MATH 105: Finite Mathematics
7-4: Conditional Probability

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Walla Walla College

Winter Quarter, 2006

Outline

1. Introduction to Conditional Probability
2. Some Examples
3. A “New” Multiplication Rule
4. Conclusion

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1 Introduction to Conditional Probability

2 Some Examples

3 A “New” Multiplication Rule

4 Conclusion

Extra Information and Probability

In 1991 the following problem caused quite a stir in the world of mathematics.

Monty Hall Problem

Monty Hall, the host of "Let's Make a Deal" invites you to play a game. He presents you with three doors and tells you that two of the doors hide goats, and one hides a new car. You get to choose one door and keep whatever is behind that door.

You choose a door, and Monte opens one of the other two doors to reveal a goat. He then asks you if you wish to keep your original door, or switch to the other door?

Play the Game

Extra Information and Probability, continued...

Monty Hall Solution

You should switch doors.

- You choose Door A and have a $\frac{1}{3}$ probability of winning.
- Monty eliminates a goat behind one of the other doors.
- Switching wins in cases 1 and 2 and looses in case 3.
- Thus, switching raises your probability of winning to $\frac{2}{3}$.

Extra Information and Probability, continued...

Monty Hall Solution

You should switch doors.

	Door A	Door B	Door C
1	goat	goat	car
2	goat	car	goat
3	car	goat	goat

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Conditional Probability

Here is another example of **Conditional Probability**.

Example

An urn contains 10 balls: 8 red and 2 white. Two balls are drawn at random without replacement.

1. What is the probability that both are red?
2. What is the probability that both are red given that the first is white?
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An urn contains 10 balls: 8 red and 2 white. Two balls are drawn at random without replacement.

1. What is the probability that both are red?

$$\frac{C(8,2)}{C(10,2)} = \frac{28}{45}$$

2. What is the probability that both are red **given** that the first is white?
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   This can't happen
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$$\frac{7}{9}$$

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In the last two questions, extra information changed the probability.

Conditional Probability

Information given about one event can effect the probability of a second event. Knowing that the first ball was white in the problem above changed the probability that both balls were red.

Conditional Probabilty

If A and B are events in a sample space then the probability of A happening given that B happens is denoted

$$\Pr[A \mid B]$$

which is read "The probabilty of A given B".

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Venn Diagrams and Conditional Probability

To help us develop a formula for $\Pr[A|B]$ we will use Venn Diagrams in the following example.

Example

The probability of an event A is 0.50. The probability of an event B is 0.70. The probability of $A \cap B$ is 0.30. Find $\Pr[A|B]$ and $\Pr[B|A]$.

$$\Pr[A|B] = \frac{0.30}{0.40 + 0.30} \approx 0.43$$

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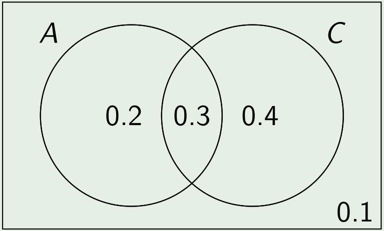

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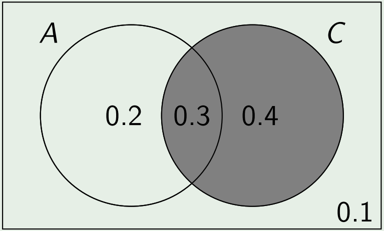

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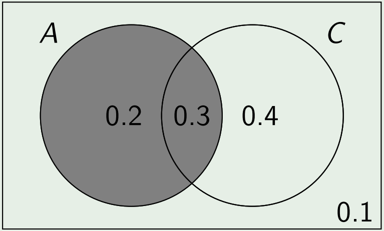

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$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

Note:

$$\frac{\Pr[A \cap B]}{\Pr[B]} = \frac{\frac{c(A \cap B)}{c(S)}}{\frac{c(B)}{c(S)}} = \frac{c(A \cap B)}{c(B)}$$

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You roll two dice and note their sum.

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$$\Pr[A|B] = \frac{c(A \cap B)}{c(B)} = \frac{1}{11}$$

On a Used Car Lot

Example

There are 4 vans, 2 SUVs, 6 compacts, and 3 motorcycles on a used car lot. One is chosen at random to be the "special sale" vehicle.

1. What is the probability the van is chosen given that the SUVs are not chosen?
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$$\Pr[\text{ compact} \mid \text{ compact or van }] = \frac{6}{10}$$

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Revising the Formula

Revised Counditional Probability Formula

We have seen the formula for conditional probability:

$$\Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

Multiplying both sides by $\Pr[B]$ yields:

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Probability on a Tree Diagram

Example

Two urns contain colored balls. The first has 2 white and 3 red balls, and the second has 1 white, 2 red, and 3 yellow balls. One urn is selected at random and then a ball is drawn. Construct a tree diagram showing all probabilities for this experiment.

More Probabilities on Tree Diagrams

Example

An experiment consists of 3 steps. First, an unfair coin with $\Pr[H] = \frac{1}{3}$ is flipped. If a heads appears, a ball is drawn from urn #1 which contains 2 white and 3 red balls. If a tails is flipped, a ball is drawn from urn #2 which contains 4 white and 2 red balls. Finally, a ball is drawn from the other urn. Construct a tree diagram to help answer the following questions.

1. What is $\Pr[HWW]$?
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The next two sections will study questions such as those below in more detail.

Example

In the previous example, find Pr[last ball red | H flipped] and Pr[last ball red | T flipped]. Does the result of the coin toss change the probability that the last ball is red?

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Again using the previous example, find Pr[H flipped | last ball red]. Can the tree diagram be used to find this probability?

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Important Concepts

Things to Remember from Section 7-4

1. Conditional Probability Formula:

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2. Using tree diagrams for probability:

$$\Pr[A \cap B] = \Pr[B] \cdot \Pr[A|B]$$

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