MATH 111 : COURSE OUTLINE

This document contains suggestions for running a Math 111 class in ways that have proven successful in the past. All of the ideas here should be understood as descriptions of what's been done and suggestions of what you might try, and nothing more.

Four chapters make up the Math 111 course materials:

(1) Problem Solving (about 3 weeks)
(2) Place Value (about 3 weeks)
(3) Number and Operations (about 4–5 weeks)
(4) Fractions (about 5–6 weeks)

The chapters are intended to be used in this order, though there is certainly flexibility. If you have a favorite topic or lesson not represented in these chapters, you should definitely use it. Perhaps you want to incorporate some history of mathematics or some ethnomathematics. These would be wonderful experiences for your students!

Be aware that the planned chapter for Math 112 include:

(1) Patterns and Algebra
(2) Fractions and Decimals
(3) Geometry

You will probably alter these suggested timelines as it suits your style and the needs of your class. If you feel students are struggling or just not getting one particular topic, it may not be fruitful to stick with it and insist on mastery by the whole class. Remember that we are trying to provide students with the tools to *eventually* develop profound understanding of fundamental mathematics, but we do not believe we can get them there in a semester or a year. Use your professional judgment about when it is worth spending more time, when to move on, and when to skip sections entirely.

The materials include reading for the students, activities for in-class use, and problem banks. Some chapters will include suggestions for self-checks on procedural skills, since these are assumed as background knowledge. As an instructor, you may wish to require that students submit to you proof of passing such assessments, or you may want to supplement the materials with some skills practice if you deem it necessary for the majority of your students.

A typical class might proceed as follows:

- Whole-class discussion about a homework problem or reading. To start the discussion, a student presents her solution to the problem or summarizes what was discussed in the reading.
- Think/pair/share[1] for a launch activity. These are labeled in the materials under this heading.
- Call on individuals or groups to present their work. Depending on the activity, there may be just one presentation or more than one. (You may want to give the groups a few minutes notice so they can prepare what to say.) Emphasize that these presentations are launch points for a class discussion, so students are not expected to give polished solutions after a partial class period.
- Debrief the launch activity. This may include student presentations or a short instructor lecture on key ideas. (In an IBL class, the content marked "Definition" or "Example" in the student materials should grow out of student work on the "Think / pair / share" activities and the problems rather than assigned as readings during class. Including these materials in the text is intended to give students a reference for later study. Working on and presenting problems, not reading the text, should be the main activity during class time.)
- The rest of class will vary depending on the activity and the class: Students may continue working on the problem with the expectation of finishing a good write-up by the end of class or for homework. There may be a short quiz or other individual assessment. Students may read from the course materials and then work on a problem or another Think/pair/share activity.

For most Math 111 classes, there will be a lot of group work both in and out of class. Students will often collaborate on their homework, though instructors should insist that write-ups are individual work except on assignments that are explicitly assigned to groups. Weaker students may be inclined to defer to group members rather than assert themselves to ask questions and make sure they follow along.

As an instructor, you can look out for these situations as the groups are working, check in with individual students, and help the groups take responsibility for all members' understanding. (One method: call randomly on students to present and assign a score to the group based on the quality of that presentation. It is therefore in the group's best

[1]This is a standard methodology used in inquiry-based learning. Read a description here: `http://theiblblog.blogspot.com/2011/08/classroom-strategy-think-pair-share.html`.

interest to insure that everyone participates and understands the solution.)

In addition to helping the groups as they work, it is useful to have regular individual assessments. Quizzes and "exit tickets"[2] can help you identify students who are struggling and help these students see for themselves how they are doing.

Choice of content. The content for Math 111 / 112 was selected to align with the Common Core State Standards[3] for grades K–5. Some content that appears in many "Math for Elementary Teachers" textbooks is not included in these materials, because it does not appear in the K–5 curriculum. This includes:

- Integers (negative numbers)
- Ratio and proportion
- Prime factorization, gcf, and lcm

Our intent is to focus on depth of coverage for the K–5 curriculum and on the process of doing mathematics and justifying solutions, rather than rushing to cover many more topics.

