MATH 008A - INTRODUCTION TO COLLEGE MATH FOR SCIENCES - NOTES

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Contents

1 Introduction **3**
1.1 Introduction . . . 3

2 MATH 008A - Precalculus **4**
2.1 Properties of Real Numbers, Exponent Properties, Right Triangles . . 4
2.1.1 Properties of Real Numbers . . . 4
2.1.2 Exponent Properties . . . 5
2.1.3 Right Triangles and Geometry . . . 5
2.2 Rational Expressions . . . 6
2.3 Word Problems . . . 7
2.4 Lines . . . 8
2.5 Graph Transformations . . . 9
2.6 Quadratic Functions . . . 10
2.7 Graphing Polynomials . . . 11
2.8 Quadratic and Rational Inequalities . . . 12
2.8.1 Quadratic Inequalities . . . 12
2.8.2 Rational Inequalities . . . 12
2.9 Complex Zeros; Exponential Functions and Equations . . . 13
2.9.1 Complex Zeros . . . 13
2.9.2 Exponential Functions and Equations . . . 13
2.10 Logarithm and Exponential Equations; Right Triangles/Trigonometry 14
2.10.1 Logarithm and Exponential Equations . . . 14
2.10.2 Right Triangles and Trigonometry . . . 15
2.11 Graphing Trig Functions . . . 16
2.12 Inverse Trig Functions; Solving Inverse Trig Equations . . . 20
2.13 Vectors; Dot Product . . . 26
2.13.1 Vectors . . . 26
2.13.2 Dot Product . . . 28
2.14 Partial Fraction Decomposition . . . 29
2.14.1 Non-repeated Linear Factors . . . 29
2.14.2 Repeated Linear Factors . . . 30
2.14.3 Irreducible Quadratic Factor . . . 32
2.14.4 Repeated Irreducible Quadratic Factor . . . 34

Chapter 1

Introduction

1.1 Introduction

This set of lecture notes is designed for the MATH 008A course in precalculus. The material covered here is not inclusive of the whole course, as I was only responsible for covering some of the material. All the topics that I gave a lecture on have been included in this file. Also included are many exercises to work, as I feel the best way to learn and master the subject material in this class is to work as many problems as you can to get good at the material. As for the exericises, I attempt to put them in order of difficulty, with the easier problems first, and the most challenging at the end. Best.

Chapter 2

MATH 008A - Precalculus

2.1 Properties of Real Numbers, Exponent Properties, Right Triangles

2.1.1 Properties of Real Numbers

Definition 2.1.1 *The* ***intersection*** *of A and B, denoted by $A \cap B$, is the set consisting of elements that belong to both A and B.*

Definition 2.1.2 *The* ***union*** *of A and B, denoted by $A \cup B$, is the set consisting of elements that belong to either A and B.*

Definition 2.1.3 *Given real numbers a, b, and c, the* ***distributive property*** *is*

$$a \cdot (b + c) = ab + ac$$

Definition 2.1.4 *Given real numbers a, b the* ***zero property*** *states: If $ab = 0$, then either $a = 0$ or $b = 0$ or both are equal to 0.*

Definition 2.1.5 *We can define* ***inequalities*** *as follows*

$$\begin{cases} a > 0 & \text{is equivalent to } a \text{ is positive} \\ a < 0 & \text{is equivalent to } a \text{ is negative} \end{cases}$$

Example 1 *On the real number line, graph all numbers x such that $x > 5$.*
On the real number line, graph all numbers x such that $x \leq -1$.

Definition 2.1.6 *The* ***absolute value*** *of a real number a is defined as $|a| = a$ if $a \geq 0$ and $|a| = -a$ if a if $a < 0$.*

Example 2 *Let $a = -5$. Then $|-5| = -(-5) = 5$.*

Definition 2.1.7 *If P and Q are two points on a real number line with coordinates a and b, then the* ***distance between*** *P and Q, is*

$$d(P, Q) = |b - a|$$

Definition 2.1.8 *The set of values that a variable may assume is called the* ***domain of the variable****.*

Example 3 *Find the domain of the following function in interval notation*

$$f(x) = \frac{5}{x-2}$$

Example 4 *Find the domain of the following function in interval notation*

$$f(x) = \frac{x-2}{x-9}$$

2.1.2 Exponent Properties

Definition 2.1.9 *The **laws of exponents** are*

$$\begin{cases} a^0 = 1 & \text{if } a \neq 0 \\ a^{-n} = \frac{1}{a^n} & \text{if } a \neq 0 \\ a^m a^n = a^{m+n} & \\ (a^m)^n = a^{mn} & \\ \frac{a^m}{a^n} = a^{m-n} = \frac{1}{a^{n-m}} & \\ (ab)^n = a^n b^n & \\ \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} & \text{if } b \neq 0 \end{cases}$$

Example 5 *Write the following expression with all positive exponents*

$$\frac{x^5 y^{-2}}{x^3 y}$$

Example 6 *Write the following expression with all positive exponents*

$$\left(\frac{x^{-3}}{3y^{-1}}\right)^{-2}$$

2.1.3 Right Triangles and Geometry

Definition 2.1.10 *The **Pythagorean Theorem** is $c^2 = a^2 + b^2$, for finding the side lengths of a right triangle.*

Example 7 *What is the hypotenuse of a right triangle with sides of 5 and 12?*

Definition 2.1.11 ***Geometry Formulas** (Here A = Area, P = Perimeter, C = Circumference, V = Volume, and SA = Surface Area.*

$$\begin{cases} A = lw, P = 2l + 2w & \text{for a rectangle} \\ A = \frac{1}{2}bh & \text{for a triangle} \\ A = \pi r^2, C = 2\pi r & \text{for a circle} \\ V = lwh, SA = 2lh + 2lw + 2wh & \text{for a rectanglular prism} \\ V = \frac{4}{3}\pi r^3, SA = 4\pi r^2 & \text{for a sphere} \\ V = \pi r^2 h, SA = 2\pi r^2 + 2\pi rh & \text{for a cylinder} \end{cases}$$

Example 8 *Find the volume and surface area of a sphere that has radius 2.*

Definition 2.1.12 *Two triangles are **congruent** if each of the corresponding angles is the same measure and each of the corresponding sides is the same length.*

Definition 2.1.13 *Two triangles are **similar** if each of the corresponding angles is the same measure and each of the corresponding sides is proportitonal.*

Example 9 *Are the trianges with side lengths 5,6, 10 and 15, 18, 30:*
(a) congruent? (b) similar?

2.2 Rational Expressions

Definition 2.2.1 *The **Cancellation Property** is*

$$\frac{ac}{bc} = \frac{a\ \not{c}}{b\ \not{c}} = \frac{a}{b}$$

Example 10 *Reduce the following*

$$\frac{x^2+4x+4}{x^2+3x+2}$$

Example 11 *Reduce the following*

$$\frac{8-2x}{x^2-x-12}$$

Definition 2.2.2 ***Multiplying Rational expressions***

$$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$$

Definition 2.2.3 ***Dividing Rational expressions***

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b}\cdot\frac{d}{c} = \frac{ad}{bc}$$

Example 12 *Reduce the following*

$$\frac{x^2-2x+1}{x^3+x}\cdot\frac{4x^2+4}{x^2+x-2}$$

Example 13 *Reduce the following*

$$\frac{\frac{x-2}{4x}}{\frac{x^2-4x+4}{12x}}$$

Definition 2.2.4 ***Adding Rational expressions***

$$\frac{a}{b}+\frac{c}{b} = \frac{a+c}{b}$$

$$\frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}$$

Example 14 *Reduce the following*

$$\frac{2x^2-4}{2x+5}+\frac{x+3}{2x+5}$$

Example 15 *Reduce the following*

$$\frac{x^2}{x^2-4}-\frac{1}{x}$$

Example 16 *Reduce the following using the least common multiple.*

$$\frac{x}{x^2+3x+2}+\frac{2x-3}{x^2-1}$$

2.3 Word Problems

1) The length of a rectangle is 3 more than twice its width. If the area of the rectangle is 119 cm^2, find its length and width.

2) John can paint a house in 4 hours while Mike can paint the same house in 6 hours. How long will it take them to paint the house if they work together?

3) The cooling system of a car has a capacity of 15 L. If the system is filled with a 40

4) An open box is to be constructed from a square piece of sheet metal by removing a square of side 3 feet and turning up the edges. If the box is to hold 300 ft^3, what should be the dimensions of thte sheet metal?

