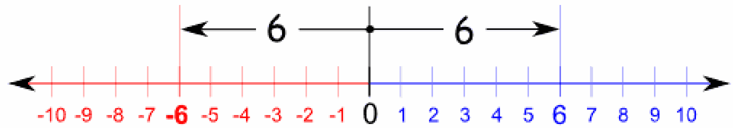
Math 1 Variable Manipulation Part 5
Absolute Value & Inequalities

ABSOLUTE VALUE REVIEW

Absolute value is a measure of distance; how far a number is from zero:



"6 is 6 away from zero, and "−6" is also 6 away from zero. So the absolute value of 6 is 6 and the absolute value of -6 is also 6. So in practice "absolute value" means to remove any negative sign in front of a number, and to think of all numbers as positive (or zero).

To show that we want the absolute value of something, we put "|" marks either side (they are called "bars"),

$|-5| = 5$ $\qquad\qquad$ $|7| = 7$

Sometimes absolute value is also written as "abs()", so abs(−1) = 1 is the same as $|-1| = 1$

When performing mathematical processes inside the absolute value sign, just figure as normal and then take the absolute value of the answer. Note that it doesn't matter in which order we do a subtraction, the absolute value will always be the same:

$|8-3| = 5$ $\qquad\qquad$ $|3-8| = 5$

$8-3 = 5$ $\qquad\qquad$ $3-8 = -5$, and $|-5| = 5$

Example: $|-3\times 6|$
Solution: $-3\times 6 = -18$, and $|-18| = 18$

Example: $-|5-2|$
Solution: Since $5-2 = 3$, multiply 3 by the first minus $= -3$

Example: $-|2-5|$
Solution: Since $2-5 = -3$, and $|-3| = 3$, and then multiply by the first minus $= -3$

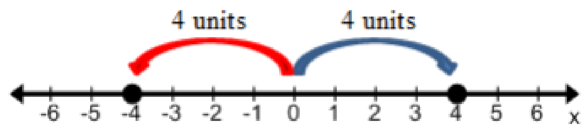
Example: $-|-12|$
Solution: Since $|-12| = 12$ and then multiply by the first minus $= -12$

Sample Questions:

1. $-|3-6|$
2. $|5-10|$
3. $|12-6| - |-8|$ is equal to:
4. $-|-24\div 3|$
5. $|-6\text{x}5|$
6. $|2-14| - |-25|$ is equal to:

SOLVING AN ABSOLUTE VALUE EQUATION

Consider the equation $|x| = 4$. This equation is essentially asking for all the numbers that are 4 units from zero. It is easier to examine this graphically:



Clearly the number 4 is 4 units from 0. But the number -4 is also 4 units from 0, only it is in the opposite direction. For absolute value equations both answers are required because both are 4 units from zero. When solving an absolute value equation need to set up and solve two separate equations.

To solve an equation that includes absolute value signs, think about the two different cases-one where what is inside the absolute value sign equals a positive number, and one where it equals a negative number. For example, to solve the equation $|x – 12| = 3$, think of it as two equations:

$x - 12 = 3$ or $x - 12 = -3$ Solving for x, will yield the two solutions: $x = 15$ or 9

Example: Solve the absolute value $|y – 11| - 2 = 0$
Solution: To find the solutions, solve the equation for y. First isolate the absolute value expression by adding 2 to both sides or $|y – 11| = 2$
Then create two different equations and the absolute value sign:
$y – 11 = 2$ and $y – 11 = -2$
$\underline{\ \ +11\ \ +11}$ $\underline{+11\ \ +11}$
$y = 13$ and $y = 9$

Example: Solve the following equation: $|\ 2x – 3\ | – 4 = 3$
Solution: Add 4 to both sides to get $|\ 2x – 3\ | = 7$
Now I'll clear the absolute-value bars by splitting the equation into its two cases, one for each sign:
$(2x – 3) = 7$ or $–(2x – 3) = 7$
$2x – 3 = 7$ or $–2x + 3 = 7$
$2x = 10$ or $–2x = 4$
$x = 5$ or $x = –2$
So the solution is $x = –2, 5$

Remember that the absolute value of any number can never be negative.
Example: How many solutions does the equation have?
$-2 = |q + 2|$
Solution: No solution because inequalities can never be negative

Sample Questions:

7. Solve the following equation: $|2x – 1| = 5$

8. What is the lowest solution for g? $8 = |g – 2|$

9. Solve: $|13 + 2x| = 5$

MORE COMPLEX ABSOLUTE VALUE EQUATIONS

Use various algebraic manipulation in conjunction with absolute value manipulation to solve more complex problems. First get the absolute value by iteself on one side of the equation and the rest on the other side of the equals sign. Then make two equations, one positive and one negative. Then solve both equations to get both possible answers.

