Division of Academic Enhancement
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MATH 1101 Chapter 5 Review

Topics Covered

Section 5.1 Exponential Growth Functions

Section 5.2 Exponential Decay Functions

Section 5.3 Fitting Exponential Functions to Data

Section 5.4 Logarithmic Functions

Section 5.5 Modeling with Logarithmic Functions

How to get the most out of this review:

1. **Watch the video and fill in the packet for the selected section. (Video links can be found at the two web addresses at the top of this page)**
2. **After each section there are some 'Practice on your own' problems. Try and complete them immediately after watching the video.**
3. **Check your answers with the key on the last page of the packet.**
4. **Go to office hours or an on-campus tutoring center to clear up any 'muddy points'.**

Section 5.1 Exponential Growth Functions

Exponential Functions
A function is called exponential if the variable appears in the exponent with a constant in the base.

Exponential Growth Function
Typically take the form $f(x) = ab^x$ where a and b are constants.

- b is the base, a positive number > 1 and often called the *growth factor*
- a is the y intercept and often called the *initial value*
- x is the variable and is in the exponent of b

Growth Factor, b
b is the growth factor and constant number greater than 1. Therefore it can be written in the form:

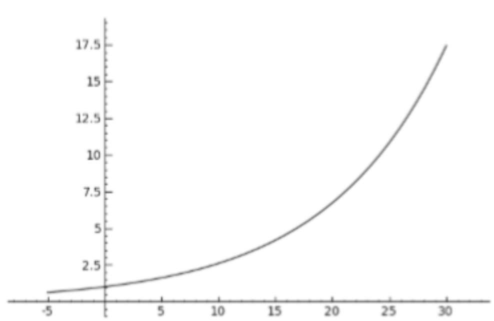
$$b = 1 + r$$

Where r is the growth rate. The rate is a percentage in decimal form. An alternative form of the exponential growth function is:

$$f(x) = a(1 + r)^x$$

Linear vs. Exponential Growth
If a quantity grows by 4 units every 2 years, this describes a linear relationship where the slope is $4\ units/2\ years = 2\ units/year$
If a quantity grows by a factor of 4 units each year, this describes an exponential relationship and the quantity increases 300% each year. $4 = 1 + 3.00$

Graphs of exponential growth functions are concave up.



Exponential growth functions ($b > 1$)
increase at an increasing rate.

Example 1

On January 1, 2000, $1000 was deposited into an investment account that earns 5% interest compounded annually. Answer the following:

(a) Find a model $A(t)$ that gives the amount in the account as a function of t. Let t be the number of years since 2000.

(b) Assuming no withdrawals, how much money will be in the account in 2010?

(c) How long until the account has a balance of $1575?

(d) What year and month did the balance hit $1575?

(e) How long does it take for the account to double in value?

Example 2

How much money must be invested in an account that earns 12% a year to have a balance of $3000 after 4 years? Round your answer to the nearest cent.

How long with it take for the account to double in value?

Finding an exponential function from 2 data points

If given two data points for an exponential growth function $(0, A_0)$ and (N, A_N), you can write the growth formula using the following

$$A(t) = A_0 \left(\frac{A_N}{A_0}\right)^{\frac{t}{N}}$$

Example 3

A certain type of bacteria was measured to have a population of 23 thousand. 4 hours later it was measured at 111 thousand. Answer the following:

(a) Write an equation $P(t)$ that models the size of the population t hours after the initial measurement.

(b) Find the hourly percentage increase in bacteria population.

(c) What is the size if the population after 10 hours?

(d) What is the doubling time of the population?

Compound Growth

The amount of money in an account, after t years, at an annual rate r and compounded n times a year can be measured with the following model

$$A(t) = A_0 \left(1 + \frac{r}{n}\right)^{nt}$$

Watch out for key vocabulary words like monthly, quarterly, daily, etc.

Make sure that r is the percentage rate written in decimal form when put into the formula.

