Math 10C - Practice Midterm #1 Solutions

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1. (6 points) Let A and B be the points with coordinates $A = (1, 3, 3)$ and $B = (2, 1, 2)$.

(a) Find a unit vector $\vec{u}$ in the direction of $\vec{AB}$.

SOLUTION: We need to first find a vector $\vec{v}$ in the direction $\vec{AB}$.

$$\begin{aligned} \vec{v} &= (2-1)\vec{i} + (1-3)\vec{j} + (2-3)\vec{k} \\ &= \vec{i} - 2\vec{j} - \vec{k} \end{aligned}$$

Now we need to make $\vec{v}$ a unit vector.

$$\begin{aligned} \vec{u} = \frac{\vec{v}}{||\vec{v}||} &= \frac{\vec{i} - 2\vec{j} - \vec{k}}{\sqrt{(1)^2 + (-2)^2 + (-1)^2}} \\ &= \frac{\vec{i} - 2\vec{j} - \vec{k}}{\sqrt{1+4+1}} \\ &= \boxed{\frac{1}{\sqrt{6}}(\vec{i} - 2\vec{j} - \vec{k})} \end{aligned}$$

(b) Let $\vec{v} = \vec{i} + \vec{j} - \vec{k}$ Find the coordinates of the point C such that $\vec{AC} = \vec{v}$.

SOLUTION: Let $C = (c_1, c_2, c_3)$ be the point such that $\vec{AC} = \vec{v} = \vec{i} + \vec{j} - \vec{k}$. Then

$$\vec{AC} = (c_1 - 1)\vec{i} + (c_2 - 3)\vec{j} + (c_3 - 3)\vec{k} = \vec{i} + \vec{j} - \vec{k}$$

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Then

$$\begin{aligned} c_1 - 1 &= 1 \\ c_2 - 3 &= 1 \\ c_3 - 3 &= -1 \end{aligned}$$

Then

$$\begin{aligned} c_1 &= 2 \\ c_2 &= 4 \\ c_3 &= 2 \end{aligned}$$

Thus, $\boxed{C = (2, 4, 2)}$

(c) Are the vectors $\vec{AC}$ and $\vec{AB}$ orthogonal? Justify your answer.

SOLUTION: We can test if $\vec{AC}$ and $\vec{AB}$ are orthogonal by taking their dot product. The two vectors are orthogonal if and only if their dot product is zero.

$$\begin{aligned} \vec{AC} \cdot \vec{AB} &= (1, 1, -1) \cdot (1, -2, -1) \\ &= 1 - 2 + 1 = 0 \end{aligned}$$

Thus, $\vec{AB}$ and $\vec{AC}$ are $\boxed{\text{orthogonal}}$.

2. (6 points) Let A, B, and C be the points with coordinates $A = (2, 0, 0)$, $B = (0, 4, 0)$, and $C = (0, 0, 3)$.

(a) Find the equation of the plane passing through the points A, B, and C.

SOLUTION: To find a plane that contains the three points, a quick method is to find a normal vector that is perpendicular to the vectors $\vec{AC}$ and $\vec{AB}$.

$$\begin{aligned} \vec{AC} &= (0 - 2)\vec{i} + (0 - 0)\vec{j} + (3 - 0)\vec{k} \\ &= -2\vec{i} + 3\vec{k} \end{aligned}$$

$$\begin{aligned} \vec{AB} &= (0 - 2)\vec{i} + (4 - 0)\vec{j} + (0 - 0)\vec{k} \\ &= -2\vec{i} + 4\vec{j} \end{aligned}$$

Now the normal to the plane (denote $\vec{n_P}$) will be found by taking the cross product of $\vec{AC}$ and $\vec{AB}$:

$$\begin{aligned}
\vec{n_P} &= \vec{AC} \times \vec{AB} \\
&= \left\| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{j} \\ -2 & 0 & 3 \\ -2 & 4 & 0 \end{array} \right\| \\
&= \vec{i} \left\| \begin{array}{cc} 0 & 3 \\ 4 & 0 \end{array} \right\| - \vec{j} \left\| \begin{array}{cc} -2 & 3 \\ -2 & 0 \end{array} \right\| + \vec{k} \left\| \begin{array}{cc} -2 & 0 \\ -2 & 4 \end{array} \right\| \\
&= \vec{i}(0-12) - \vec{j}(0+6) + \vec{k}(-8-0) \\
&= -12\vec{i} - 6\vec{j} - 8\vec{k}
\end{aligned}$$

Now we need to make sure that it contains the points. It is sufficient to make sure that it contains one of the points. Let's choose $A = (2,0,0)$

$$\begin{aligned}
-12(x-2) - 6(y-0) - 8(z-0) &= 0 \\
\implies -12x + 24 - 6y - 8z &= 0 \\
\implies -12x - 6y - 8z &= -24 \\
\implies 6x + 3y + 4z &= 12
\end{aligned}$$

(In the last step, I divided by two, just to make the final equation cleaner)

You can test for yourself whether this equation is correct by plugging in the three points, and testing to see whether the vector is perpendicular to the vectors made by the three points.