Example: $5|3 + 7m| + 1 = 51$
Solution: First, subtract 1 from both sides and then divide by 5 to get
$|3 + 7m| = 50/5 = 10$
Write two equations:
$3 + 7m = 10$ and $3 + 7m = 10$
Subtract three from both sides and then divide by 7 for each of the equations to get
$m = 1$ and $m = -13/7$

Example: For real numbers a and b, when is the equation $|a + b| = |a – b|$ true?

a. Always
b. Only when $a = b$
c. Only when $a = 0$ and $b = 0$
d. Only when $a = 0$ or $b = 0$
e. Never

Solution: The answer to this one is easiest seen by plugging in the options
If always is true, then $a = 1$ and $b = 2$ would be true. However, $|1 + 2| = 3$ and $|1 – 2| = 1$, so not true.
If $a = b$, then $2a = 0$, which is not true.
If a and b are both 0 then, $0 = 0$, which is true, but there could be other options.
If $a = 0$, but not b, then $0 = 0$ and If $b = 0$ but not a, then $0 = 0$. So this is always true.
We already know that c and d are true, so Never cannot be true.
The answer is d.

Sample Questions:

10. $5 + 8|-10n – 2| = 101$

11. $3 - \frac{1}{2}\left|\frac{1}{2}x - 4\right| = 2$

12. If $|x| = x + 12$, then x = ?

a. -12
b. -6
c. 0
d. 6
e. 12

13. $-6|2x – 14| = 42$

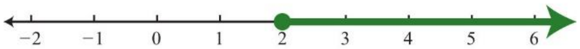
14. $|3(x – 2) + 7| = 6$

15. $\frac{1}{3}|6x + 5| = 7$

INEQUALITY AND INTERVAL NOTATIONS

An algebraic inequality, such as x ≥ 2, is read "x is greater than or equal to 2." This inequality has infinitely many solutions for x. Some of the solutions are 2, 3, 3.5, 5, 20, and 20.001. Since it is impossible to list all of the solutions, a system is needed that allows a clear communication of this infinite set. Two common ways of expressing solutions to an inequality are by graphing them on a number line and using interval notation.
To express the solution graphically, draw a number line and shade in all the values that are solutions to the inequality. Interval notation is textual and uses specific notation as follows:

"*x is greater than or equal to* 2" $x \geq 2$

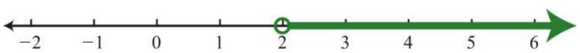


Corresponding interval notation: $[2, \infty)$

Determine the interval notation after graphing the solution set on a number line. The numbers in interval notation should be written in the same order as they appear on the number line, with smaller numbers in the set appearing first. In this example, there is an inclusive inequality, which means that the lower-bound 2 is included in the solution. Denote this with a closed dot on the number line and a square bracket in interval notation. The symbol (∞) is read as infinity and indicates that the set is unbounded to the right on a number line. Interval notation requires a parenthesis to enclose infinity. The square bracket indicates the boundary is included in the solution. The parentheses indicate the boundary is not included. Infinity is an upper bound to the real numbers, but is not itself a real number: it cannot be included in the solution set.

Now compare the interval notation in the previous example to that of the strict, or non-inclusive (does not equal), inequality that follows:

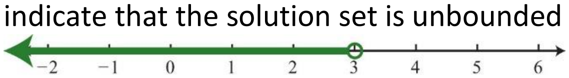
"*x is strictly greater than* 2" $x > 2$



Corresponding interval notation: $(2, \infty)$

Strict inequalities imply that solutions may get very close to the boundary point, in this case 2, but not actually include it. Denote this idea with an open dot on the number line and a round parenthesis in interval notation.

Example: Graph and give the interval notation equivalent: x < 3.
Solution: Use an open dot at 3 and shade all real numbers strictly less than 3. Use negative infinity (−∞) to indicate that the solution set is unbounded to the left on a number line.



Answer: Interval notation: (−∞, 3)

Example: Graph and give the interval notation equivalent: x ≤ 5.
Solution: Use a closed dot and shade all numbers less than and including 5.