Example 4

Determine the value of an account where \$1500 is invested earning 2% annually and compounded the following ways. Round your answers to the nearest cent.

(a) Annually

(b) Semiannually

(c) Quarterly

(d) Monthly

(e) Weekly

(f) Daily

<u>Effective Annual Yield</u>

The more times interest is compounded per year it creates more opportunities to earn interest on the balance AND the interest you have already earned that year. The rate equivalent to the same amount of yield you would have earned if you only compounded annually is called the effective annual yield. It is the investments exponential growth rate and can be found using:

$$EAY = r = \left(1 + \frac{APR}{n}\right)^n - 1$$

Example 5

Determine the EAY for the account in Example 10. Round your answer to 4 decimal places

(a) $n = 1$

(b) $n = 12$

(c) $n = 52$

Practice on Your Own

1. Is this a linear function, exponential function, some other type of function, or not a function?

x	0	1	2	3	4	5
y	3.500	.350	.954	1.956	2.80	3.54

2. Is this a linear function, exponential function, some other type of function, or not a function?

x	0	1	2	3	4	5
y	7.100	3.950	.800	2.350	5.500	8.650

3. Is this a linear function, exponential function, some other type of function, or not a function?

x	0	1	2	3	4	5
y	3.100	3.720	4.464	5.357	6.428	7.7137

4. Is this a linear function, exponential function, some other type of function, or not a function?

$$y = 934(1 + 0.39)^x$$

5. Is this a linear function, exponential function, some other type of function, or not a function?

$$y = 27(1 + 3x)^2$$

6. Makebelievia Inc. is making automatic homework doers, they can make 2000 the first year, increasing by 5%per year.

   (a) What is the growth rate?

   (b) What is the growth factor?

(c) Find a model, $D(t)$ for the production level, as a function of the number of years after the first production.

(d) Find the amount produced 7 years later.

(e) Find the time when they first produce over 3000. (Watch your rounding)

(f) Find the decadal growth rate (growth rate over a decade).

7. You invest \$1700 in a Ponzi scheme paying 3.7% compounded annually.

   (a) Find a model A(t) that gives the amount you believe you have, in dollars, as a function of years after initial deposit.

   (b) Find the amount in the account after 27 years.

   (c) Find the doubling time.

   (d) Find the monthly growth rate.

8. A colony of ants has moved into your roommate's sock drawer, on Feb. 17 the population was 1200, by Feb 20 the population had grown to 2300. Assume exponential growth.

   (a) Find a model P(t) that gives the number of ants as a function of days after Feb. 17th.

   (b) Find the amount of ants on Feb 29.

(c) Find the day when the number of ants is exactly 3000. (Extra credit: find the hour in the day) (d) Find the daily growth factor.

(e) Find the daily growth rate.

9. A group of bacteria doubles in size in 8 hours, if there are 1600 at noon, how many are there at 5:00 pm.

10. Find the effective annual yield for each of these accounts:

   (a) $1000 invested at 5% compounded annually

   (b) $2172 invested at 4.5% compounded daily

   (c) $829 invested at 3.2% compounded monthly

Section 5.2 Exponential Decay Functions

Exponential Functions
A function is called exponential if the variable appears in the exponent with a constant in the base.

Exponential Decay Function
Typically take the form $f(x) = ab^x$ where a and b are constants.

