical situations. ■ Multiply vectors by a scalar.
AM.6.2	Demonstrate an understanding of the dot product and cross product of vectors to solve problems.	■ Define the dot product and explore its properties. ■ Apply the dot product to find the angle between two vectors. ■ Define the cross product and explore its properties. ■ Apply the right-hand rule to determine the direction of the cross product vector. ■ Represent the cross product in Cartesian form. ■ Apply the cross product to problems involving contexts such as the area of a parallelogram and torque.

AM.6.3	Develop and apply the vector equation of a line.	■ Write the vector equation of a line using the direction vector.
		■ Write the parametric and Cartesian forms of the vector equation of a line.
		■ Determine whether a point is on a line.
		■ Determine the point of intersection of two lines in two and three dimensions using vectors.
		■ Apply the vector equation of a line to kinematics such as:
		    ▪ determining the speed of an object given its equation
		    ▪ determining whether objects will collide given the paths of the objects
		    ▪ determining the distance between objects at given times

Topic: Conic Sections

Big Idea:

- Thinking of conic sections geometrically makes certain properties of conic sections easier to see than thinking of them algebraically, and vice versa.

Overview: The curves associated with conic sections are the circle, the ellipse, the parabola, and the hyperbola. Conics can be defined in terms of distance between points or distance to a line and a point. This topic extends students' learning of circles and parabolas. Conics are an aspect of analytic geometry and have many applications in calculus and the sciences, especially physics and astronomy.

Specific Learning Outcomes: *It is expected the students will…*		**Achievement Indicators:**

AM.7.1	Represent and analyze conic sections algebraically and geometrically.	■ Define conic sections algebraically and geometrically:
		  ▪ circle
		  ▪ parabola
		  ▪ ellipse
		  ▪ hyperbola
		■ Analyze the conics in terms of their characteristics (as applicable), such as centre, vertices, axes of symmetry, length of major and minor axes, and asymptotes.
		Note: Teachers are encouraged to relate the conic sections to cross-sectional slices of a pair of cones (one inverted). The conics are restricted to those with equations of the form: $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, *where* $B = 0$.

AM.7.2	Demonstrate an understanding of focal points in a conic section.	■ Determine the focal points of any conic section. ■ Determine the equation of a conic section given its focal point(s). ■ Determine the directrix of a parabola and its relationship to the focal point.
AM.7.3	Analyze a conic section in terms of its eccentricity.	■ Define eccentricity and its relationship to each of the conic sections. ■ Given the eccentricity, identify the conic section. ■ Evaluate the eccentricity of a conic section. ■ Determine the equation of a conic, given eccentricity and other characteristics.

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