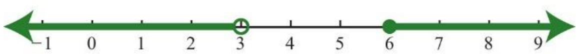
Answer: Interval notation: (−∞, 5]

It is important to see that 5 ≥ x is the same as x ≤ 5. Both require values of x to be smaller than or equal to 5. To avoid confusion, it is good practice to rewrite all inequalities with the variable on the left. Also, when using text, use "inf" as a shortened form of infinity. For example, (−∞, 5] can be expressed textually as (−inf, 5].

COMPOUND INEQUALITIES

A compound inequality is actually two or more inequalities in one statement joined by the word "and" or by the word "or." Compound inequalities with the logical "or" require that either condition must be satisfied. Therefore, the solution set of this type of compound inequality consists of all the elements of the solution sets of each inequality. When we join these individual solution sets it is called the union, denoted ∪. For example, the solutions to the compound inequality x < 3 or x ≥ 6 can be graphed as follows:

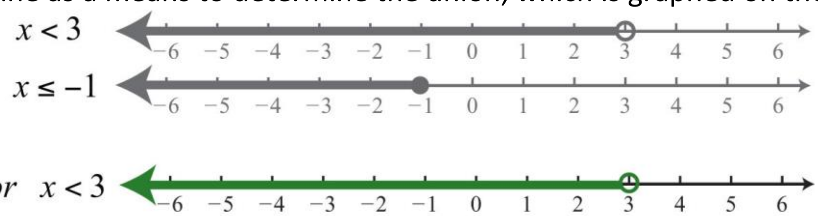
$x < 3 \;\; or \;\; x \geq 6$



Corresponding interval notation: $(-\infty, 3) \cup [6, \infty)$

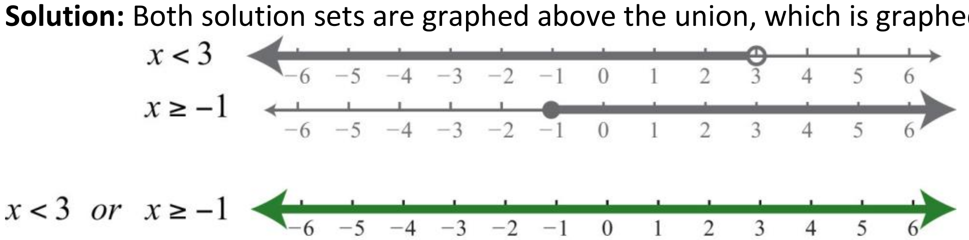
Sometimes we encounter compound inequalities where the separate solution sets overlap. In the case where the compound inequality contains the word "or," we combine all the elements of both sets to create one set containing all the elements of each.

Example: Graph and give the interval notation equivalent: x ≤ −1 or x < 3.
Solution: Combine all solutions of both inequalities. The solutions to each inequality are sketched above the number line as a means to determine the union, which is graphed on the number line below.



Answer: Interval notation: (−∞, 3)
Any real number less than 3 in the shaded region on the number line will satisfy at least one of the two given inequalities.

Example: Graph and give the interval notation equivalent: x<3 or x≥−1.
Solution: Both solution sets are graphed above the union, which is graphed below.

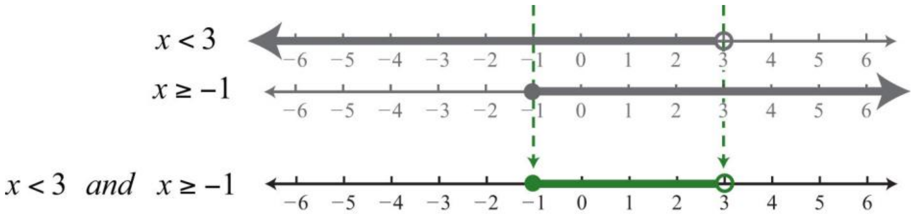


Answer: Interval notation: R = (−∞, ∞)
When you combine both solution sets and form the union, you can see that all real numbers satisfy the original compound inequality.

An inequality such as: $-1 \leq x < 3$ reads "−1 one is less than or equal to x and x is less than three." This is a compound inequality because it can be decomposed as $-1 \leq x \; and \; x < 3$

The logical "and" requires that both conditions must be true. Both inequalities are satisfied by all the elements in the intersection, denoted ∩, of the solution sets of each.