We encourage you to spend time in class letting students struggle and find their own way. Ask questions rather than explaining. Remember that your students will eventually have to make sense of mathematical ideas without your guidance. They will have to become the mathematics experts in their own classrooms. It is far more important that we affect their view of mathematics and of themselves as learners and doers of mathematics than that we cover any particular piece of the elementary curriculum.

Notation. The student materials use notation like $>$, $<$, and fractions without comment. We assume these ideas are familiar to students, even if they are perhaps a bit fuzzy. Through repeated exposure to the ideas, working in collaborative groups to help unpack meaning and fill in the gaps, this notation will become second nature.

The use of variables to express relationships rather than as "unknowns" where we solve an equation is really at the heart of algebra. But most students' algebra backgrounds have focused on procedural

[2]Usually this is a short, often anonymous, questionnaire done at the end of class that students must turn in before leaving. Typical questions: "One thing I understand from today's class is ____________. One thing I do not understand well is ____________. One question or concern I have is ____________." You can read other ideas here: `http://www.adlit.org/strategies/19805/`.

[3]`http://www.corestandards.org/Math`

questions (solve for x) rather than structural ideas (write an expression to describe this relationship). Carolyn Kieran found that when students were asked to express the meaning of $a + 3$, they couldn't because there is no equals sign and no number on the other side.

By using variables in this more procedural way throughout the course, we in fact will bolster students' overall algebra skills and their ability to (in the words of the Common Core State Standards) "reason abstractly and quantitatively." We encourage you to reinforce this use of algebra in your students' work. Ask them to give names to quantities when they want to refer to them, ask how something could be written in symbols, and so on. This is part of our mathematical language, and through use (not through repeated dill exercises), students will acquire the language more naturally and become fluent in its use.

1. Chapter 1: Problem Solving

Starting with the "Problem Solving" chapter allows you to set the tone for the class. The message to students is intended to be:

- Mathematics is about solving problems and explaining your work to others.
- You should always be able to explain *why* your answer is correct for any mathematical problem (or exercise).

Most of the problems in this chapter require little mathematical background, though some of them use ideas which some students may find "familiar but forgotten" (examples: prime numbers, the symbols $<$ and $>$, use of variables in explanations, and so on.)

This is intentional. The reason for couching many of the activities in the "Think/pair/share" structure is twofold:

(1) It creates a classroom atmosphere in the first few weeks that insists on students talking to each other about mathematics, and
(2) it encourages students to ask clarifying questions without fear of reproach.

It also reinforces the idea that a mathematical problem doesn't ask you what you already know. Sometimes, in working on a problem, you need to build some mathematical background (or remind yourself of it) in order to proceed. Many of the ideas will appear in later chapters, so there's no need to spend a lot of time on them now.

Reassure students (and yourself) that if they're not quite solid on some of the ideas discussed in class, they'll have opportunities to master them when those ideas come back around. The focus is on building a collaborative environment and developing a toolbox of problem solving strategies.

The chapter contains a short warm-up activity, followed by seven sections:

Problem or Exercise?: This short in-class activity is designed to help students understand the difference between building skills and solving mathematical problems. Both activities are important, but the point of the skill-building is to improve our ability to solve problems and reason carefully. Basic skills themselves are not the end-goal of learning mathematics.

Problem Solving Strategies: Used as in-class activities, this could take several days of class. Alternately (and maybe preferably?) your students can generate their own list of strategies by working on problems in class. After each problem, encourage students to articulate how they solved it, and especially how they got started. You might wish to keep a running list of your students' own problem-solving strategies, either on the board or a poster or online where students can access it an add to it. The whole section could then be a homework reading assignment *after* students have created their own list of strategies.

Beware of Patterns!: This is an optional section, but we have found that students (and practicing teachers) tend to believe that if an "obvious" pattern fits a problem, then that pattern simply must continue. This section is designed to dispel that myth through a surprising "pattern-breaking" problem. The remainder of the section focuses on tying apparent patterns back to the problem context. It may be heavy-going for some classes early in the term, and may best be left to a later date.