5) A motorboat heads upstream a distance of 24 miles on a river whose current is running at 3 mi/hr. The trip back takes 6 hours. Assuming that the motorboat maintained a constant speed relative to the water, what was its speed?

6) A Tennis court has an area of 2808 ft^2 and the length is 6 ft longer than 2 times the width. Find the dimensions of the court.

2.4 Lines

Definition 2.4.1 *Given two points* (x_1, y_1) *and* (x_2, y_2)*, the* ***slope*** *of the line that passes through these points is*

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{rise}{run}$$

Note: Slope is undefined for vertical lines ($x_1 = x_2$)*. Slope is zero for horizontal lines* ($y_1 = y_2$)*.*

Example 17 *Find the slope of the line that passes through* $(1, 2)$ *and* $(5, -3)$*.*

Definition 2.4.2 *The* ***Point Slope Formula*** *for a line is*

$$y - y_1 = m(x - x_1)$$

Definition 2.4.3 *The* ***Slope intercept formula*** *for a line is*

$$y = mx + b$$

Definition 2.4.4 *The* ***General Form*** *for a line is*

$$Ax + By = C$$

Example 18 *Find the equation for a line passing through* $(1, 2)$ *with slope* $m = 3$*.*

Definition 2.4.5 *To find the* ***x-intercepts****, plug in 0 for* y *and solve for* x*. Similarly, to find the* ***y-intercepts****, plug in 0 for* x *and solve for* y*.*

Example 19 *Find the* x *and* y *intercepts of* $x^2 + 16y^2 = 4$*.*

Definition 2.4.6 ***Parallel lines*** *have the* **same** *slope.*
Perpendicular lines *have a slope that is the* **negative reciprocal** *(Take the first slope, invert the number, and multiply by -1).*

Example 20 *Find the equation of the line that passes through* $(1, -2)$ *and is perpendicular to the line* $x + 3y = 6$*.*

Definition 2.4.7 *The* **equation of the circle** *is*

$$(x - h)^2 + (y - k)^2 = r^2$$

where the center is the coordinate (h, k) *and the radius is* r*.*

Definition 2.4.8 *The* ***General Form*** *of a circle is*

$$x^2 + y^2 + ax + by + c = 0$$

Example 21 *Find the equation of the circle that has the general form*

$$x^2 + y^2 + 4x - 6y + 12 = 0$$

and give the center of the circle, the radius, and graph the circle.

Example 22 *Find the equation of the circle that has the general form*

$$x^2 + y^2 - 2x - 4y - 4 = 0$$

and give the center of the circle, the radius, and graph the circle.

Example 23 *Find the equation of the circle that has the general form*

$$2x^2 + 2y^2 - 2x + 16y - 10 = 0$$

and give the center of the circle, the radius, and graph the circle.

2.5 Graph Transformations

Below is a chart that summarizes all of the transformations we covered in class. The general formula is in the second column, and there is an example given for each case for the function $f(x) = |x|$ (the absolute value function). You can use the second column as a guide to work with any function that you are given.

Transformations of the Absolute Value Parent Function f(x)=|x|

Transformation	F (x) Notation	Examples	
Vertical translation	f(x) + k	Y = |x| + 3 Y = |x| - 4	3 units up 4 units down
Horizontal translation	f(x – h)	Y = |x-2| Y = |x+1|	2 units right 1 unit left
Vertical stretch/ compression	af(x)	Y = 6|x| Y = ½ |x|	vertical stretch by 6 vertical compression by 1/2
Horizontal stretch/compression	F(1/bx)	Y = |1/5x| Y= |3x|	horizontal stretch by 5 horizontal compression by 1/3
Reflection	-f(x) f(-x)	Y = -|x| Y = |-x|	across x-axis across y-axis

Figure 2.1: Transformation Chart - From http://hellermaayanotmath.wikispaces.com

Below are a few examples of using transformations that may be helpful for studying.

Example 24 *Apply a horizontal translation 3 units left and a vertical translation of 4 units down for the graph of* $f(x) = x^2$*, and write the formula for this graph.*

Example 25 *Apply a horizontal translation 2 units right, a vertical translation of 5 units up, and a reflection over the x-axis for the graph of* $f(x) = |x|$*, and write the formula for this graph.*

Example 26 *Sketch the graph of* $f(x) = (x + 5)^5 - 2$ *using graph transformations.*

Example 27 *Sketch the graph of* $f(x) = -|x - 3| + 4$ *using graph transformations.*

Example 28 *Sketch the graph of* $f(x) = -2(x+1)^2 - 5$ *using graph transformations.*

2.6 Quadratic Functions

Definition 2.6.1 *A **quadratic function** is of the form*

$$f(x) = ax^2 + bx + c$$

Definition 2.6.2 *The maximum or minimum of the parabola is called a **vertex**.*

Definition 2.6.3 *The vertical line that passes through the vertex is called the **axis of symmetry**.*

The following proof is to show how to get the vertex form of the parabola in general. We will have to complete the square.

$$\begin{aligned} f(x) &= ax^2 + bx + c \\ &= a\left(x^2 + \frac{b}{a}x\right) + c \\ &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) \\ &= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \\ &= a(x-h)^2 + k \end{aligned}$$

where we define $h = -\frac{b}{2a}$ and $k = \frac{4ac-b^2}{4a}$. So then we have that

$$\text{Vertex} = \left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$$

$$\text{Axis of Symmetry} \Rightarrow x = -\frac{b}{2a}$$

Definition 2.6.4 *If we have a quadratic function $f(x) = ax^2 + bx + c$, then*
(i) $a > 0$ implies that the vertex is a minimum.
(ii) $a < 0$ implies that the vertex is a maximum.

Definition 2.6.5 *If we have a quadratic function $f(x) = ax^2 + bx + c$, then*
(i) If $b^2 - 4ac > 0 \Rightarrow$, then f has 2 real roots.
(ii) If $b^2 - 4ac = 0 \Rightarrow$, then f has 1 real root.
(iii) If $b^2 - 4ac < 0 \Rightarrow$, then f has no real roots (does not cross the x-axis).

Below are a few examples of writing the quadratic in vertex form.

Example 29 *Rewrite the function $f(x) = x^2 - 8x + 7$ in vertex form.*

Example 30 *Rewrite the function $f(x) = x^2 + 6x + 1$ in vertex form.*

Example 31 *Rewrite the function $f(x) = 3x^2 + 12x + 2$ in vertex form.*

Example 32 *Rewrite the function $f(x) = 4x^2 - 40x + 13$ in vertex form.*

2.7 Graphing Polynomials

Note: Remember to always write your answers in interval notation.

Example 33 *Solve* $x^2 - 4x - 12 \leq 0$

Example 34 *Solve* $2x^2 < x + 10$

Example 35 *Solve* $x^2 + x > 12$

Example 36 *Solve* $-x^2 - x + 6 < 0$

Note: Remember to always write your answers in interval notation. Also, remember that when moving the parts of the expression, **ONLY** use addition and subtraction.

Example 37 *Solve* $\frac{4x+5}{x+2} \geq 3$

Example 38 *Solve* $\frac{x+2}{x-4} \geq 1$

Example 39 *Solve* $\frac{5}{x-3} > \frac{3}{x+1}$

Example 40 *Solve* $\frac{(x+3)(x-5)}{3(x-1)} > 0$

Example 41 *Solve* $\frac{x^2+4x-45}{x+1} \leq 0$

2.8 Quadratic and Rational Inequalities

2.8.1 Quadratic Inequalities

Note: Remember to always write your answers in interval notation.

Example 42 *Solve* $x^2 - 4x - 12 \leq 0$

Example 43 *Solve* $2x^2 < x + 10$

Example 44 *Solve* $x^2 + x > 12$

Example 45 *Solve* $-x^2 - x + 6 < 0$

2.8.2 Rational Inequalities

Note: Remember to always write your answers in interval notation. Also, remember that when moving the parts of the expression, **ONLY** use addition and subtraction.

Example 46 *Solve* $\frac{4x+5}{x+2} \geq 3$

Example 47 *Solve* $\frac{x+2}{x-4} \geq 1$

Example 48 *Solve* $\frac{5}{x-3} > \frac{3}{x+1}$

Example 49 *Solve* $\frac{(x+3)(x-5)}{3(x-1)} > 0$

Example 50 *Solve* $\frac{x^2+4x-45}{x+1} \leq 0$

2.9 Complex Zeros; Exponential Functions and Equations

2.9.1 Complex Zeros

Note: Remember the Conjugte Pairs Theorem.

Example 51 *Find polynomial of degree 3 with roots* 3 *and* $2-5i$.