$\boxed{6x + 3y + 4z = 12}$

(b) Find a vector that is perpendicular to the plane in part (a).

SOLUTION: We already have from part (a) that $\vec{v} = 6\vec{i} + 3\vec{j} + 4\vec{k}$ is perpendicular to the plane.

$\boxed{\vec{v} = 6\vec{i} + 3\vec{j} + 4\vec{k}}$

(c) Does the vector $\vec{w} = \frac{3}{2}\vec{i} - \frac{2}{3}\vec{j} - 2\vec{k}$ lie in the plane from part (a)?

SOLUTION: We can test if the vector lies in the plane by testing if the vector is perpendicular to the normal of the plane. We test this by taking the dot product of the two vectors:

$$\begin{aligned}
\vec{w} \cdot \vec{v} &= (\frac{3}{2}, -\frac{2}{3}, -2) \cdot (6,3,4) \\
&= 9 - 2 - 8 = -1 \neq 0
\end{aligned}$$

So $\vec{w}$ $\boxed{\text{does not lie in the plane}}$.

3. (6 points) Let $\vec{v} = 2\vec{i} - \vec{j} + 3\vec{k}$ and $w = -\vec{i} + \vec{j} + \vec{k}$.

(a) Find the vector $\vec{v} \times \vec{w}$.

SOLUTION:

$$\begin{aligned}
\vec{v} \times \vec{w} &= \left\| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{j} \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{array} \right\| \\
&= \vec{i} \left\| \begin{array}{cc} -1 & 3 \\ 1 & 1 \end{array} \right\| - \vec{j} \left\| \begin{array}{cc} 2 & 3 \\ -1 & 1 \end{array} \right\| + \vec{k} \left\| \begin{array}{cc} 2 & -1 \\ -1 & 1 \end{array} \right\| \\
&= \vec{i}(-1-3) - \vec{j}(2+3) + \vec{k}(2-1) \\
&= -4\vec{i} - 5\vec{j} + \vec{k}
\end{aligned}$$

$\boxed{\vec{v} \times \vec{w} = -4\vec{i} - 5\vec{j} + \vec{k}}$

(b) Are $\vec{v}$ and $\vec{w}$ parallel? Justify your answer.

SOLUTION: Saying that two vectors are parallel is equivalent to saying that the vectors are just scalar multiples of eachother. Suppose there exists a scalar c such that $\vec{v} = c\vec{w}$. Then

$$\begin{aligned}
2\vec{i} - \vec{j} + 3\vec{k} &= c(-\vec{i} + \vec{j} + \vec{k}) \\
&= -c\vec{i} + c\vec{j} + c\vec{k}
\end{aligned}$$

By equating the $\vec{i}, \vec{j}$ and $\vec{k}$ components, we get that $c = -2$, $c = -1$, and $c = 3$. So, there does not exist a single scalar such that $\vec{v} = c\vec{w}$. So, $\vec{v}$ and $\vec{w}$ are $\boxed{\text{not parallel}}$.

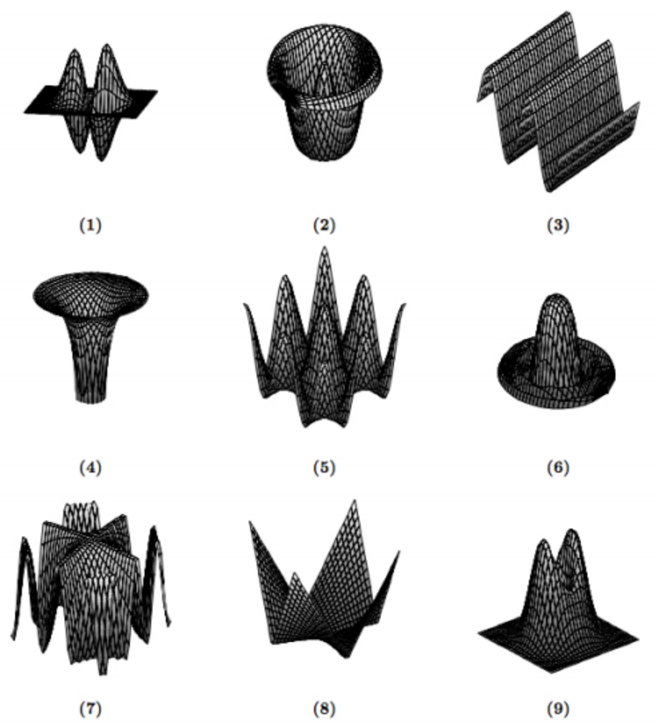
(c) Find a unit vector perpendicular to both $\vec{v}$ and $\vec{w}$