Problem Bank: These problems are intended to be used for homework or additional in-class work. There are too many problems here, so do not expect or plan to have students tackle them all! The idea is to pick-and-choose (or to allow students to do so). We also have a suggestion for using these problems as part of a whole-class writing activity. See later in these notes for ideas. You may even wish to continue assigning problems from this section as students work through the other three chapters, to keep problem solving as a major theme in the course.

Careful Use of Language in Mathematics: The focus throughout the entire Math 111 / 112 course is on sense-making and explaining mathematics. This requires students to develop better mathematical language skills. This section starts them on that path. These ideas should be covered at some point during Math 111, but could be put off until they come up more naturally in class. This is a text-heavy section, so look for ways to make it more interactive and engaging. For example, in the "All About the Benjamins" problem, you want to create envelopes with Monopoly money (or just slips of paper, though you'll need to change the statements accordingly) and have volunteers justify to the class whether the given statements are true or false.

Explaining Your Work: Clear communication is essential in all mathematics classes, but it is especially important that future teachers develop the ability to explain mathematical ideas well. The activity presented of critiquing student work on a problem is particularly powerful, and one we suggest using with regularity (drawing on your own students' work). Looking at other students' work with a critical eye helps students begin to do the same with their own work. See the notes below for an additional writing activity to accompany this section. This section could also be put off until a later point in class, when you begin requiring students to write more explanations and justifications. It is perfectly fine to keep the first chapter short and focused on the idea of solving problems rather than on the write-ups.

The Last Step: This section encourages students to become not just problem solvers but also problem posers. Getting in the habit of asking questions is powerful for mathematics learners, especially since they will probably quickly stumble upon unsolved or very difficult problems if they do this. Again, the hope is to develop the capacity for asking "what if" questions in these future teachers, in the hope that they encourage questioning and curiosity about mathematics in their future students.

1.1. **Writing Activity.** This activity seems to help students better understand what is expected in a good write-up of a mathematics problem. Remember that this is a process, and their write-ups are not likely to be very good at first. But clearly articulating expectations and then revisiting those expectations throughout the semester helps them improve over time. This group activity is just a first step.

Before class: Ask students to read all of Section 6 "Explaining Your Work." You may also want to decide on the groups of three students who will work on the activity together and pre-assign each group a problem from Section 4 "Problem Bank." (Some pairs or groups of four is OK, but three is usually optimal.) You may wish to group stronger students together and assign those groups a more challenging problem, or you may opt for more random groupings. We have had success with both approaches.

(Alternately, you can have students self-select their groups and their problems. While this may make students feel more empowered and invested in their group and their problem, it can cause some classroom chaos and make it difficult to ensure that several groups are working on different problems.)

During class: Put students in their groups and ask them to compare their scores for the different solutions presented in Section 6. Follow this with a whole class discussion about those solutions and what makes a good mathematical explanation. Focus on clarity, justifications for their solutions, and making things easier for the reader.

The activity proceeds this way (of course, make whatever modifications suit your teaching and your class):

- Each group chooses a "code name" and gives that to the instructor who records it.
- Each group is assigned a problem from the "Problem Bank" and a rubric on which their write-up will be scored (one suggested rubric appears below).
- The groups will have time during class that day to work together on the problem. You may decide to give additional class time to the groups during a second class period, or you may require them to find time to work as a group to complete the problem outside of class time.
- They are to have a solution write-up ready by a pre-set due date. It is the responsibility of the whole group to ensure that the write-up is complete and on time. They must be aware that their write-up will be graded based on the rubric by someone who has not worked on the same problem. So it is imperative that the statement of the problem is clear, and the solution must be convincing.
- On the due date, they will reconvene in their groups. The write-ups will be distributed so that each group will read and assess another group's write-up. Each group will receive a rubric on which they should base their assessment. The groups should provide clear and honest feedback about the write-up, with a focus on clarity and correctness.
- Before the end of class, the write-ups and rubrics are returned to the groups that worked on the problem. Those groups make a plan for revising their work based on the feedback.
- During the next class, the instructor collects everything: the original writeup, the feedback from the other group, and the new writeup. Students will receive a group score on the activity, and each group's score will be based on the quality of their writeup, the quality of their feedback and honest scoring, and the improvement shown in the revised write-up.