Example 52 *Find polynomial of degree 3 with roots* -5 *and* $1+2i$.

Example 53 *Find polynomial of degree 3 with roots* 2 *and* $-3+i$.

Example 54 *Find polynomial of degree 3 with roots* -1 *and* $4-2i$.

2.9.2 Exponential Functions and Equations

Definition 2.9.1 *An* ***exponential function*** *has the form*

$$f(x) = Ca^x$$

where C *is a constant and* $a > 0$ *and* $a \neq 1$.

Definition 2.9.2 ***Properties for the exponential function*** $f(x) = a^x$*:*
(i) Domain is all reals, $(-\infty, \infty)$.
(ii) Range is all positive reals, $(0, \infty)$.
(iii) No x*-intercepts.*
(iv) The y*-intercept is the coordinate* $(0, 1)$.
(v) There is an asymptote at the x*-axis,* $y = 0$.

Example 55 *Graph* $f(x) = 2^x$.

Example 56 *Graph* $f(x) = (\frac{1}{2})^x$.

Note: It is **strongly recommended** that you work more problems similar to these in order to get good at these types of problems as they are very likely to show up on quizzes and tests.

Example 57 *Solve* $3^{x+1} = 81$

Example 58 *Solve* $4^{2x-1} = 8^{x+3}$

Example 59 *Solve* $3^{x^2-7} = 27^{2x}$

Example 60 *Solve* $e^{-x^2} = (e^x) \cdot \frac{1}{e^3}$

Example 61 *Solve* $4^{x+1} = \frac{1}{64}$

Example 62 *Solve* $8^{x-2} = \sqrt{8}$

2.10 Logarithm and Exponential Equations; Right Triangles/Trigonometry

2.10.1 Logarithm and Exponential Equations

Definition 2.10.1 *So we define the logarithm as*

$$y = \log_a(x) \Leftrightarrow x = a^y$$

for $a > 0$ *and* $a \neq 1$*. We also have that if*

$$\log_a(M) = \log_a(N) \Rightarrow M = N$$

Note: Some notation that to be aware of:

$$\log = \log_{10}$$
$$\ln = \log_e$$

Note: The log is the inverse of the exponential function. Also, you **CANNOT** plug in negative numbers into a logarithm!

Definition 2.10.2 ***Properties for the logarithm function*** $\log_a(x)$*:*

(i) $r\log_a(M) = \log_a(M^r)$
(ii) $\log_a(M) + \log_a(N) = \log_a(MN)$
(iii) $\log_a(M) - \log_a(N) = \log_a(\frac{M}{N})$

Here are some basic examples that you should understand how to compute. vspace.5cm

Example 63 *Find the value of* $\log_4(64)$.

Example 64 *Find the value of* $\log_3(\frac{1}{9})$.

Example 65 *Find the value of* $\log(100)$.

Example 66 *Find the value of* $\ln(e^{10})$.

Here are some basic examples that you should understand how to compute.

Example 67 *Solve* $2\log_5(x) = \log_5(9)$

Example 68 *Solve* $\log_5(x+6) + \log_5(x+2) = 1$

Example 69 *Solve* $\ln(x) = \ln(x+6) - \ln(x-4)$

Example 70 *Solve* $-2\log_4(x) = \log_4(9)$

Example 71 *Solve* $\ln(x-3) + \ln(x-2) = \ln(2x+24)$

Example 72 *Solve* $\log(8x) - \log(1+\sqrt{x}) = 2$

2.10.2 Right Triangles and Trigonometry

Definition 2.10.3 *We define the trig functions as following using the diagram above.*

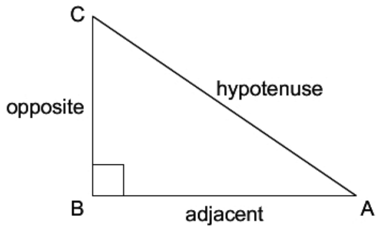

Figure 2.2: Right Triangle

$$\sin(\theta) = \frac{opp}{hyp} = \frac{b}{c} \qquad \csc(\theta) = \frac{hyp}{opp} = \frac{c}{b}$$

$$\cos(\theta) = \frac{adj}{hyp} = \frac{a}{c} \qquad \sec(\theta) = \frac{hyp}{adj} = \frac{c}{a}$$

$$\tan(\theta) = \frac{opp}{adj} = \frac{b}{a} \qquad \cot(\theta) = \frac{adj}{opp} = \frac{a}{b}$$

Example 73 *Find the values of all the trig functions given the triangle with sides* $3, 4, 5$.

Example 74 *Find the values of all the trig functions given the triangle with sides* $5, 12, 13$.

2.11 Graphing Trig Functions

Definition 2.11.1 *We can define the* ***cosine*** *function as a horizontal shift of* $\sin(x)$*, since we know the graph of this function already:*

$$f(x) = \cos(x) = \sin(x + \frac{\pi}{2})$$

where the horizontal shift is $\frac{\pi}{2}$ *units to the left. The graph is given in the plot below. Note the zeros are at* $\frac{\pi}{2}$ *and* $\frac{3\pi}{2}$*.*

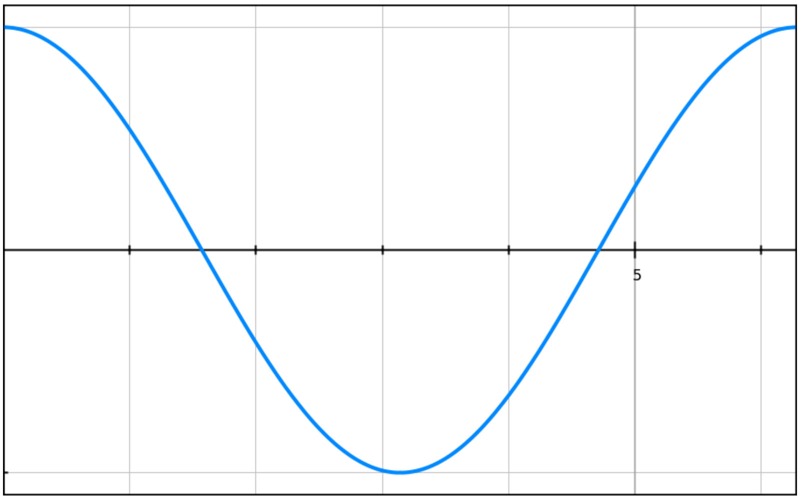

Figure 2.3: Graph of $f(x) = \cos(x)$

Example 75 *Sketch a complete graph of* $f(x) = -4\cos(2x) - 2$

***Solution*:** We want to do this graph by the graph transformations that are given in a previous lecture. First we calculate 3 quantities: Amplitude, Period, and vertical shift. Recall that the function has the general form: $f(x) = A\cos(\omega x) + b$. So then we have that $|A| = 4$ is the amplitude, $P = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$, and $b = -2$. From this, we can deduce the graph. Starting out with the parent graph above in figure 1, we apply transformations as follows:

