MAT137

Problem Set #1

Sample Solutions

1. Negate the following statement without using any negative words (no, not, none, zero, etc.):

   "At each zoo, there is some animal that visitors want to pet and which is either too dangerous to see visitors or which runs away from people."

   Sample solution:

   There is some zoo where if visitors want to pet an animal, that animal is safe enough to visit with and either comes towards people or stays put.

2. In this problem, we will deal with subsets $A \subseteq \mathbb{R}$. We define two new properties a subset A might have:

   - We say A is *naturally decreasing* if:
   $$\forall x \in A, \exists y \in A \text{ s.t. } x \neq y \text{ and } x - y \in \mathbb{N}$$
   - We say A is *spottily increasing* if:
   $$\exists x \in A \text{ s.t. } \forall y \in A,\ y - x \in \mathbb{N}$$

   Below are three claims. Which ones are true and which ones are false? If a claim is true, prove it. If a claim is false, show it with a counterexample.

   a) If A is a naturally decreasing set, then A is not spottily increasing.

   Sample solution: This statement is true.

   Proof. Suppose A is a naturally decreasing set.

   We wish to show that $\exists x \in A$ s.t. $\forall y \in A$, $y - x \in \mathbb{N}$ is false, so we will show that its negation $\forall x \in A$, $\exists y \in A$ s.t. $y - x \notin \mathbb{N}$ is true.

   Let $x \in A$. Since A is naturally decreasing, $\exists y \in A$ s.t. $x \neq y$ and $x - y \in \mathbb{N}$. So $x - y > 0$. However, this then gives that $y - x < 0$ and so $y - x \notin \mathbb{N}$.

   Therefore, $\forall x \in A$, $\exists y \in A$ s.t. $y - x \notin \mathbb{N}$, and so A is not spottily increasing. □

   b) If $A \subseteq \mathbb{Z}$ and A is not a spottily increasing set, then A is naturally decreasing.

   Sample solution: This statement is true.

   Proof. Since A is not spottily increasing, $\forall x \in A$, $\exists y \in A$ s.t. $y - x \notin \mathbb{N}$.

   Now, since $A \subset \mathbb{Z}$, $y - x \in \mathbb{Z}$. So, $y - x < 0$. As such, $x - y > 0$ and $x - y \in \mathbb{Z}$. So $x - y \in \mathbb{N}$ and $x \neq y$.

   Therefore, $\forall x \in A$, $\exists y \in A$ s.t. $x \neq y$ and $x - y \in \mathbb{N}$. So A is naturally decreasing. □

c) If $0 \in A$ and A is naturally decreasing, then $\{-x | x \in \mathbb{N}\} \subseteq A$.

Sample solution: This statement is false.

Consider the following counter example:

$$A = \{-2k | k \in \mathbb{N}\}$$

To show that this is a counterexample, we need to show that it verifies the assumptions of the implication (i.e. $0 \in A$ and A is naturally decreasing), while failing the conclusion of the implication (i.e., $\{-x|x \in \mathbb{N}\} \subseteq A$).

Well, since $-1 \notin A$, we know that $\{-x|x \in \mathbb{N}\} \not\subseteq A$.

Also, since $0 \in \mathbb{N}$, $-2(0) = 0 \in A$.

So we need only verify that A is naturally decreasing. Let $x \in A$. Then $x = -2k$ for some $k \in \mathbb{N}$. Let $y = -2(k+1)$. Since $k+1 \in \mathbb{N}$, $y \in A$. Also, $x - y = 2 \in \mathbb{N}$ and $x \neq y$. So, $\forall x \in A, \exists y \in A$ s.t. $x \neq y$ and $x - y \in \mathbb{N}$. So A is naturally decreasing.

3. Let $a \in \mathbb{N} \setminus \{0\}$ be a positive integer and $b \in \mathbb{Z}$ be any integer. We say that a *divides* b if there is an integer k with $b = ka$.

   For which n does 5 divide $3^n - 2^n$? Prove your claim using induction, the definition given, and basic arithmetic.

Sample solution: The statement holds for $n = 2k$, with $k \in \mathbb{N}$. It does not hold for $n = 2k + 1$ with $k \in \mathbb{N}$.

As such, we will need to give two proofs here: one for when $n = 2k$ and one for when $n = 2k + 1$.

Claim 1: n divides $3^n - 2^n$ when $n = 2k$ for some $k \in \mathbb{N}$:

Proof. We proceed by induction:

Base case: When $n = 0$, $3^n - 2^n = 1 - 1 = 0$. Now, $0 = 5 \times 0$, so 5 divides 0. So the claim is true when $n = 0$.

Induction Hypothesis: Suppose 5 divides $3^n - 2^n$ for some $n = 2k$ with $k \in \mathbb{N}$.

Induction Step: We proceed to show that 5 divides $3^{n+2} - 2^{n+2}$.

Notice that $3^{n+2} - 2^{n+2} = 9(3^n) - 4(2^n) = 9(3^n - 2^n) + 5(2^n)$. Now, since 5 divides $3^n - 2^n$ by the induction hypothesis, there exists $a \in \mathbb{Z}$ so that $3^n - 2^n = 5a$. This gives us that:

$$3^{n+2} - 2^{n+2} = 9(3^n - 2^n) + 5(2^n) = 9(5a) + 5(2^n) = 5(9a + 2^n)$$

and since $9a + 2^n \in \mathbb{Z}$, we have that 5 divides $3^{n+2} - 2^{n+2}$.