Problem Solving Write-up Rubric

Code name of group solving the problem:

Code name of group evaluating the write-up:

Directions: Please fill in a score from 0 (should have done it but did not) to 5 (perfect!) for each category. If a category does not apply to the problem you are reading, you may write N/A (not applicable) instead of a numerical score.

Does this write-up:

Clearly (re)state the problem?	
State the (correct) answer clearly?	
Clearly label diagrams, tables, graphs, or other visual representations of the math (if these are used)?	
Define all variables used and explain all equations?	
Have correct spelling, grammar, and punctuation?	
Have clear and easy-to-ready writing and formatting?	

Comments: (Is the answer correct? Are you convinced that it's right or not sure? Are the answer and the justification clear and easy to read? Did the group actually solve the problem as stated? Could the explanation be more clear? Would a picture have helped you understand? Is the write-up too long? Too short? Just right? What could be done to improve it?)

1.2. **Quizzes.** In a problem-solving chapter, it can be difficult to give a short quiz that accurately reflects the goals of the chapter. We provide here some possible quiz problems with ideas for how to grade them. Also remember that revisions can be a powerful way to help students learn from mistakes made on a quiz. The goal is to get students to say something more than "I tried some numbers and it didn't work."

(1) Find three **consecutive** numbers that add up to 95 or explain why it's not possible. Show all of your work.
This problem is a natural follow-up the clock problem in the chapter. Students might attack the problem in a few ways: They might observe that all of the numbers should be about the same size, so they should be about $95/3 \approx 32$. They can try combinations of numbers in the right range and show that some choices sum to less than 95 and some more. Other students might observe that the sum of three consecutive numbers is always divisible by 3, and 95 is not.

(2) Jeff said that when he opened his book, the product of the two page numbers was 409. Find the two page numbers, or explain how you know Jeff got it wrong.
There are a few approaches to this problem. Students might realize that since $20^2 = 400$, the answer should be somewhere in the twenties. They can try several pairs of consecutive numbers and show through systematic checking that none of them work. Or they may explain that the product of two consecutive numbers is even (and say how they know this), so this is impossible.

2. Chapter 2: Place Value

One of the major ideas in elementary mathematics is understanding the base 10 system and using properties of the base 10 system to algorithmically carry out computations. Of course, if one deeply understands place value, there is absolutely no difference between working in base 10 and working in some other base. We use the context of representing numbers in different bases to challenge these future teachers to think more deeply about the material they will teach.

This chapter uses the successful "Dots and Boxes" approach of James Tanton[4] to introduce the idea of representing numbers in different bases. The "Dots and Boxes" model will be reprised in Chapter 3 when we use it to compute in base 10 and in other bases, and it will come back in Math 112 when we use it to talk about place value and decimal numbers.

Using this unfamiliar "game" to set up the mathematics allows your students to develop their understanding of regrouping ("carrying") without being bogged down by what they already know (or think they know) form working in base 10. In section 2, they will make the connection between this context and familiar base 10 numbers in what we hope is an "aha!" moment for the whole class.

The chapter contains seven sections along with a "Problem Bank":

Dots and Boxes: This section sets explains the game and has students practice what is essentially creating binary representations of base 10 numbers. This will all be made more formal in future sections, so there's no need to treat it as anything other than a game at this point.

Other Rules: This section builds on the game we set up, getting students to work in different bases. (Again, that language is not used yet, but it will be. Don't feel the need to introduce it too early, before students are comfortable with "the rules of the game.") At the end of this section, students will see the connection to familiar base 10 numbers.