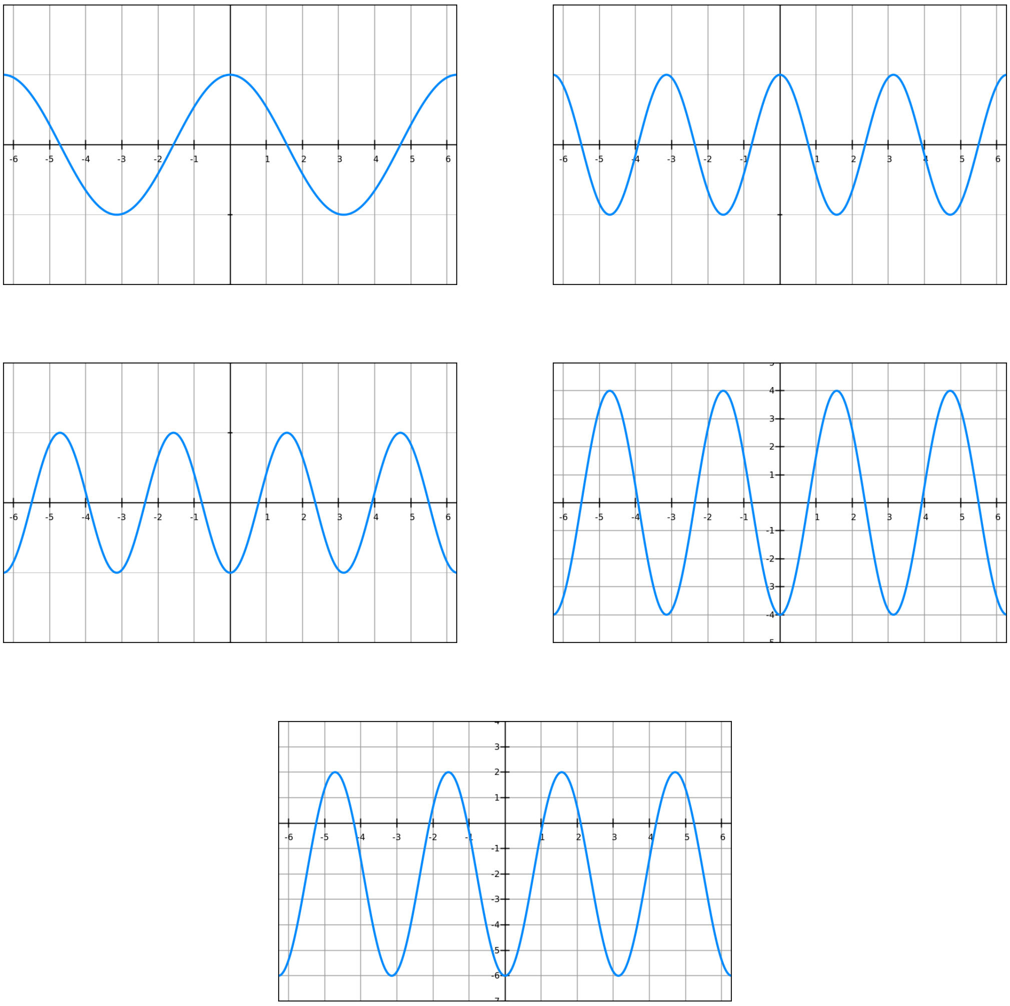

Figure 2.4: Graph Transformations: (1) $f(x) = \cos(x)$, (2) $f(x) = \cos(2x)$, (3) $f(x) = -\cos(2x)$, (4) $f(x) = -4\cos(2x)$, (5) $f(x) = -4\cos(2x) - 2$. From left to right and top to bottom.

□

Example 76 *Sketch a complete period:* $y = 4\sin(4x) + 1$

Example 77 *Sketch a complete period:* $y = 2\cos(6x) - 3$

Example 78 *Sketch a complete period:* $y = 3\sin(2x) + 1$

Definition 2.11.2 *We can define the **tangent** function using* $\sin(x)$ *and* $\cos(x)$*, since we know the function values of these trig functions already:*

$$f(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$$

So we will have vertical asymptotes where $\cos(x) = 0$ *and zeros where* $\sin(x) = 0$*. The graph is shown below for the zoomed in version and the periodic zoomed out version.*

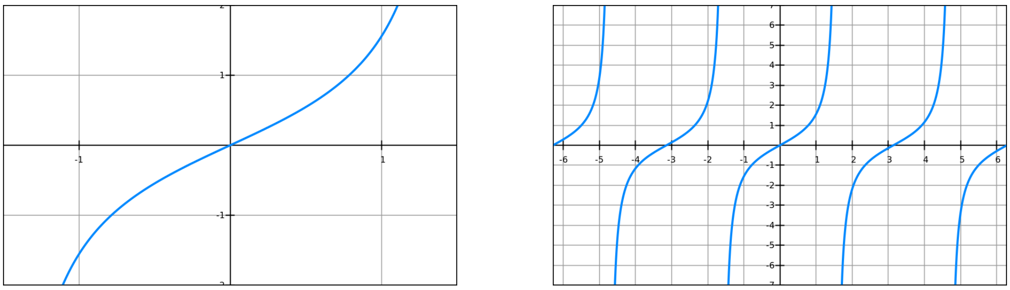

Figure 2.5: Graph of $f(x) = \tan(x)$ zoomed in on left. Graph of $f(x) = \tan(x)$ zoomed out on right.

Definition 2.11.3 *We can define the **cotangent** function using* $\sin(x)$ *and* $\cos(x)$*, since we know the function values of these trig functions already:*

$$f(x) = \cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$$

So we will have vertical asymptotes where $\sin(x) = 0$ *and zeros where* $\cos(x) = 0$*. The graph is shown below for the periodic zoomed out version.*

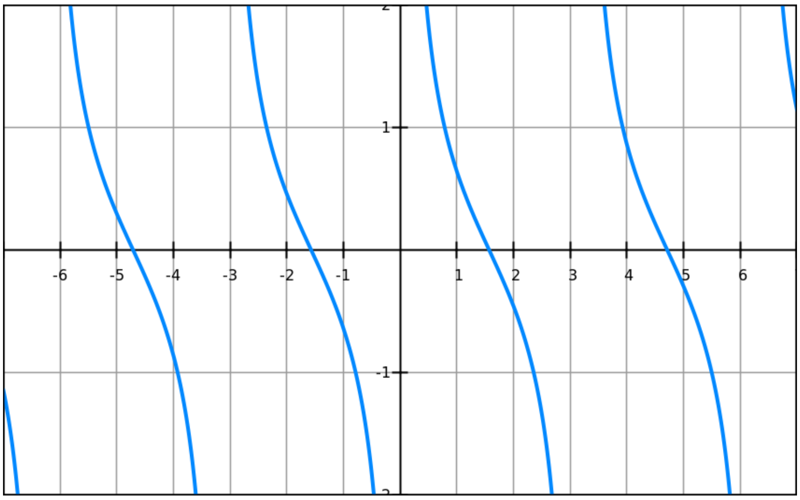

Figure 2.6: Graph of $f(x) = \cot(x)$ zoomed out.

Definition 2.11.4 *We can define the* ***cosecant*** *function using* $\sin(x)$*, since we know the function values of this trig functions already:*

$$f(x) = \csc(x) = \frac{1}{\sin(x)}$$

So we will have vertical asymptotes where $\sin(x) = 0$*. The graph is shown below for the periodic zoomed out version.*

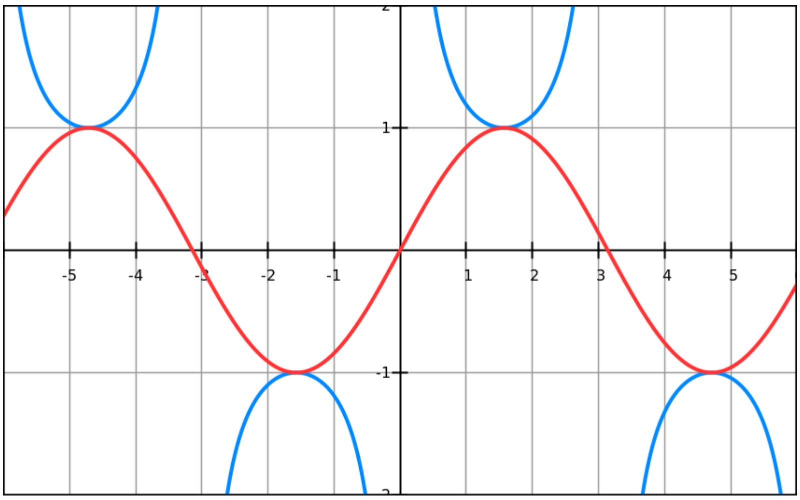

Figure 2.7: Graph of $f(x) = \csc(x)$ in blue. Graph of $f(x) = \sin(x)$ in red.

Definition 2.11.5 *We can define the* ***secant*** *function using* $\cos(x)$*, since we know the function values of this trig functions already:*

$$f(x) = \sec(x) = \frac{1}{\cos(x)}$$

So we will have vertical asymptotes where $\cos(x) = 0$*. The graph is shown below for the periodic zoomed out version.*

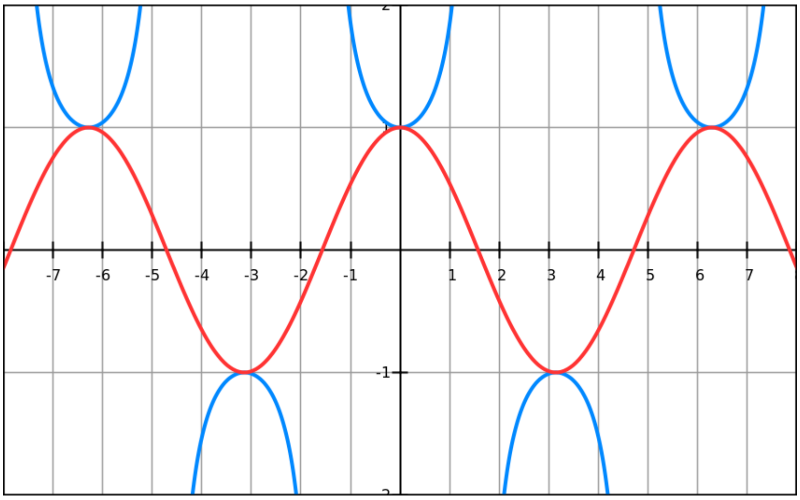

Figure 2.8: Graph of $f(x) = \sec(x)$ in blue. Graph of $f(x) = \cos(x)$ in red.