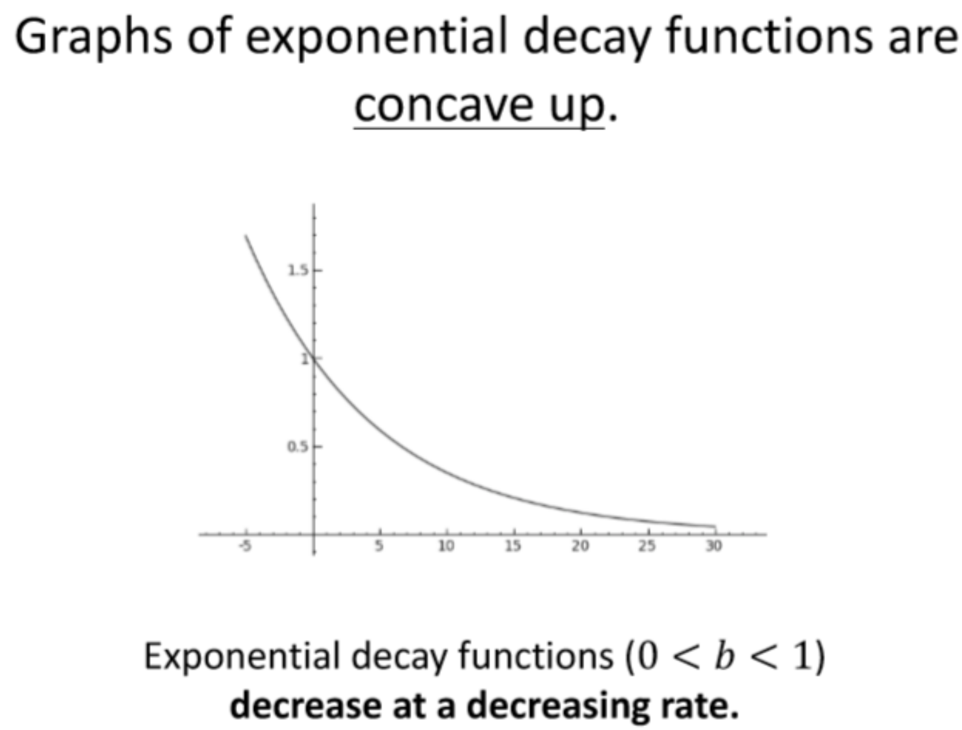
- b is the base where $0 > b > 1$ and often called the *decay factor*
- a is the y intercept and often called the *initial value*
- x is the variable and is in the exponent of b

Decay Factor, b
b is the decay factor and constant number between 0 and 1. Therefore it can be written in the form:

$$b = 1 - r$$

Where r is the decay rate. The rate is a percentage in decimal form. An alternative form of the exponential decay function is:

$$f(x) = a(1 - r)^x$$



Example 6
Determine the model $f(t)$ for a substance that decays at 12% per hour and an initial value of 2500.

Example 7

The town of Pawnee had a population of 242.3 thousand in 2005 and began decreasing at a rate of 0.3% after Little Sebastian was kidnapped by Eagleton residents during Pawneefest that year. Assuming this trend continues for several years, answer the following:

(a) Find a model $P(t)$ that gives the population of Pawnee as a function of t, the number of years since 2005.

(b) Predict when the population of Pawnee will fall to 236 thousand.

(c) Predict the population of Pawnee in 2011.

(d) What is the annual decay factor?

(e) The 10 year decay factor?

(f) The 10 year growth rate? How do we interpret this value?

(g) What is the monthly growth rate?

Example 8

The town of Eagleton had a population of 185 thousand in 2005 and a population of 162 thousand in 2010 after the city coffers went bankrupt. Answer the following.

(a) Find a model $P(t)$ that gives the population of Eagleton as a function of t, the number of years after 2005.

(b) What is the decay rate of Eagleton's population?

(c) When will Eagleton's population fall to 100 thousand?

Practice on Your Own

1. Makebelievia's national reserves, in billions, can be modeled by the function $D(t) = 273(.8432)^t$ where t is the number of years after January 1, 2000.

   (a) Find the country's annual percent growth/decay rate.

   (b) The country's reserves [increase / decrease] by __________ each __________.

   (c) Find the per-decade growth/decay factor, round to four decimal places (extra credit, why four places?)

   (d) Find the average rate of change during President Phakie's term, (2002 to 2009)

   (e) The country's actual reserves in 2009 were $54.71 billion, find the model's error.