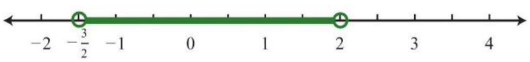
Example: Graph and give the interval notation equivalent: x < 3 and x ≥ −1.
Solution: Determine the intersection, or overlap, of the two solution sets. The solutions to each inequality are sketched above the number line as a means to determine the intersection, which is graphed on the number line below.



Here x = 3 is not a solution because it solves only one of the inequalities.
Answer: Interval notation: [−1, 3)

Alternatively, we may interpret −1 ≤ x < 3 as all possible values for x between or bounded by −1 and 3 on a number line. For example, one such solution is x = 1. Notice that 1 is between −1 and 3 on a number line, or that −1 < 1 < 3. Similarly, we can see that other possible solutions are −1, −0.99, 0, 0.0056, 1.8, and 2.99. Since there are infinitely many real numbers between −1 and 3, we must express the solution graphically and/or with interval notation, in this case [−1, 3).

Example: Graph and give the interval notation equivalent: −32 < x < 2.
Solution: Shade all real numbers bounded by, or strictly between, −32 = −112 and 2.



Answer: Interval notation: (−32, 2)

Example: Graph and give the interval notation equivalent: −5 < x ≤ 15.
Solution: Shade all real numbers between −5 and 15, and indicate that the upper bound, 15, is included in the solution set by using a closed dot.

Answer: Interval notation: (−5, 15]

In the previous two examples, we did not decompose the inequalities; instead we chose to think of all real numbers between the two given bounds.

Sample Questions:

16. Graph and give the interval notation equivalent: −3 < x ≤ 10

17. Graph and give the interval notation equivalent: x < 5 and x ≥ −3

18. Graph and give the interval notation equivalent: x < 2 or x ≥ 9

SOLVING AN INEQUALITY

Use the following properties when solving inequalities.

Properties of Inequalities (a, b and c are real numbers)	**Example**
Addition Property of Inequality Adding the same number to each side of an equation produces an equivalent inequality.	If $a > b$ then $a + c > b + c$
Subtraction Property of Inequality Subtracting the same number from each side of an equation produces an equivalent inequality.	If $a < b$ then $a - c > b - c$
Multiplication Property of Inequality Multiplying each side of the inequality by the same *positive* number produces an equivalent inequality. Multiplying each side of the inequality by the same *negative* number produces an equivalent inequality when the inequality sign is *reversed*.	If $a \geq b$ and $c > 0$ then $ac \geq bc$ If $a \geq b$ and $c < 0$ then $ac \leq bc$
Division Property of Inequality Dividing each side of the inequality by the same *positive* number produces an equivalent inequality. Dividing each side of the inequality by the same *negative* number produces an equivalent inequality when the inequality sign is *reversed*.	If $a \leq b$ and $c > 0$ then $\frac{a}{c} \leq \frac{b}{c}$ If $a \leq b$ and $c < 0$ then $\frac{a}{c} \geq \frac{b}{c}$

Solving inequalities works just the same as solving equations--do the same thing to both sides, until the variable you are solving for is isolated--with one exception: When you **multiply or divide both sides by a negative number**, you must **reverse the inequality sign**. This may seem strange, but you can prove that it is true by trying out values. It can also be explained by thinking about the inequality and absolute values. If a number x > 5 then –x < 5 (not greater than). The signs must be switched to make it correct.

Example: 5x - 3 < 7

Solution: Add 3 to both sides to get 5x < 10. Now divide both sides by 5 to get x < 2.

Example: -5x + 7 < -3

Solution: Subtract 7 from both sides to get -5x < -10. Now divide both sides by -5, remembering to reverse the inequality sign: x > 2. Note that when you divide by a negative number, you flip the direction of the inequality sign.

Example: Solve the compound inequality 4x – 3 > 9 or -2x > 2. Write the answer in interval notation.

Solution: Solve each inequality separately.

4x – 3 > 9	or	
4x > 12	-2x > 2	
x > 3	x < -1 (remember to switch direction of inequality)	

Since this is an "or" compound inequality, it is a union of the solutions

(-∞, -1)∪(3, ∞)

Note that x < -1. If we did not switch the direction of the inequality, it would state x > -1. Let's try it out. 0 is a number greater than -1. If we substitute 0 into the equation -2x > 2, we would get that 0 > 2, which is not true. However, the correct answer is x < -1. If we substitute -2 (which is less than -1), into the equation, we get -4 < 2 which is true.