So, n divides $3^n - 2^n$ whenever $n = 2k$ for some $k \in \mathbb{N}$ by induction. □

Claim 2: n does not divide $3^n - 2^n$ when $n \neq 2k$ for any $k \in \mathbb{N}$:

Proof. Before we prove this, note that $n \in \mathbb{N}$ and $n \neq 2k$ for any $k \in \mathbb{N}$ means that $n = 2k + 1$ for some $k \in \mathbb{N}$. We proceed by induction:

Base case: When $k = 0$ we have $n = 1$. Well, $3^1 - 2^1 = 1$, which is not divisible by 5. So the claim is true when $n = 1$.

Induction Hypothesis: Suppose 5 does not divide $3^n - 2^n$ for some $n = 2k + 1$ with $k \in \mathbb{N}$.

Induction Step: We proceed to show that 5 does not divide $3^{n+2} - 2^{n+2}$.

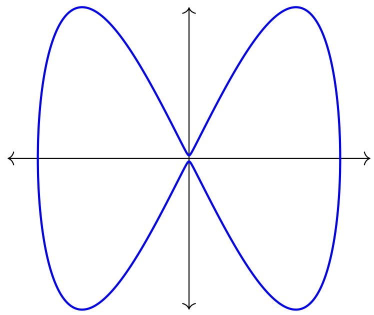
Notice that $3^{n+2} - 2^{n+2} = 9(3^n) - 4(2^n) = 9(3^n - 2^n) + 5(2^n)$. We can rearrange this to give us:

$$9(3^n - 2^n) = 3^{n+2} - 2^{n+2} - 5(2^n)$$

If 5 divides $3^{n+2} - 2^{n+2}$, then we have that $3^{n+2} - 2^{n+2} = 5a$ for some $a \in \mathbb{Z}$. In particular, this then tells us that $9(3^n - 2^n) = 5(a - 2^n)$. But we assumed that $9(3^n - 2^n)$ is not divisible by 5 (since 5 is prime, and it doesn't divide either 9 or $3^n - 2^n$). That gives us a contradiction, since $9(3^n - 2^n) = 5b$ for some $b \in \mathbb{Z}$. So it must be that $3^{n+2} - 2^{n+2}$ is also not divisble by 5.

So, if $n = 2k + 1$ for some $k \in \mathbb{Z}$, 5 does not divide $3^n - 2^n$ by induction. □

4. The equation $x^4 - 100x^2 + 25y^2 = 1$ defines a curve that looks as follows:



Consider the following argument, which contains several mistakes:

Proof. Since the slope of the tangent line to a curve is given by $\frac{dy}{dx}$, the curve has a vertical tangent when $\frac{dy}{dx}$ does not exist.
Now, we know that $\frac{dy}{dx}$ does not exist when the curve has a sharp corner at x. We see from the graph that this occurs when $x = 0$, and $y = \pm\frac{1}{5}$. So, the curve only has vertical tangent lines at the two points $(0, \pm\frac{1}{5})$. □

a) Find all of the errors in the above argument and explain why they are errors.

Sample solution: There are a couple key errors:

- It is not true that there is a vertical tangent line when $\frac{dy}{dx}$ does not exist. In fact, it is the converse that is true: if there is a vertical tangent at a, then $\frac{dy}{dx}\Big|_{x=a}$ does not exist.

  There are several other ways that the derivative can fail to exist: sharp corners, oscillation, discontinuities, etc.
- I have missed two places where the derivative does not exist: when $y = 0$.
- The graph does not have a sharp corner at the points $\left(0, \frac{\pm 1}{5}\right)$. These points are on the graph, but they are actually places where $\frac{dy}{dx} = 0$!
- Tying into the last point: a proof by picture is not a proof. Pictures lie. This picture lied. It looks like a corner but is not.

Some things that are not errors:

- There is nothing wrong with the way the proof is written. In fact, writing in full sentences is preferred. Writing a proof is the same as writing an essay. You want it to be human readable, and we're trained to consume ideas in sentences.
- Discussing $\frac{dy}{dx}$ without introducing the notation is not an error. We have defined the derivative, and x, y are introduced in the question (you're given an equation in terms of x, y!)
- $\left(0, \frac{\pm 1}{5}\right)$ are actually on the curve.

b) Explain how to locate the points on this curve with vertical tangent line, and then do so.

Sample solution: There a couple of ways of finding these points:

- Look for places where $\frac{dy}{dx}$ does not exist. If you take this approach, you have to show that these points do actually give you a vertical tangent though. In particular, if this occurs at some point $(a, 0)$, then you would need to discuss how $\lim_{x\to a} \frac{dy}{dx} = \pm\infty$. Since we hadn't talked about limits yet, this approach doesn't fully work.
- View x as a function of y. If we think in this context (and view the graph with y as the independent axis), then it becomes clear that vertical tangents occur when $\frac{dx}{dy} = 0$.

We'll take the second approach. If we implicitly differentiate with respect to y, we get:

$$4x^3\frac{dx}{dy} - 200x\frac{dx}{dy} + 50y = 0$$

Rearranging gives:

$$\frac{dx}{dy} = \frac{-50y}{4x^3 - 200x}$$

From here, we see that $\frac{dx}{dy} = 0$ when $y = 0$ and $4x^3 - 200x \neq 0$. When $y = 0$, we have $x^4 - 100x^2 = 1$. The quadratic formula gives us that:

$$x^2 = \frac{100 \pm \sqrt{100^2 + 4}}{2}$$

And so $x = \pm\sqrt{50 \pm \sqrt{2501}}$. Since $50 < \sqrt{2501} = \sqrt{50^2 + 1}$, we see that $\sqrt{50 - \sqrt{2501}}$ is not defined, and so actually $x = \pm\sqrt{50 + \sqrt{2501}}$. One can quickly verify that for these x, $4x^3 - 200x \neq 0$.

So, the curve has vertical tangents at the two points: $(\pm\sqrt{50 + \sqrt{2501}}, 0)$.