2.12 Inverse Trig Functions; Solving Inverse Trig Equations

Definition 2.12.1 *We define **inverse sine** function to be*

$$y = \sin^{-1}(x) \text{ means } x = \sin(y)$$
$$\text{for } -1 \leq x \leq 1 \text{ and } -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

*What $y = \sin^{-1}(x)$ means is that "y is the angle whose sine is equal to x." Or in other words, what angle y gives you the value of x. **NOTE:** $y = \sin^{-1}(x)$ does **NOT** mean $y = \frac{1}{sin(x)}$! We know that this function is cosecant. The -1 is purely notation, if you like, you can write the expression as $y = \sin^{-1}(x) = \arcsin(x)$, which is another name for the inverse sine function.*

Using the notion of "what angle y gives you the value x" in the formula $y = \sin^{-1}(x)$, we can figure out the values from the table below of $\sin(\theta)$:

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin(\theta)$	-1	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1

Note that for $f^{-1}(x) = \sin^{-1}(x)$, the domain is $[-1, 1]$ and the range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, reverse of $\sin(x)$. Try to do the following exercises without the chart, using the method: "what angle y gives you the value x."

Example 79 *Find the exact value of* $\sin^{-1}(-\frac{1}{2})$.

Example 80 *Find the exact value of* $\sin^{-1}(-1)$.

Example 81 *Find the exact value of* $\sin^{-1}(\frac{\sqrt{3}}{2})$.

Example 82 *Find the exact value of* $\sin^{-1}(\frac{\sqrt{2}}{2})$.

Sometimes we can use the fact that $\sin(x)$ and $\sin^{-1}(x)$ are inverses directly, but you **MUST** be careful! The values need to be in the domain and range of $\sin^{-1}$ to be well defined to be able to apply the following rules.

Definition 2.12.2 *For values x in the domain and range of $\sin^{-1}$ to cancel the $\sin$ and $\sin^{-1}$:*

$$f^{-1}(f(x)) = \sin^{-1}(\sin(x)) = x \text{ for } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$
$$f(f^{-1}(x)) = \sin(\sin^{-1}(x)) = x \text{ for } -1 \leq x \leq 1$$

Here are some examples where you may or may not be able to use this definition. Always check to see if the values are in the above intervals!

Example 83 *Find the exact value of $\sin^{-1}(\sin(\frac{\pi}{8}))$.*

Example 84 *Find the exact value of $\sin^{-1}(\sin(\frac{5\pi}{8}))$.*

Example 85 *Find the exact value of $\sin(\sin^{-1}(\frac{1}{2}))$.*

Example 86 *Find the exact value of $\sin(\sin^{-1}(1.8))$.*

Definition 2.12.3 *We define **inverse cosine** function to be*

$$y = \cos^{-1}(x) \text{ means } x = \cos(y)$$
$$\text{for } -1 \leq x \leq 1 \text{ and } 0 \leq y \leq \pi$$

What $y = \cos^{-1}(x)$ means is that "y is the angle whose cosine is equal to x." Or in other words, what angle y gives you the value of x. ***NOTE:*** *$y = \cos^{-1}(x)$ does **NOT** mean $y = \frac{1}{\cos(x)}$! We know that this function is secant. The -1 is purely notation, if you like, you can write the expression as $y = \cos^{-1}(x) = \arccos(x)$, which is another name for the inverse cosine function.*

Using the notion of "what angle y gives you the value x" in the formula $y = \cos^{-1}(x)$, we can figure out the values from the table below of $\cos(\theta)$:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\cos(\theta)$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Note that for $\sin^{-1}(x)$, the domain is $[-1, 1]$ and the range is $[0, \pi]$. Try to do the following exercises without the chart, using the method: "what angle y gives you the value x."

Example 87 *Find the exact value of* $\cos^{-1}(-\frac{1}{2})$.

Example 88 *Find the exact value of* $\cos^{-1}(0)$.

Example 89 *Find the exact value of* $\cos^{-1}(\frac{\sqrt{3}}{2})$.

Example 90 *Find the exact value of* $\cos^{-1}(\frac{\sqrt{2}}{2})$.

Sometimes we can use the fact that $\cos(x)$ and $\cos^{-1}(x)$ are inverses directly, but you **MUST** be careful! The values need to be in the domain and range of $\cos^{-1}$ to be well defined to be able to apply the following rules.

Definition 2.12.4 *For values x in the domain and range of* $\cos^{-1}$ *to cancel the* $\cos$ *and* $\cos^{-1}$*:*

$$f^{-1}(f(x)) = \cos^{-1}(\cos(x)) = x \text{ for } 0 \leq x \leq \pi$$
$$f(f^{-1}(x)) = \cos(\cos^{-1}(x)) = x \text{ for } -1 \leq x \leq 1$$

Here are some examples where you may or may not be able to use this definition. Always check to see if the values are in the above intervals!

Example 91 *Find the exact value of* $\cos^{-1}(\cos(\frac{\pi}{12}))$.

Example 92 *Find the exact value of* $\cos(\cos^{-1}(-0.4))$.

Example 93 *Find the exact value of* $\cos^{-1}(\cos(-\frac{2\pi}{3}))$.

Example 94 *Find the exact value of* $\cos(\cos^{-1}(\pi))$.

Definition 2.12.5 *We define **inverse tangent** function to be*

$$y = \tan^{-1}(x) \text{ means } x = \tan(y)$$
$$\text{for } -1 \le x \le 1 \text{ and } 0 \le y \le \pi$$

*What $y = \tan^{-1}(x)$ means is that "y is the angle whose tangent is equal to x." Or in other words, what angle y gives you the value of x. **NOTE:** $y = \tan^{-1}(x)$ does **NOT** mean $y = \frac{1}{tan(x)}$! We know that this function is cotangent. The -1 is purely notation, if you like, you can write the expression as $y = \tan^{-1}(x) = \arctan(x)$, which is another name for the inverse tangent function.*

Using the notion of "what angle y gives you the value x" in the formula $y = \tan^{-1}(x)$, we can figure out the values from the table below of $\tan(\theta)$:

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan(\theta)$	undefined	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undefined

Note that the domain is $(-\infty, \infty)$ and the range is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Try to do the following exercises without the chart, using the method: "what angle y gives you the value x."

Example 95 *Find the exact value of* $\tan^{-1}(-\sqrt{3})$.

Example 96 *Find the exact value of* $\tan^{-1}(1)$.

Example 97 *Find the exact value of* $\tan^{-1}(\frac{\sqrt{3}}{3})$.

Example 98 *Find the exact value of* $\tan^{-1}(0)$.

Sometimes we can use the fact that $\tan(x)$ and $\tan^{-1}(x)$ are inverses directly, but you **MUST** be careful! The values need to be in the domain and range of $\tan^{-1}$ to be well defined to be able to apply the following rules.

Definition 2.12.6 *For values x in the domain and range of* $\tan^{-1}$ *to cancel the* tan *and* $\tan^{-1}$*:*

$$f^{-1}(f(x)) = \tan^{-1}(\tan(x)) = x \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$f(f^{-1}(x)) = \tan(\tan^{-1}(x)) = x \text{ for } -\infty < x < \infty$$

Here are some examples where you may or may not be able to use this definition. Always check to see if the values are in the above intervals!

Example 99 *Find the exact value of* $\tan^{-1}(\tan(-\frac{3\pi}{8}))$.

Example 100 *Find the exact value of* $\tan(\tan^{-1}(4))$.

Example 101 *Find the exact value of* $\tan^{-1}(\tan(-\frac{2\pi}{3}))$.

Example 102 *Find the exact value of* $\tan(\tan^{-1}(\pi))$.