Example: Solve the compound inequality $-4 < x + 1 < 6$. Write the answer in interval notation.
Solution: Write the compound inequality as two inequalities and Solve each inequality separately.

$-4 < x + 1$
$-4 - 1 < x + 1 - 1$
$-5 < x$

$x + 1 < 6$
$x + 1 - 1 < 6 - 1$
$x < 5$

Put the compound inequality back together. This is an "and" compound inequality so it is the intersection of the two solutions.
$-5 < x < 5$ so $(-5, 5)$ in interval notation.

Example: Solve the compound inequality $5v + 10 \le -4v - 17 < 9 - 2v$. Write the answer in interval notation.
Solution: Write the compound inequality as two inequalities and Solve each inequality separately.

$5v + 10 \le -4v - 17$
$5v + 4v \le -17 - 10$
$9v \le -27$
$v \le -3$

$-4v - 17 < 9 - 2v$
$-4v + 2v < 9 + 17$
$-2v < 26$
$v > -13$

Put the compound inequality back together.
$-13 < v \le -3$ so $(-13, -3)$ in interval notation.

Sample Questions:

19. $-3 + 2b < 13$

20. $5x - 7(x + 1) > 9$

21. $3 < 2x + 11 < 17$

22. $8x < 1$ or $x - 9 > -5$

23. $7v - 5 \ge 65$ or $-3v - 2 \ge -2$

24. $5x - (x + 2) < -5(1 + x) + 3$

25. $8x + 8 \ge -64$ and $-7 - 8x \ge -79$

ABSOLUTE VALUE (LESS THAN)

Inequalities can also contain absolute values. There are two forms, less than and greater than with absolute value. In general, if an inequality is in the form: $|a| < b$, then the solution will be in the form: $-b < a < b$ for any argument a.

Example: $|a| < 3$.

Solution: For that inequality to be true, what values could a have?

Geometrically a is less than 3 units from 0.

Therefore, $-3 < a < 3$. The inequality will be true if a has any value between -3 and 3.

Example: For which values of x will this inequality be true?

$|2x - 1| < 5$.

Solution: The argument, $2x - 1$, will fall between -5 and 5:

$-5 < 2x - 1 < 5$.

We must isolate x. First, add 1 to each term of the inequality:

$-5 + 1 < 2x < 5 + 1$

$-4 < 2x < 6$

Now divide each term by 2:

$-2 < x < 3$.

The inequality will be true for any value of x in that interval.

Sample Questions:

26. If $x > |y|$, which of the following is the solution statement for x when $y = -4$?
    a. x is any real number.
    b. $x > 4$
    c. $x < 4$
    d. $-4 < x < 4$
    e. $x > 4$ or $x < -4$

27. Solve this inequality for x: $|x + 2| < 7$

28. Solve this inequality for x: $|3x - 5| < 10$.

29. Solve this inequality for x: $|1 + 2x| < 9$.

ABSOLUTE VALUE (GREATER THAN)

If the inequality is in the form $|a| > b$ (and $b > 0$), then $a > b$ or $a < -b$.

Example: $|a| > 3$

For which values of a will this be true?

Solution: Geometrically, $a > 3$ or $a < -3$.

Sample Questions:

30. Solve for x: $|x| > 5$.

31. For which values of x will this be true? $|x + 2| > 7$.

32. Solve for x: $|2x + 5| > 9$.

33. If $|6 - 2x| > 9$, which of the following is a possible value of x?
    a. -2
    b. -1
    c. 0
    d. 4
    e. 7

34. If $|5 - 2x| > 5$, which of the following is a possible value of x?
    a. 2
    b. 3
    c. 4
    d. 5
    e. 6

OTHER VARIABLE MANIPULATION QUESTIONS WITH INEQUALITIES AND/OR ABSOLUTE VALUE

Many variable manipulation type problems can contain inequalities. Plug in, solve for x, translate into English, fractions, absolute value, etc. Just solve the problems as regular equations with one exception: When you **multiply or divide both sides by a negative number**, remember to **reverse the inequality sign**.

