MAS113 Introduction to Probability and Statistics

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2020–21

Studying probability theory

There are (at least) two ways to think about the study of probability theory.

Pure mathematical approach

Probability theory is a branch of Pure Mathematics.

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Probability theory is a subject in its own right, that we can study (and appreciate!) without concerning ourselves with any possible applications.

We can also observe links to other areas of mathematics.

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We must think carefully about when and how the theory can be applied in practice.

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Applications of probability theory are important and interesting, but as a mathematician, you should also develop an understanding and an appreciation of the theory in its own right.

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- A government is considering a CO_2 emissions target. What would the effect of a 20% cut in emissions be on global mean temperatures in 20 years' time?

Describing uncertainty

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Consider the following example.

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The test result is positive. How certain are you that you really have the disease?

Quantifying uncertainty

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In other words, we would like to *measure* how likely it is that something will happen, or how likely it is that some statement about the world turns out to be true.

Quantifying uncertainty continued

We will introduce a theory for measuring uncertainty: probability theory.

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We have two ingredients: **set theory**, which we will use to describe what sort of things we want to measure our uncertainty about, and, appropriately, **measure theory**, which sets out some basic rules for how to measure things in a sensible way.

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Suppose all the outcomes in an experiment are equally likely.

The probability of an event A is defined to be

$$P(A) := \frac{\text{number of outcomes in which } A \text{ occurs}}{\text{total number of possible outcomes}}. \qquad (1)$$

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The ‘definition’ can work for examples such as this (with a finite number of equally likely outcomes), but we soon run into difficulties:

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(For example, consider spinning a roulette wheel, and measuring the angle between the horizontal axis (viewing the wheel from above) and a line from the centre of the wheel that bisects the zero on the wheel).
What is the probability that the angle is between 0 and $\sqrt{2}$ radians?

Harder example continued

You might guess that the answer should be $\frac{\sqrt{2}}{2\pi}$, but the definition in (1) causes us problems!

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There are infinitely many possible angles that could be generated, and there are infinitely many angles between 0 and $\sqrt{2}$ radians. Clearly, we can't divide infinity by infinity and come up with $\frac{\sqrt{2}}{2\pi}$.

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We will resolve this by constructing a more general definition of probability, using the tools of set theory and a special type of function called a **measure**, and we'll show how choosing different types of measure will give us sensible answers in both the above examples.

Set theory and probability

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(Here we use the word experiment in a loose sense to mean observing something in the future, or discovering the true status of something that we are currently uncertain about.)

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Definition

A **sample space** is a set which lists all the possible outcomes of an 'experiment'.

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In the context of probability, the sample space will play the role of the universal set S.

Examples

Example

Examples of sample spaces

Events

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Examples of events

Set operations and events

Given a sample space S and two subsets/events A and B, the set operations union, intersection, complement and difference all define further events.

Union

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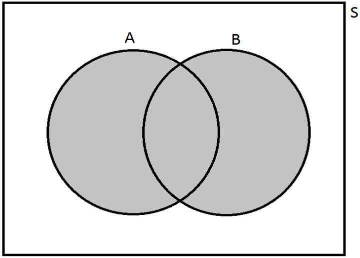
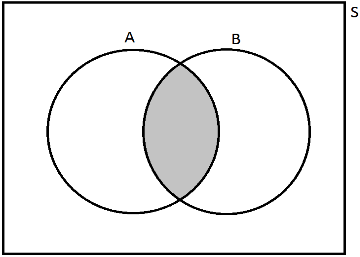
We can visualise this using a *Venn diagram*. The rectangle represents the universal set, the two circles represent the subsets A and B, and the shaded area represents the set $A \cup B$.



Intersection

The **intersection** $A \cap B$ corresponds to both A and B occurring.