Here is a chart to summarize the above information. This is how to use the chart. When encountering problems of the form:

$$\begin{array}{ll} \textbf{(A)} & \sin^{-1}(\sin(\theta)) = \phi \\ \textbf{(B)} & \sin(\sin^{-1}(x)) = y \end{array}$$

Look at the function on the outside, then reference the table. For the **(A)**, look at the first three rows, if θ is in the interval in the second column, then you can just cancel the trig functions and $\phi = \theta$. For the **(B)**, look at the last three rows, if x is in the interval in the second column, then you can just cancel the trig functions and $y = x$. For case **(A)**, if the value θ is not in the interval, you must do more work and find the correct angle. The answer will **NOT** be the one in parentheses. For case **(B)**, if the value x is not in the interval, the value will not be defined. The answer will **NOT** be the one in parentheses.

Trig Function on the outside	Interval
$\sin^{-1}$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$
$\cos^{-1}$	$[0, \pi]$
$\tan^{-1}$	$(-\frac{\pi}{2}, \frac{\pi}{2})$
sin	$[-1, 1]$
cos	$[-1, 1]$
tan	$(-\infty, \infty)$

Here I will do one worked example of each of the two last topics. The first is finding the inverse function of a trigonometric function. The second is solving an equation involving an inverse trigonometric function.

Example 103 *Find the inverse function f^{-1} of $f(x) = 2\sin(x) - 1$ on $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. Find the range of f and the domain and range of f^{-1}.*

***Solution*:** Using our previous knowledge, we switch x and y and solve for y.

$$\begin{aligned} y &= 2\sin(x) - 1 \\ x &= 2\sin(y) - 1 \\ x + 1 &= 2\sin(y) \\ \frac{x+1}{2} &= \sin(y) \\ y &= \sin^{-1}\left(\frac{x+1}{2}\right) \end{aligned}$$

So the inverse function is $f^{-1}(x) = \sin^{-1}\left(\frac{x+1}{2}\right)$. Now we have to find the range of f. To do this, we solve $y = 2\sin(x) - 1$ for $\sin(x)$. So we have that $\sin(x) = \frac{y+1}{2}$. We know that $-1 \le \sin(x) \le 1$, so then we must have the following

$$\begin{aligned} &-1 \le \frac{y+1}{2} \le 1 \\ &-2 \le y + 1 \le 2 \\ &-3 \le y \le 1 \end{aligned}$$

So in interval notation, the range is $[-3, 1]$. Therefore, for f^{-1}, we switch the domain and the range. So then we have that the domain for f^{-1} is $[-3, 1]$ and the range is $[-\frac{\pi}{2}, \frac{\pi}{2}]$. □

Example 104 *Solve the equation:* $3\sin^{-1}(x) = \pi$.

***Solution*:** To solve this type of question, isolate the trig function and use the definition from above:

$$\begin{aligned} 3\sin^{-1}(x) &= \pi \\ \sin^{-1}(x) &= \frac{\pi}{3} \\ x &= \sin(\frac{\pi}{3}) \\ x &= \frac{\sqrt{3}}{2} \end{aligned}$$

□

Here are some examples to practice of these two ideas.

Example 105 *Find the inverse function* f^{-1} *of* $f(x) = 5\sin(x) + 2$ *on* $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$. *Find the range of* f *and the domain and range of* f^{-1}.

Example 106 *Find the inverse function* f^{-1} *of* $f(x) = -2\cos(x)$ *on* $0 \le x \le \frac{\pi}{3}$. *Find the range of* f *and the domain and range of* f^{-1}.

Example 107 *Find the inverse function* f^{-1} *of* $f(x) = 3\sin(2x)$ *on* $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$. *Find the range of* f *and the domain and range of* f^{-1}.

Example 108 *Solve the equation:* $-6\sin^{-1}(x) = \pi$.

Example 109 *Solve the equation:* $3\tan^{-1}(x) = \pi$.

Example 110 *Solve the equation:* $3\cos^{-1}(2x) = 2\pi$.

2.13 Vectors; Dot Product

2.13.1 Vectors

Definition 2.13.1 *A* ***vector*** *is a "quantity" that has both* ***magnitude*** *and* ***direction****. We will use the notion of an arrow with some length (length defines the magnitude) pointing in a certain direction to represent a vector.*

Definition 2.13.2 *Vectors have the following properties:*
Commutative: $\boldsymbol{v} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{v}$
Associative: $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{u} + \boldsymbol{v}) + \boldsymbol{w}$
Zero Vector: $\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{v} = \boldsymbol{v}$
Additive Inverse: $\boldsymbol{v} + (-\boldsymbol{v}) = \boldsymbol{0}$

Definition 2.13.3 *If α is a scalar (in our case, a real number), and $\boldsymbol{v}$ is a vector, then a* ***scalar multiple****, which is $\alpha\boldsymbol{v}$, is defined different depending on α.*

(a) If $\alpha > 0$, then the vector $\boldsymbol{v}$ changes in magnitude by α and the vector is pointing in the same direction. For magnitude: If $0 < \alpha < 1$, the vector becomes shorter. If $\alpha = 1$, the vector remains the same length. If $\alpha > 1$, the vector becomes longer.
(b) If $\alpha < 0$, then the vector $\boldsymbol{v}$ changes in magnitude by α and the vector is pointing in the opposite direction. For magnitude: If $0 > \alpha > -1$, the vector becomes shorter. If $\alpha = -1$, the vector remains the same length. If $\alpha < -1$, the vector becomes longer.
(c) If $\alpha = 0$, or $\boldsymbol{v} = \boldsymbol{0}$, then $\alpha\boldsymbol{v} = 0$.

Definition 2.13.4 *Scalar multiples have the following properties:*

$$\begin{cases} 0\boldsymbol{v} = \boldsymbol{0} \\ 1\boldsymbol{v} = \boldsymbol{v} \\ -1\boldsymbol{v} = -\boldsymbol{v} \\ (\alpha + \beta)\boldsymbol{v} = \alpha\boldsymbol{v} + \beta\boldsymbol{v} \\ \alpha(\boldsymbol{v} + \boldsymbol{w}) = \alpha\boldsymbol{v} + \alpha\boldsymbol{w} \\ \alpha(\beta\boldsymbol{v}) = (\alpha\beta)\boldsymbol{v} \end{cases}$$

Definition 2.13.5 *The symbol $||\boldsymbol{v}||$ denotes the magnitude of a vector $\boldsymbol{v}$. It has the following properties:*

$$\begin{cases} ||\boldsymbol{v}|| \geq 0 \\ ||\boldsymbol{v}|| = 0 \text{ if and only if } \boldsymbol{v} = 0 \\ ||-\boldsymbol{v}|| = ||\boldsymbol{v}|| \\ ||\alpha\boldsymbol{v}|| = |\alpha|||\boldsymbol{v}|| \end{cases}$$

Definition 2.13.6 *An* ***algebraic vector****, say $\boldsymbol{v}$, is given by $\boldsymbol{v} = \langle a, b\rangle$, where a and b are points in the plane. If $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ and the vector points from P_1 to P_2, then $\boldsymbol{v} = \langle x_2 - x_1, y_2 - y_1\rangle$.*

NOTE*: If we have $\langle a, b\rangle$, then to draw this vector, start at the origin. Then go a units in the x direction and b units in the y direction. Then connect the origin to that point with an arrow.*

Definition 2.13.7 *Here are some properties for vector addition (*$\boldsymbol{i}$ *and* $\boldsymbol{j}$ *represent the* i^{th} *and* j^{th} *components, or* x *and* y *components). Let* $\boldsymbol{v} = a_1\boldsymbol{i} + b_1\boldsymbol{j} = \langle a_1, b_1\rangle$ *and* $\boldsymbol{w} = a_2\boldsymbol{i} + b_2\boldsymbol{j} = \langle a_2, b_2\rangle$*, and* α *be a scalar. Then we have*

$$\begin{cases} \boldsymbol{v} + \boldsymbol{w} = (a_1 + a_2)\boldsymbol{i} + (b_1 + b_2)\boldsymbol{j} = \langle a_1 + a_2, b_1 + b_2\rangle \\ \boldsymbol{v} - \boldsymbol{w} = (a_1 - a_2)\boldsymbol{i} + (b_1 - b_2)\boldsymbol{j} = \langle a_1 - a_2, b_1 - b_2\rangle \\ \alpha\boldsymbol{v} = (\alpha a_1)\boldsymbol{i} + (\alpha b_1)\boldsymbol{j} = \langle \alpha a_1, \alpha b_1\rangle \\ ||\boldsymbol{v}|| = \sqrt{a_1^2 + b_1^2} \end{cases}$$

Example 111 *If* $\boldsymbol{v} = \langle 2, 3\rangle$ *and* $\boldsymbol{w} = \langle 3, -4\rangle$*. Find* ***v****+****w****,* ***v*** *-* ***w****,* $||\boldsymbol{v}||$*, and* $||\boldsymbol{w}||$*.*

Example 112 *If* $\boldsymbol{v} = \langle 3, 4\rangle$ *and* $\boldsymbol{w} = \langle -1, 3\rangle$*. Find* ***v****+****w****,* ***v*** *-* ***w****,* $||\boldsymbol{v}||$*, and* $||\boldsymbol{w}||$*.*

Definition 2.13.8 *The* ***unit vector*** *is defined to be*

$$\boldsymbol{v} = \frac{\boldsymbol{v}}{||\boldsymbol{v}||}$$

To find the magnitude of a vector $\mathbf{v} = \langle a, b\rangle$, all that is required is to compute $||\mathbf{v}||$. The direction, θ, which is an angle in the plane, can be found by using the formula

$$\tan(\theta) = \frac{b}{a}$$

NOTE: When you get to the step of $\tan(\theta) = z$, where z is a number, ask yourself which value of θ in $[0, 2\pi)$ gives you the value of z. I think this is the easiest way to understand this. Use the unit circle and draw the picture of the graph. The angle you get should make sense with the picture you draw.