Sample Questions:

35. If 5 times a number x is subtracted from 15, the result is negative. Which of the following gives the possible value(s) for x?
    a. All $x < 3$
    b. All $x > 3$
    c. 10 only
    d. 3 only
    e. 0 only

36. Which of the following inequalities defines the solution set for the inequality $23 - 6x \geq 5$?
    a. $x \geq -3$
    b. $x \geq 3$
    c. $x \geq 6$
    d. $x \leq 3$
    e. $x \leq -6$

37. Which of the following is the set of all real numbers x such that $x - 3 < x - 5$?
    a. The empty set
    b. The set containing only zero
    c. The set containing all nonnegative real numbers
    d. The set containing all negative real numbers
    e. The set containing all real numbers

38. For which values of x will $3(x + 4) \geq 9(4 + x)$?

39. What is the solution set of $|3a - 2| \leq 7$?
    a. $\{a: a \leq 3\}$
    b. $\{a: -\frac{5}{3} \leq a \leq 3\}$
    c. $\{a: -\frac{5}{3} \geq a \geq 3\}$
    d. $\{a: -\frac{5}{3} \leq a \geq 3\}$
    e. $\{a: -\frac{5}{3} \geq a \leq 3\}$

40. Which of the following intervals contains the solution to the equation $x - 2 = \frac{2x+5}{3}$?
    a. $-6 < x < 11$
    b. $11 \leq x < 15$
    c. $6 < x \leq 10$
    d. $-5 < x \leq -3$
    e. $-11 \leq x \leq -2$

41. If 3 times a number x is added to 12, the result is negative. Which of the following gives the possible value(s) for x?
    a. All $x > 4$
    b. All $x < -4$
    c. 36 only
    d. 4 only
    e. 0 only

42. What is the largest integer value of t that satisfies the inequality $\frac{24}{30} > \frac{t}{24}$?

43. How many different integer values of a satisfy the inequality $\frac{1}{11} < \frac{2}{a} < \frac{1}{8}$?

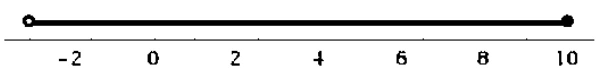
44. The temperature, t, in degrees Fahrenheit, in a certain city on a certain spring day satisfies the inequality $|t - 34| \leq 40$. Which of the following temperatures, in degrees Farenheit, is NOT in this range?
    a. 74
    b. 16
    c. 0
    d. -6
    e. -8

45. It costs *a* dollars for an adult ticket to a reggae concert and s dollars for a student ticket. The difference between the cost of 12 adult tickets and 18 student tickets is \$36. Which of the following equations represents this relationship between *a* and s?
    a. $\frac{12a}{18s} = 36$
    b. $216as = 36$
    c. $|12a - 18s| = 36$
    d. $|12a + 18s| = 36$
    e. $|18a - 12s| = 36$

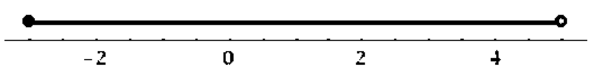
46. What is the smallest possible integer for which 15% of that integer is greater than 2.3?

Answer Key

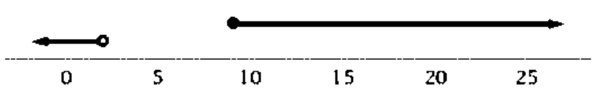
1. -3
2. 5
3. -2
4. -8
5. 30
6. -13
7. 3 and -2
8. 10 and -6 so lowest is -6
9. -4 and -9
10. -7/5 and 1
11. 12 and 4
12. B = -6
13. 7/2 and 21/2
14. 5/3 and -7/3
15. 8/3 and -13/3
16. (-3, 10]



17. [-3, 5)



18. $(-\infty, 2) \cup [9, \infty)$



19. $b < 8$
20. $x < -8$
21. $-4 < x < 3$
22. $x < 1/8$ or $x > 4$
23. $v \geq 10$ or $v > 0$
24. $x < 0$
25. $x \geq -9$ and $x \leq 9$ which is $-9 \leq x \leq 9$
26. $x > 4$
27. $-9 < x < 5$
28. $-5/3 < x < 5$
29. $-5 < x < 4$
30. $x > 5$ or $x < -5$
31. $x > 5$ or $x < -9$
32. $x > 2$ or $x < -7$
33. -2 (A)
34. 6 (E)
35. All $x > 3$ (B)
36. $x \leq 3$ (D)
37. No solution or The empty set (A)
38. $x \leq -4$
39. $\{a: -\frac{5}{3} \leq a \leq 3\}$ (B)
40. $11 \leq x < 15$ (B)
41. All $x < -4$ (B)
42. 19
43. 5 different integers
44. -8 (E)
45. $|12a - 18s| = 36$ (C)
46. 16