Binary Numbers: We formalize the game in the first section, introduce the notion of binary numbers, and ask students to come up with methods to convert between binary numbers and base 10 (in both directions). Formal methods to convert between numbers in different bases will be introduced in a later section, so don't feel the need to correct or optimize students'

[4]You can find the original "Dots and Boxes" activities and lots of other stuff at James Tanton's website: `http://www.jamestanton.com/`.

methods. If they draw out (hundreds of) dots and work by "exploding" or "unexploding," that's perfectly fine. The tedium of that work will help them to appreciate more concise methods introduced later.

Other Bases: We introduce the idea of an arbitrary (positive integer) base and notation for numbers written in different bases. We describe general methods for converting between base 10 and base b for positive integer bases b. The general methods may be difficult for many students owing to their recursive nature and heavy use of variables. As usual, it is preferable is to develop these procedures in class as refinements to students' own methods, and then assign the readings afterwards to reinforce the ideas.

Number Systems: This section is a short read about additive versus positional number systems and Fibonacci's role in bringing the arabic positional system to the Western world. It is optional. You could opt to assign the reading and give a short fact-based quiz (we *do not* encourage asking students to memorize any of the historical additive number systems), or you could call on students to summarize the reading for the class. If students are familiar with, for example, Roman numerals, you could ask them to perform some calculations (multiplication is particularly difficult!) using Roman numerals without translating to our base-10 system. They will probably quickly appreciate the benefits of a positional number system.

Even Numbers: Students often have misconceptions about properties of numbers of numbers. For example, they will describe even numbers as ones where the units digit is even (a property of the *representation*) rather than saying it can evenly be divided into two parts (a property of the *quantity* itself). This demonstrates a *lack* of profound understanding of fundamental mathematics. They are conflating the base-10 rule for testing if a number is even with the property of "being even." In this section, students review the real meaning of "even" and explore what even numbers look like in different bases.

Orders of Magnitude: The problems in this section allow students to explore the multiplicative nature of place value, specifically understanding the ideas of million, billion, and trillion and appreciating the vast chasm between those magnitudes. A follow-up section introduces students to the idea of Fermi problems (estimating using reasonable guesses of quantities and calculations to answer questions like the famous, "How many piano tuners are there in Chicago?" (though we use questions closer to our students' experiences).

Problem Bank: These problems can be used for homework or additional in-class work. There are too many problems here! You should not try to assign (and certainly not grade!) all of them. Pick and choose the problems that appeal to you, and encourage students to work on other problems for additional practice and to reinforce the ideas.

Exploration: This section introduces some natural extensions to the "Dots and Boxes" game, including a base-one system and a fractional base. This is an optional section, but would make a fun culminating activity or extra credit assignment. These are the kinds of experiences that students should be given at least occasionally during Math 111 and 112 — opportunities to really stretch their thinking and explore unfamiliar territory.

Most of the problems in the "Problem Bank" are not applicable until students complete Section 3. Until then, you probably want to assign reading and problems from the sections for homework, then have students discuss their work with a partner and present their solutions in class before moving on. You could also assign a problem or two from the Chapter 1 "Problem Bank," keeping students focused on solving problems and explaining their work.

2.1. **Materials.** You may want to bring materials for students to use during the dots and boxes games — these could be pennies, unit cubes[5], buttons, or just about anything else. Having physical materials encourages students who are more tactile learners and mimics how students in elementary schools work with base 10 blocks[6].

In fact, if you have access to base 10 blocks, you may want to bring them in during one of the later classes and have students do some base 10 "dots and boxes" type problems with those materials instead. In that case, "exploding" becomes "grouping," with ten ones becoming a "long," ten longs becoming a "flat" and so on.

You may want to incorporate some video of young students working on ideas similar to the ones tackled here as part of your class. For example, you might want your class to watch Deborah Ball's videos of third grade mathematics students discussing even and odd numbers[7] after completing the section on "Even Numbers" in this chapter.