Example 113 *Find the direction* θ *of* $\boldsymbol{v} = \langle 4, -4\rangle$*.*

Example 114 *Find the direction* θ *of* $\boldsymbol{v} = \langle 1, \sqrt{3}\rangle$*.*

Example 115 *Find the direction* θ *of* $\boldsymbol{v} = \langle 3, 3\rangle$*.*

Example 116 *Find the direction* θ *of* $\boldsymbol{v} = \langle -3\sqrt{3}, 3\rangle$*.*

Definition 2.13.9 *Let* $\boldsymbol{v}$ *be a vector. If we are given the magnitude of* $\boldsymbol{v}$ *(this is* $||\boldsymbol{v}||$*) and a direction angle* θ*, then we can find what the vector looks like using the following formula*

$$\boldsymbol{v} = ||\boldsymbol{v}||\langle \cos(\theta), \sin(\theta)\rangle$$

Example 117 *If the magnitude of a vector* $\boldsymbol{v}$ *is 10 and has direction angle* $\theta = \frac{\pi}{3}$*, write the vector* $\boldsymbol{v}$ *in component form.*

Example 118 *If the magnitude of a vector* $\boldsymbol{v}$ *is 4 and has direction angle* $\theta = \frac{2\pi}{3}$*, write the vector* $\boldsymbol{v}$ *in component form.*

Example 119 *If the magnitude of a vector* $\boldsymbol{v}$ *is 8 and has direction angle* $\theta = \frac{5\pi}{6}$*, write the vector* $\boldsymbol{v}$ *in component form.*

Example 120 *If the magnitude of a vector* $\boldsymbol{v}$ *is 5 and has direction angle* $\theta = \frac{3\pi}{4}$*, write the vector* $\boldsymbol{v}$ *in component form.*

2.13.2 Dot Product

Definition 2.13.10 *Let* $\boldsymbol{v} = a_1\boldsymbol{i} + b_1\boldsymbol{j}$ *and* $\boldsymbol{w} = a_2\boldsymbol{i} + b_2\boldsymbol{j}$*. Then the* ***dot product*** *is defined by* $\boldsymbol{v}\cdot\boldsymbol{w}$ *is*

$$\boldsymbol{v} \cdot \boldsymbol{w} = a_1a_2 + b_1b_2$$

Here are some properties for the dot product

Definition 2.13.11

$$\begin{cases} \boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{v} \\ \boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w} \\ \boldsymbol{u} \cdot \boldsymbol{u} = ||\boldsymbol{u}||^2 \\ \boldsymbol{0} \cdot \boldsymbol{u} = 0 \end{cases}$$

Definition 2.13.12 ***(Angle between vectors)*** *- If* $\boldsymbol{u}$ *and* $\boldsymbol{v}$ *are non-zero vectors, the angle* θ*, such that* $0 \leq \theta \leq \pi$*, between the vectors is defined to be*

$$\cos(\theta) = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{||\boldsymbol{u}||||\boldsymbol{v}||}$$

Definition 2.13.13 *If the angle* θ *between two vectors is* 0 *using the above formula, then the vectors are* ***parallel****. If the angle* θ *between two vectors is* $\frac{\pi}{2}$ *using the above formula, then the vectors are* ***perpendicular*** *or* ***orthogonal****.*

Example 121 *Find the dot product of* $\boldsymbol{u}\cdot\boldsymbol{v}$ *and the angle between the vectors for* $\boldsymbol{u} = \boldsymbol{i} - \boldsymbol{j}$ *and* $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j}$*.*

Example 122 *Find the dot product of* $\boldsymbol{u}\cdot\boldsymbol{v}$ *and the angle between the vectors for* $\boldsymbol{u} = \boldsymbol{i} - \boldsymbol{j}$ *and* $\boldsymbol{v} = -\boldsymbol{i} + \boldsymbol{j}$*.*

Example 123 *Find the dot product of* $\boldsymbol{u}\cdot\boldsymbol{v}$ *and the angle between the vectors for* $\boldsymbol{u} = 2\boldsymbol{i} - \boldsymbol{j}$ *and* $\boldsymbol{v} = \boldsymbol{i} - 2\boldsymbol{j}$*.*

Example 124 *Find the dot product of* $\boldsymbol{u}\cdot\boldsymbol{v}$ *and the angle between the vectors for* $\boldsymbol{u} = \sqrt{3}\boldsymbol{i} - \boldsymbol{j}$ *and* $\boldsymbol{v} = \boldsymbol{i} + \boldsymbol{j}$*.*

2.14 Partial Fraction Decomposition

The process of partial fraction decomposition is used to break down a complex fraction expression into a sum of fractions. For example we will learn how to break down an expression like the following

$$\frac{5x-1}{x^2+x-12} = \frac{3}{x+4} + \frac{2}{x-3}$$

Each fraction has the form of $\frac{P}{Q}$. There are 4 different cases depending on what the polynomial Q looks like. The polynomial Q can have (a) Non-repeated Linear Factors, (b) Repeated Linear Factors, (c) Non-repeated Irreducible Quadratic Factor, and (d) Repeated Irreducible Quadratic Factor. We will consider these cases individually. Below there is one worked example of each kind of problem and each section has some example problems that you can work out.

2.14.1 Non-repeated Linear Factors

If Q has only non-repeated linear factors, then Q looks like

$$Q(x) = (x-a_1)(x-a_2)...(x-a_n)$$

where $a_1, a_2, ..., a_n$ are roots of the polynomial Q. Then we can break up the fraction as

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ... + \frac{A_n}{x-a_n}$$

where A_1, A_2, ... , A_n are expressions we need to find.

Example 125 *Find the partial fraction decomposition of*

$$\frac{x}{x^2-5x+6}$$

Solution: *First we have to decompose the denominator into pieces:*

$$x^2-5x+6 = (x-2)(x-3)$$

Then we rewrite the expression in the form above and find a common denominator

$$\frac{x}{x^2-5x+6} = \frac{A}{x-2} + \frac{B}{x-3}$$
$$\frac{x}{x^2-5x+6} = \frac{A(x-3)+B(x-2)}{(x-2)(x-3)}$$

Now we can set the numerators equal to get

$$x = A(x-3) + B(x-2)$$

The easiest way to solve for A and B is by plugging "special" numbers into the above equation. Notice that if we plug in $x = 3$ and $x = 2$, we easily get the values of A and B. Here is the work:

$$\begin{aligned} x &= A(x-3) + B(x-2) \\ 3 &= A(3-3) + B(3-2) \\ 3 &= B \end{aligned}$$

and

$$\begin{aligned} x &= A(x-3) + B(x-2) \\ 2 &= A(2-3) + B(2-2) \\ 2 &= -A \\ -2 &= A \end{aligned}$$

So then we have the final expression

$$\frac{x}{x^2 - 5x + 6} = \frac{-2}{x-2} + \frac{3}{x-3}$$

□

Example 126 *Find the partial fraction decomposition of*

$$\frac{3x}{(x+2)(x-1)}$$

Example 127 *Find the partial fraction decomposition of*

$$\frac{x}{(x-2)(x-1)}$$

Example 128 *Find the partial fraction decomposition of*

$$\frac{3x}{(x+2)(x-4)}$$

Example 129 *Find the partial fraction decomposition of*

$$\frac{4}{2x^2 - 5x - 3}$$

2.14.2 Repeated Linear Factors

If Q has a repeated linear factor, then Q has some term of the form

$$(x-a)^n$$

where a is a root and $n \geq 2$. Then we can break up the fraction as

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + ... + \frac{A_n}{(x-a)^n}$$

where A_1, A_2, ... , A_n are expressions we need to find.