   (f) Predict the reserves in 2018.

   (g) In what year will the reserves reach 4,380 million? (watch your units)

2 My laptop battery loses 4% of its runtime every month, find the time when it's down to half of it's initial runtime.

Section 5.3 Fitting Exponential Functions to Data

Exponential Functions

A function is called exponential if the variable appears in the exponent with a constant in the base.

Exponential Growth Function

Typically take the form $f(x) = ab^x$ where a and b are constants.

- b is the base, a positive number > 1 and often called the *growth factor*
- a is the y intercept and often called the *initial value*
- x is the variable and is in the exponent of b

Exponential Decay Function

Typically take the form $f(x) = ab^x$ where a and b are constants.

- b is the base where $0 > b > 1$ and often called the *decay factor*
- a is the y intercept and often called the *initial value*
- x is the variable and is in the exponent of b

How do we build models off data sets that have exponential growth or decay?

If the data points show that the function behaves like an exponential function, we can build a model using the exponential regression feature in the TI-83/84 calculator.

Exponential Regression

A computational technique that produces an exponential model that best fits the data set. The model of best fit is determined by minimizing the SSE (sum of squared errors) and AE (average error).

Model Error

Since the actual data points may or may not lie on the curve of best fit, the model error for a specific data point is calculated by finding the vertical distance between the point and the curve.

$$Error = actual\ value - predicted\ value$$

The *actual value* is the y value of the data point for a given x and the *predicted value* is the y value the model gives for the same value x.

- Data points that lie above the curve will produce positive errors and the model *under predicts*
- Data points that lie below the curve will produce negative errors and the model *over predicts*
- The average error (AE) of the model is found by using the SSE (Sum of Squared Errors) where n is the total number of data points in the set

$$AE = \sqrt{\frac{SSE}{n}}$$

Finding the Curve of Best Fit in the Calculator

1. Go to STAT→Edit…
2. Enter the x values in L_1
3. Enter the y values in L_2
4. Go to STAT→CALC→ExpReg
5. Type L_1, L_2, Y_1 (Y_1 is under VARS)
6. You will see the values for a, b and the line will be stored in Y_1
7. The model is of the form $f(x) = a(b)^x$

Example 9

The following data set is the population of a city (y) measured in thousands where x is the number of years after a major flood. Find the curve of best fit for data set 1.

Data Set 1							
x	0	5	8	11	15	18	22
y	179.5	168.7	158.1	149.2	141.7	134.6	125.4

Finding the SSE in the Calculator

1. Go to STAT→Edit…
2. Enter the x values in L_1
3. Enter the y values in L_2
4. Enter $Y_1(L_1)$ in L_3
5. Enter $L_2 - L_3$ in L_4
6. Enter L_4^2 in L_5
7. Go to 2nd →STAT→MATH→sum(L_5)

Example 10

Find the SSE and AE for data set 1. Round your answers to 2 decimal places.

SSE:

AE:

Example 11

What is the correlation coefficient for data set 1? What does it say about this model?

Example 12
What is the decay factor for this model? Round your answer to three decimal places.

Example 13
By what percentage does the population decline each year? Round your answer to two decimal places.

Example 14
How long will it take the population to fall to 100 thousand? If the flood happened on January 1, 1870, when does the population fall to 100 thousand? Give the year and month.

Example 15
What is the predicted population of the city at 10, 20 and 30 years after the flood?

Example 16
What is the error for population measurements at 5 and 8 years after the flood?

Newton's Law of Cooling

When an object at temperature T_0 is placed in an environment at constant temperature T_s, where $T_s < T_0$, the object will cool until it reaches T_s and no further. We can model the temperature of the object at any given time t by

$$T(t) = PTD(t) + T_s$$

where $PDT(t)$ is the positive temperature difference and is modeled by a decreasing exponential function. T_s is the temperature of the surroundings.