SOLUTION: In part (a), we found $\vec{v} \times \vec{w}$, which is perpendicular to $\vec{v}$ and $\vec{w}$. We just need a unit vector $\vec{u}$ that goes in the same direction. So,

$$\begin{aligned}
\vec{u} &= \frac{\vec{v} \times \vec{w}}{||\vec{v} \times \vec{w}||} \\
&= \frac{\vec{v} \times \vec{w}}{\sqrt{(-4)^2 + (-5)^2 + (1)^2}} \\
&= \frac{1}{\sqrt{42}}(-4\vec{i} - 5\vec{j} + \vec{k})
\end{aligned}$$

Thus, $\boxed{\vec{u} = \frac{1}{\sqrt{42}}(-4\vec{i} - 5\vec{j} + \vec{k})}$

4. (6 points) Write in your Blue Book the letter of each equation lettered (a)–(f) below. Next to each letter, write the number of the corresponding graph from among the graphs numbered (1)–(9) below. You need not provide any explanation; however, you must clearly list your choices. (Note: Yes, there are more graphs than equations.)



SOLUTION: (a) $z = -\dfrac{1}{x^2+y^2}$: We see that as $(x, y) \to (0, 0)$, $z \to -\infty$. Looking at the graphs, we see that the only graph that diverges to negative infinity is (4).

(b) $z = sin(y)$: Notice z is only dependent on y, which means that the graph is the equation $z = \sin y$ in the zy plane, and then stretched out along the x-axis. The graph (3) is the only graph that satisfies this.

(c) $z = cos^2(x)cos^2(y)$: Try fixing $x = 0$. Then $z = cos^2(y)$. So along the y-axis ($x = 0$), the graph of z looks like $cos^2(y)$. Now try fixing $y = 0$. Then $z = cos^2(x)$. So along the x-axis ($y = 0$), the graph of z looks like $cos^2(x)$. The graph (5) is the only graph that satisfies this.

(d) $z = \dfrac{sin(x^2+y^2)}{(x^2+y^2)}$: Along the circle $x^2+y^2=c$ for some constant c, the graph should be the same. So, the graph should be circularly symmetrical about the origin. Also notice that as $(x,y)\rightarrow(\infty,\infty)$, $z\rightarrow 0$, since the denominator tends to infinity, while the numerator is always between -1 and 1. Finally, we notice that as $(x,y)\rightarrow(0,0)$, $z\rightarrow 1$, (using the handy fact from 10a that $\lim_{x\rightarrow 0}\dfrac{\sin x}{x}$=1). It looks like the graph (6) fits the bill.

(e) $z=|x||y|$: Notice that for $x=0$ (y-axis), $z=0$, and for $y=0$ (z-axis), $z=0$. The only graphs that appear to satisfy this are (1), (7), and (8). Notice also that as $(x,y)\rightarrow(\infty,\infty)$, $z\rightarrow\infty$. However, graph (1) approaches $z=0$ as x and y approach infinity, so it can't be graph (1). Now notice that graph (7) is entirely negative, but equation (e) is entirely positive. Thus, we choose graph (8).

(f) $z=xye^{-(x^2+y^2)}$: Notice the equation can be written as $z=\dfrac{xy}{e^{x^2+y^2}}$. Then we notice that for $x=0$ (y-axis), $z=0$, and for $y=0$ (z-axis), $z=0$. The only graphs that appear to satisfy this are graphs are (1), (7), and (8). Notice that as $(x,y)\rightarrow(\infty,\infty)$, $z\rightarrow 0$, since the denominator ($e^{x^2+y^2}$, exponential) grows much faster than the numerator (xy, quadratic). The only graph that satifies this is graph (1).

$\boxed{a=4,\ b=3,\ c=5,\ d=6,\ e=8,\ f=1}$