Example 130 *Find the partial fraction decomposition of*

$$\frac{x+2}{x^3 - 2x^2 + x}$$

***Solution*:** *First we have to decompose the denominator into pieces:*

$$x^3 - 2x^2 + x = x(x-1)^2$$

Then we rewrite the expression in the form above along with the first term that comes from the case we already did. Then we find a common denominator and get

$$\begin{aligned} \frac{x+2}{x^3 - 2x^2 + x} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\ \frac{x+2}{x^3 - 2x^2 + x} &= \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2} \end{aligned}$$

Now we can set the numerators equal to get

$$x + 2 = A(x-1)^2 + Bx(x-1) + Cx$$

The easiest way to solve for A, B, and C is by plugging "special" numbers into the above equation as before. Notice that if we plug in $x = 0$ and $x = 1$, we easily get the values of A and C. Here is the work:

$$\begin{aligned} x + 2 &= A(x-1)^2 + Bx(x-1) + Cx \\ 0 + 2 &= A(0-1)^2 + B(0)(0-1) + C(0) \\ 2 &= A \end{aligned}$$

and

$$\begin{aligned} x + 2 &= A(x-1)^2 + Bx(x-1) + Cx \\ 1 + 2 &= A(1-1)^2 + B(1)(1-1) + C(1) \\ 3 &= C \end{aligned}$$

Now so far we have

$$x + 2 = 2(x-1)^2 + Bx(x-1) + 3x$$

At this point, we can plug in ANY value for x (except 0 and 1 of course) and solve for B. We choose 2, so then

$$\begin{aligned} x + 2 &= 2(x-1)^2 + Bx(x-1) + 3x \\ 2 + 2 &= 2(2-1)^2 + B(2)(2-1) + 3(2) \\ 4 &= 8 + 2B \\ -2 &= B \end{aligned}$$

So then we have the final expression

$$\frac{x+2}{x^3 - 2x^2 + x} = \frac{2}{x} + \frac{-2}{x-1} + \frac{3}{(x-1)^2}$$

□

Example 131 *Find the partial fraction decomposition of*

$$\frac{x+1}{x^2(x-2)}$$

Example 132 *Find the partial fraction decomposition of*

$$\frac{x-3}{(x+2)(x+1)^2}$$

Example 133 *Find the partial fraction decomposition of*

$$\frac{x^2+x}{(x+2)(x-1)^2}$$

Example 134 *Find the partial fraction decomposition of*

$$\frac{x^2}{(x-1)^2(x+1)^2}$$

2.14.3 Irreducible Quadratic Factor

If Q has an irreducible quadratic factor, then Q has some term of the form ax^2+bx+c that we cannot break down any further. In this case we attempt to use the form

$$\frac{Ax+B}{ax^2+bx+c}$$

where A and B are expressions we need to find.

Example 135 *Find the partial fraction decomposition of*

$$\frac{3x-5}{x^3-1}$$

***Solution*:** *As before, the first step is to reduce the denominator as much as we can. This means*

$$x^3-1=(x-1)(x^2+x+1)$$

We can find this easily by noticing that $x=1$ is a root of x^3-1, so we can do synthetic division to get the other polynomial. The quadratic term above is irreducible, so we cannot break this up any further. Now we can use the definition above to write out the expression

$$\frac{3x-5}{x^3-1}=\frac{A}{x-1}+\frac{Bx+C}{x^2+x+1}$$

where the first term comes from Case 1.1 above and the second part from the definition we defined here in 1.3. Then we follow the same procedure as above to get

$$3x-5=A(x^2+x+1)+(Bx+C)(x-1)$$

by finding the common denominator and setting the numerators equal. Here, we can also be clever in choosing values of x to plug in to easily solve for the A, B, and C. Notice that if we pick $x=1$, we get

$$\begin{aligned}3x-5&=A(x^2+x+1)+(Bx+C)(x-1)\\3(1)-5&=A(1^2+1+1)+(B(1)+C)(1-1)\\-2&=3A\\A&=-\frac{2}{3}\end{aligned}$$

Now we have a value for A, so we have

$$3x-5=-\frac{2}{3}(x^2+x+1)+(Bx+C)(x-1)$$

Now notice that if we choose $x=0$, we get

$$\begin{aligned}
3x-5 &= -\frac{2}{3}(x^2+x+1)+(Bx+C)(x-1)\\
3(0)-5 &= -\frac{2}{3}(0^2+0+1)+(B(0)+C)(0-1)\\
-5 &= -\frac{2}{3}-C\\
C &= 5-\frac{2}{3}\\
C &= \frac{13}{3}
\end{aligned}$$

So far our expression is

$$3x-5=-\frac{2}{3}(x^2+x+1)+(Bx+\frac{13}{3})(x-1)$$

Now we have reached the point where we can plug in ANY value for x (not the ones we already chose) to solve for B. So let's pick $x=2$. Then we get

$$\begin{aligned}
3x-5 &= -\frac{2}{3}(x^2+x+1)+(Bx+\frac{13}{3})(x-1)\\
3(2)-5 &= -\frac{2}{3}(2^2+2+1)+(B(2)+\frac{13}{3})(2-1)\\
1 &= -\frac{2}{3}(7)+(2B+\frac{13}{3})(1)\\
1 &= -\frac{14}{3}+2B+\frac{13}{3}\\
1 &= -\frac{1}{3}+2B\\
2B &= \frac{4}{3}\\
B &= \frac{2}{3}
\end{aligned}$$

So now we have all the values, and our final solution has the form

$$\frac{3x-5}{x^3-1}=\frac{-\frac{2}{3}}{x-1}+\frac{\frac{2}{3}x+\frac{13}{3}}{x^2+x+1}$$

□

Example 136 *Find the partial fraction decomposition of*

$$\frac{1}{x^2+1}$$

Example 137 *Find the partial fraction decomposition of*

$$\frac{1}{(x+1)(x^2+4)}$$

Example 138 *Find the partial fraction decomposition of*

$$\frac{x+4}{x^2(x^2+4)}$$

Example 139 *Find the partial fraction decomposition of*

$$\frac{x^2-11x-18}{x(x^2+3x+3)}$$

2.14.4 Repeated Irreducible Quadratic Factor

If Q has a repeated irreducible quadratic factor, then Q has some term of the form

$$(ax^2+bx+c)^n$$

and $n \geq 2$. Then we attempt to use the form

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+...+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

where A_1, A_2, ... , A_n and B_1, B_2, ... , B_n are expressions we need to find.

Example 140 *Find the partial fraction decomposition of*

$$\frac{x^3+x^2}{(x^2+4)^2}$$

Solution: *Using the above definition to rewrite our expression, we get*

$$\frac{x^3+x^2}{(x^2+4)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{(x^2+4)^2}$$

Finding a common denominator, and setting the numerator equal, we get

$$x^3+x^2=(Ax+B)(x^2+4)+Cx+D$$
$$x^3+x^2=Ax^3+Bx^2+4Ax+4B+Cx+D$$
$$x^3+x^2=Ax^3+Bx^2+(4A+C)x+D$$

Now we set the coefficients on the left and right hand sides equal. By that we mean, the coefficients of the x^3 terms must be the same, the coefficients of the x^2 terms must be the same, and so on. Then we get the system

$$\begin{cases} A=1 \\ B=1 \\ 4A+C=0 \\ 4B+D=0 \end{cases}$$

Since we already know what A and B are, we can use them to solve for C and D using equations 3 and 4 above. So now we just have to solve

$$\begin{cases} 4 + C = 0 \\ 4 + D = 0 \end{cases}$$

From this, it is clear that $C = -4$ and $D = -4$. So then our final solution is

$$\frac{x^3 + x^2}{(x^2+4)^2} = \frac{x+1}{x^2+4} + \frac{-4x-4}{(x^2+4)^2}$$

□

Example 141 *Find the partial fraction decomposition of*

$$\frac{x^2 + 2x + 3}{(x^2+4)^2}$$

Example 142 *Find the partial fraction decomposition of*

$$\frac{(x^3 + 1)}{(x^2+16)^2}$$

Example 143 *Find the partial fraction decomposition of*

$$\frac{x^3}{(x^2+16)^3}$$

Example 144 *Find the partial fraction decomposition of*

$$\frac{x^2}{(x^2+4)^3}$$