Example 17

You buy a hot cup of coffee on a cold day and accidentally leave it outside. The coffee was 160°F at $t = 0$ and the temperature outside is 37°F. The following data set are the actual temperatures (y) after sitting outside for x minutes. Answer the following:

Data Set 2							
x	0	5	10	15	20	25	30
y	160	138	120	104	91	81	74

(a) Predict the temperature of the coffee at 40 min. Round your answer to two decimal places.

(b) When will the coffee cool to 40°F? Round your answer to the nearest minute.

(c) What is the SSE, AE and correlation coefficient of the model? Round your answers to two decimal places.

Newton's Law of Heating

When an object at temperature T_0 is placed in an environment at constant temperature T_s, where $T_s > T_0$, the object will heat until it reaches T_s and no further. We can model the temperature of the object at any given time t by

$$T(t) = T_s - PTD(t)$$

where $PDT(t)$ is the positive temperature difference and is modeled by an increasing exponential function. T_s is the temperature of the surroundings.

Example 18

To cook your favorite holiday turkey, you bake it in the oven at 425°F until it reaches an internal temperature of 165°F. To help you, your mom gave you the following data set from her turkey last year. x is the time in hours after the turkey was placed in the oven and y is the temperature of the turkey taken with an internal thermometer.

Data Set 3: Turkey					
x	0	1	2	3	4
y	40	105	128	155	170

(a) If you put the turkey in at 1PM, what time will it reach 165°F? Round your answer to the nearest minute.

(b) If you fall asleep and forget to take it out in time, what temperature will it be at 6PM? Round your answer to the nearest degree.

(c) When will the turkey reach 350°F? Round your answer to the nearest minute.

Example 19

Fortunately, your turkey was successful, but you realize after the fact that your mom's model was incorrect due to the difference in size of your respective turkeys. At 1PM your turkey was 40°F and by 1:30PM your turkey had heated to only 52°F.

(a) If you buy the same size turkey next year, what model should you use if you still plan bake it at 425°F?

(b) How long will it take to reach an internal temperature of 165°F? Round your answer to two decimal places.

(c) If you start the turkey at 1PM, what time will it be done? Round your answer to the nearest minute.

Practice on Your Own

1. The following table shows Makebelievia's Vibranium production for the years between 2000 and 2005. Let t=0 represent the year 2000.

Year	2000	2001	2002	2003	2004	2005
Vibranium	39	63	101	144	199	280

(a) Find the linear model that best fits this data

(b) Find the exponential model that best fits this data. What is the initial value, what is the growth factor, what is the growth rate?

(c) Which of these models is a better choice? Use it for the rest of these questions.

(d) The model [overpredicts — underpredicts — exactly predicts] 2002's vibranium production.

(e) What is the SSE, average error and correlation coefficient?

(f) When does the vibranium production exceed 1000?

(g) Predict the vibranium production in 2015.

2. A mathematician cooks a cake in a 350 degree oven. The table below shows the internal temperature of the cake.

Time (minutes)	0	5	10	15	20	25
"Cake" Temperature (F)	67.7	114.8	154.1	186.6	214.9	236.1

(a) Find the Newton's Law of Heating equation for this data.

(b) Find the SSE, average error, and correlation coefficient for this model.

(c) Predict the cake's temperature when the mathematician pulls it out at 45 minutes. (Extra credit: Predict the look on the mathematician's face when he pulls this "cake" out)

(d) What is the exact time this cake passes 212 degrees F.

Section 5.4 Logarithmic Functions

Compound Interest

An account with interest rate r compounded n times a year will grow using the following equation:

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

However, as n gets *really* big (like real big), the units of time get so small that we can model the growth of the account with the *continuous* compound interest formula:

$$A = A_0 e^{rt}$$

where r is the annual interest rate and t is the time in years.

