Math 104: Improper Integrals (With Solutions)

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Tuesday March 12, 2013

Outline

Improper integrals

Definite integrals $\int_a^b f(x)dx$ were required to have

- finite domain of integration $[a, b]$
- finite integrand $f(x) < \pm\infty$

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Improper integrals

1. Infinite limits of integration
2. Integrals with vertical asymptotes i.e. with infinite discontinuity

Infinite limits of integration

Definition

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- convergent if the limit is finite and that limit is the value of the improper integral.
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Each integral on the previous page is defined as a limit.

If the limit is finite we say the integral **converges**, while if the limit is infinite or does not exist, we say the integral **diverges**.

Convergence is good (means we can do the integral); divergence is bad (means we can't do the integral).

Example 1

Find

$$\int_0^\infty e^{-x}\ dx.$$

(if it even converges)

Solution:

$$\int_0^\infty e^{-x}\ dx = \lim_{b\to\infty}\int_0^b e^{-x}\ dx = \lim_{b\to\infty}\Big[-e^{-x}\Big]_0^b$$
$$= \lim_{b\to\infty} -e^{-b} + e^0 = 0 + 1 = 1.$$

So the integral converges and equals 1.

Example 2

Find

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

(if it even converges)

Solution: By definition,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx = \int_{-\infty}^{c} \frac{1}{1+x^2}\, dx + \int_{c}^{\infty} \frac{1}{1+x^2}\, dx,$$

where we get to pick whatever c we want. Let's pick $c = 0$.

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\, dx = \lim_{b\to-\infty} \Big[\arctan(x)\Big]_b^0 = \lim_{b\to-\infty} [\arctan(0) - \arctan(b)]$$

$$= 0 - \lim_{b\to-\infty} \arctan(b) = \frac{\pi}{2}$$

Example 2, continued

Similarly,

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2}.$$

Therefore,

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Example 3, the p-test

The integral

$$\int_1^\infty \frac{1}{x^p}\,dx$$

1. Converges if $p > 1$;
2. Diverges if $p \leq 1$.

For example:

$$\int_1^\infty \frac{1}{x^{3/2}}\,dx = \lim_{b\to\infty} -\left[\frac{2}{x^{1/2}}\right]_1^b = 2,$$

while

$$\int_1^\infty \frac{1}{x^{1/2}}\,dx = \lim_{b\to\infty}\left[2\sqrt{x}\right]_1^b = \lim_{b\to\infty} 2\sqrt{b} - 2 = \infty,$$

and

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln(x)\right]_1^b = \lim_{b\to\infty} \ln(b) - 0 = \infty.$$

Convergence vs. Divergence

In each case, if the limit exists (or if both limits exist, in case 3!), we say the improper integral **converges**.

If the limit fails to exist or is infinite, the integral **diverges**. In case 3, if either limit fails to exist or is infinite, the integral diverges.

Example 4

Find

$$\int_0^2 \frac{2x}{x^2-4}\,dx.$$

(if it converges)

Solution: The denominator of $\frac{2x}{x^2-4}$ is 0 when $x = 2$, so the function is not even defined when $x = 2$. So

$$\int_0^2 \frac{2x}{x^2-4}\,dx = \lim_{c\to 2-}\int_0^c \frac{2x}{x^2-4}\,dx = \lim_{c\to 2-}\Big[\ln|x^2-4|\Big]_0^c$$
$$= \lim_{c\to 2-}\ln|x^2-4| - \ln(4) = -\infty,$$

so the integral diverges.

Example 5

$$\text{Find} \int_0^3 \frac{1}{(x-1)^{2/3}}\, dx, \quad \text{if it converges.}$$

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$$\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx = \left[3(x-1)^{1/3}\right]_0^3,$$

but this is not okay: The function $f(x) = \frac{1}{(x-1)^{2/3}}$ is undefined when $x = 1$, so we need to split the problem into two integrals.

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\, dx = \int_0^1 \frac{1}{(x-1)^{2/3}}\, dx + \int_1^3 \frac{1}{(x-1)^{2/3}}\, dx.$$

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The two integrals on the right hand side both converge and add up to $3[1+2^{1/3}]$, so $\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3[1+2^{1/3}]$.

Tests for convergence and divergence

The gist:

1. If you're smaller than something that converges, then you converge.
2. If you're bigger than something that diverges, then you diverge.

Theorem

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^\infty f(x)\,dx$ *converges if* $\int_a^\infty g(x)\,dx$ *converges.*
2. $\int_a^\infty g(x)\,dx$ *diverges if* $\int_a^\infty f(x)\,dx$ *diverges.*

Example 6

Which of the following integrals converge?

$$(a)\ \int_1^\infty e^{-x^2}\ dx, \qquad (b)\ \int_1^\infty \frac{\sin^2(x)}{x^2}\ dx.$$

Solution: Both integrals converge.
(a) Note that $0 < e^{-x^2} \leq e^{-x}$ for all $x \geq 1$, and from example 1 we see $\int_1^\infty e^{-x}\ dx = \frac{1}{e}$, so $\int_1^\infty e^{-x^2}\ dx$ converges.

(b) $0 \leq \sin^2(x) \leq 1$ for all x, so

$$0 \leq \frac{\sin^2(x)}{x^2} \leq \frac{1}{x^2}$$

for all $x \geq 1$. Since $\int_1^\infty \frac{1}{x^2}\ dx$ converges (by p-test), so does $\int_1^\infty \frac{\sin^2(x)}{x^2}\ dx$.

Limit Comparison Test

Theorem

If positive functions f and g are continuous on $[a, \infty)$ and

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\ dx \quad \text{and} \quad \int_a^\infty g(x)\ dx$$

BOTH converge or BOTH diverge.

Example 7: Let $f(x) = \frac{1}{\sqrt{x}+1}$; consider

$$\int_1^\infty \frac{1}{\sqrt{x}+1}\ dx.$$

Does the integral converge or diverge?

Example 7, continued

Solution: We note that f looks a lot like $g(x) = \frac{1}{\sqrt{x}}$, and $\int_1^\infty g(x)\ dx$ diverges by the p-test. Furthermore,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{x}+1} = 1,$$

so the LCT says $\int_1^\infty \frac{1}{\sqrt{x}+1}\ dx$ diverges.