Empty set

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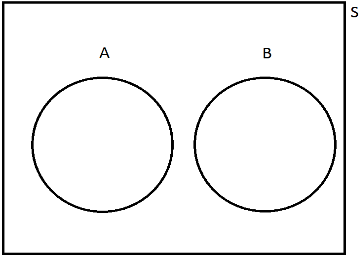
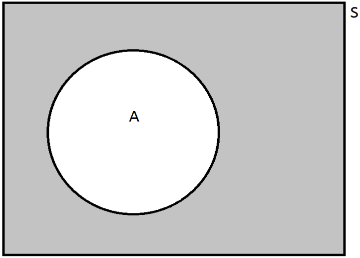
In this case we say that A and B are **mutually exclusive**.



Complement

The **complement** of A, $\overline{A}$, corresponds to A not occurring.



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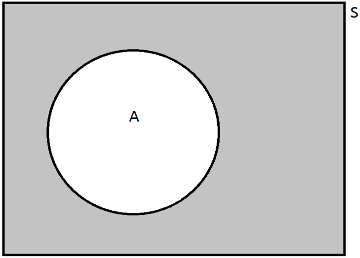

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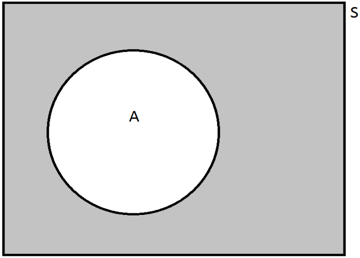

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Note that $\varnothing = \bar{S}$, and that the event $\varnothing$ is one which cannot happen, as it contains no elements.

Set difference

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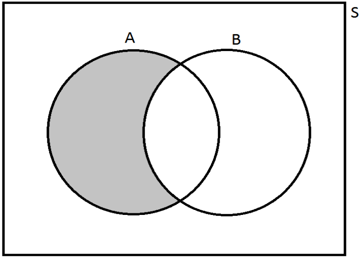

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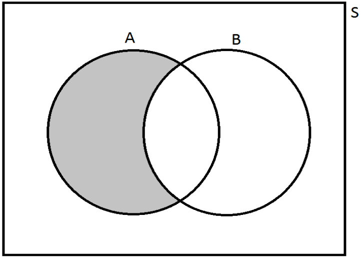

Note that

$$A \setminus B = A \cap \bar{B}.$$

De Morgan's Laws

Theorem

(De Morgan's Laws) For any two sets A and B,

$$\begin{aligned}\overline{(A \cup B)} &= \bar{A} \cap \bar{B},\\ \overline{(A \cap B)} &= \bar{A} \cup \bar{B}.\end{aligned}$$

Proof: an exercise in MAS110.

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Combinations of events

The difference between an outcome and an event

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Spain v. Germany

Suppose Spain play Germany at football this evening.

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So the sample space is the set of such pairs, $\mathbb{N}_0 \times \mathbb{N}_0$.

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- An event could be a single outcome (you could bet on "Spain win 2-1").

Notation

For a sample space S, outcome a and event A, we would use the notation $a \in S$ and $A \subset S$. Make sure you understand when to use the symbol $\in$ and when to use the symbol $\subset$ or $\subseteq$.

Set functions

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Not all set functions make sense for all possible sets, so when defining a set function, we'll first say what the sample space is, and then define set functions that operate on subsets of a sample space.

Set functions continued

Example

Examples of set functions

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Examples of set functions

Formally, we can think of a set function as a function whose domain is a set whose elements are subsets of S.

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We first define a **measure** as a special type of set function.

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For any two disjoint subsets A and B of S (ie $A \cap B = \varnothing$),

$$m(A \cup B) = m(A) + m(B). \tag{2}$$

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If we take two non-overlapping (disjoint) sets, we can either combine the sets and then apply the measure, or apply the measure to each set and then sum the results, and we will get the same value either way.

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(M3) $m(\varnothing) = 0$

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(M6) *If* m *and* n *are both measures, the function* $h(A) = m(A) + n(A)$ *is also a measure.*

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If S is a sample space, one of the events in $\mathcal{E}$ *must* occur, but no two events can occur simultaneously.

Partitions continued

From (2), we have

$$m(S) = \sum_{i=1}^{n} m(E_i).$$