Example 20

You deposit $3000 in an account with an annual rate of 5%,

(a) Find the balance of the account after 3 years. Round your answer to the nearest cent.

(b) When will your account balance be $4000? Assume no withdrawals are made. Round your answer to the nearest cent.

Effective Annual Yield

The effective annual yield is computed using the following:

$$EAY = e^r - 1$$

Example 21

Find the EAY for the account in example 12. What does this mean?

Continuous Growth and Decay

If a quantity continuously grows or decays, we can model it by the following:

$$A(t) = A_0 e^{rt}$$

- $r > 0$ (positive) if there is continuous growth
- $r < 0$ (negative) if there is continuous decay
- t can be measured in any unit of time
- A_0 is the initial quantity when $t = 0$

Example 22

Jen really hates spiders. Her house is infested and her exterminator tells her that the pesticide has a continuous kill rate of 0.90%.

(a) If the exterminator estimates that there are 10,000 spiders in her basement (gross), how long until half of the spiders are gone? Round your answer to two decimal places.

(b) How many spiders are in her basement 24 hours after he sprays his pesticide? Round your answer to the nearest spider.

(c) Find the hourly percent decay rate. Round your answer to two decimal places.

Inverse Functions
If $g(x)$ and $f(x)$ are functions where $g(x)$ is the inverse of $f(x)$ and vice versa, then $g\big(f(x)\big) = x = f\big(g(x)\big)$.

Logarithmic Functions
Logarithmic functions are the inverse of exponential functions. The two most common are

- Common Log (base 10)
  - $\log(10^x) = x$ for all x
  - $10^{\log x} = x$ for $x > 0$
  - $\log(b) = x \rightarrow 10^x = b$
- Natural Log (base e)
  - $\ln(e^x) = x$ for all x
  - $e^{\ln x} = x$ for $x > 0$
  - $\ln(b) = x \rightarrow e^x = b$

Example 23

$$30e^{0.25t} = 120$$

(a) Solve for t. Round your answer to two decimal places.

(b) What is the growth rate and continuous growth rate? Round your answer to two decimal places.

$$100(0.90)^t = 500$$

(c) Solve for t. Round your answer to two decimal places.

(d) What is the decay and continuous decay rate? Round you answer to two decimal places.

Practice on Your Own

1. Makebelievia Credit Union offers a savings account with 3.8% interest compounded monthly. Pietro opens an account with $2123.
   (a) How much is the account worth in 6 months?

   (b) When does the account reach $3000?

   (c) What is the expected annual yield?

2. Makebelievia Credit Union also offers a Certificate of Deposit with 7.3% interest compounded continuously. Wanda opens an account with $6382.
   (a) How much is the account worth in 2 years?

   (b) When does the account reach $10,000?

   (c) What is the expected annual yield?

3. Which of these accounts has the best return on investment?

Interest Rate	8.72%	8.32%	8.07%	7.94%	7.61%	7.50%
Compounded	annually	quarterly	monthly	weekly	daily	continuously

4. Palladium-103 has a half-life of 17 days, The reactor that powers the capital of Makebelievia has 23.1 kg of Palladium on January 1, 2016.
   (a) How much palladium is there on March 1st (60 days later)?

   (b) When does the amount of palladium fall below 7 kg?

   (c) What is the daily decay rate?

   (d) What is the continuous daily decay rate?

   (e) What is the monthly (30 days) decay rate?

Section 5.5 Modeling with Logarithmic Functions

How do you decide to model with an exponential or logarithmic function?

- If the data set is increasing nonlinearly, we look at the concavity to determine which model is appropriate
    - Exponential growth functions are concave up
    - Logarithmic growth functions are concave down
- If the data set is decreasing nonlinearly, both exponential AND logarithmic decay functions are concave up. Technically both methods would work. However, the best model can be determined by the following:
    - Comparing their respective SSE values. The best model will have the smallest SSE
    - Comparing their respective AE values. The best model will have the smallest AE
    - Compare their respective r^2 values. The best model will have the largest r^2.

Example 24

Which is the best model for the following data set?

t	2	12	22	32
A	186.71	98.06	69.06	56.23

Exponential Model?

Logarithmic Model?

Which one is better? Why?

Practice on Your Own

1. Does the following data match a linear function, exponential function, logarithmic function, or none of these?

x	0	1	2	3	4	5
y	7.100	3.950	.800	-2.350	-5.500	-8.650

2. Does the following data match a linear function, exponential function, logarithmic function, or none of these?

x	0	1	2	3	4	5
y	3.100	3.720	4.464	5.357	6.428	7.7137

3. Does the following data match a linear function, exponential function, logarithmic function, or none of these?

x	1	2	3	4	5	6
y	2.460	3.257	3.723	4.054	4.311	4.521

Answers to the Practice on Your Own problems

Section 5.1

1. Some other type of function
2. Linear function
3. Exponential function
4. Exponential function
5. Some other type of function
6. (a) 5%
   (b) 1.05
   (c) $D(t) = 2000(1 + .05)^t$
   (d) 2814
   (e) $t = 8.31$, but round up to $t = 9$
   (f) 62.88%
7. (a) $A(t) = 1700(1 + .037)^t$
   (b) $4533.96
   (c) $t = 19.08$
   (d) 0.30%
8. (a) $P(t) = 1200(2300/1200)^{t/3}$ or $P(t) = 1200(1.2422)^t$
   (b) 16195
   (c) $t = 4.23$ or Feb 21, 5:24 am
   (d) 1.2422
   (e) 24.22%
9. 2468 (round to a whole number)
10. Effective yield
    (a) 5%
    (b) 4.60%
    (c) 3.25%

Section 5.2

1. (a)

$$d(t) = \begin{cases} 60t + 0, 0 \leq t \leq 3 \\ 0t + 180, 3 < t \leq 4 \\ 55t - 40, 4 < t \leq 6 \\ 20t + 170, 6 < t \leq 6.5 \end{cases}$$

   (b) Domain: [0,6.5], Range: [0,720]
   (c) $d(5.5) = 262.5$
   (d) $t = 1.67\ hours$, 1 hr 40 min

2. (a) $27
   (b) $75
   (c) $905
   (d) $392 kWh
   (e) 0.15
   (f) There is a 15 dollar monthly charge and every kWh costs $0.15,

Section 5.3

1. Vibranium
   (a) $V(t) = 47.3143t + 19.3809$
   (b) $V(t) = 42.273 * 1.47746^t$, initial value = 42.273, growth factor = 1.47746, growth rate = 47.746%
   (c) Exponential (r^2 is closer to 1)
   (d) Under predicts
   (e) SSE = 461.66, average error = 9.61, correlation coefficient = .9959
   (f) $t = 8.11$ so 2009
   (g) 14750.10
2. Cake
   (a) $D(t) = 282.12 * .9641^t$
   (b) Find the SSE = 1.2307, average error = .4961, correlation coefficient = −.9999
   (c) D = 54.65, T = 295.35. (Extra credit: :'( )
   (d) t = 19.61

Section 5.4

1. (a) $2163.66
   (b) t = 9.11, 9 years and 3 months
   (c) 3.867%
2. (a) $7385.22
   (b) t = 6.15
   (c) 7.573%
3. E.A.Y.'s:

Interest Rate	8.72%	8.32%	8.07%	7.94%	7.61%	7.50%
Compounded	annually	quarterly	monthly	weekly	daily	continuously
E.A.Y.	8.72%	8.58%	8.37%	8.26%	7.91%	7.79%

   Annual is best.

4. (a) 2.0 kg
   (b) t = 29.28
   (c) r = −3.995%
   (d) r = 4.077%
   (e) r = −79.544

Section 5.5

1. linear
2. exponential
3. logarithmic