[5] `http://www.mathtacular.com/gram-cm-multifix-cubes/`

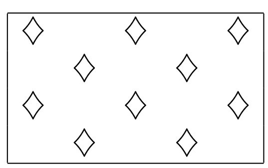
[6] `http://www.basetenblocks.com/`

[7] `http://deepblue.lib.umich.edu/handle/2027.42/65013` and `http://deepblue.lib.umich.edu/handle/2027.42/65012`

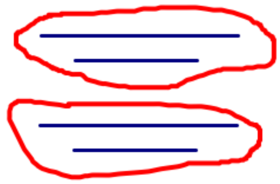
Our preservice teachers are often surprised at the sophisticated work that elementary students are able to do. They find classrooms like the one presented in this video tremendously exciting and motivating. You might decide to show these videos in class, or ask students to watch them for homework. You can then ask for either a brief written reaction to the video, describing the content and your students' impressions of the class. Or you may wish to lead a whole-class discussion about what they saw. One interesting conversation: what knowledge and skills does the *teacher* need in order to lead the class discussion that they saw? Does the teacher not know the answer? If she does know the answer, why does she let the students wrestle with the ideas instead of just explaining the right way to think about it?

2.2. **In Class Activity — Math Talks.** One activity you may consider for occasional in-class warm-ups is a Dot Card Math Talk. This short activity proceeds as follows:

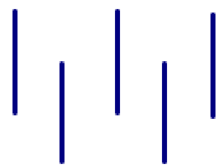
- Give students directions: They are to count the number of objects on the card you show without writing anything down and without counting one-by-one. When they have the answer, they are to simply give you a subtle "thumbs up" signal. (Don't ask them to raise their hands. This can be intimidating for students who work more slowly, if they notice lots of hands are up around them. They can simply hold a thumb up close to their chest.)
- Show a card with some number of a shape repeated in a pattern. Here is one example:



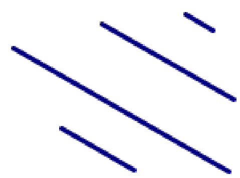
  Make sure the card and the design are large enough to be seen by all students in the class, and leave it up until everyone has given the thumbs-up signal.
- Call on a student (this is a good time to randomly pull a name from the class list) to give their answer and explain how they thought about it. You can draw a schematic of their thinking along with a calculation showing how the student thought about "counting without counting" the dots. For example, one student may have seen two groups of five dots, so you might write this abstract drawing on the board along with the calculation $5 + 5$.



Another student may have seen vertical lines of two, each shifted up or down from the previous. So the schematic looks like this, with the calculation $2 + 2 + 2 + 2 + 2$:



Someone else might see diagonal lines and the calculation $4 + 3 + 2 + 1$:



Lots of variations on these ideas are possible, of course. You might be surprised at how many different strategies your students will have for such a simple task.

- Ask for a show of hands for students who thought about the problem in exactly the same way. Then ask if anyone else thought about it differently, and have one of those students share a different strategy. Keep going until you have several different strategies to talk about.

Why spend time on an activity like these "Math Talks"? Here are some answers to that question from teacher Cathy Humprheys.

> Cards with configurations of objects, that we often call "dot" card number talks, establish important new principles for mathematics classes. While it may seem that these arrangements of shapes are only for young children, we have found that they are critical for older students because they help to lay the groundwork for changing how students think about mathematics. Dot cards do not suggest procedures that students are "supposed" to follow; instead, they encourage students to think about

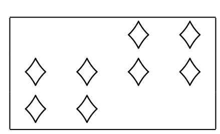
what they “see” rather than what they are supposed to “do.” This frees up students for learning new ways of interacting in math class.

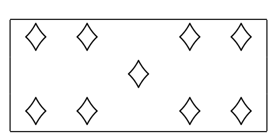
Some of the things they can learn from dot card number talks:

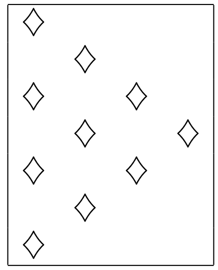
- Just as people “see” things differently, there are often many ways to approach any mathematical problem.
- Explaining ones thinking clearly is important. This requires that students retrace the steps of their answers and learn to use academic language, where possible, to describe what they did to solve the problem.
- It is important for students not only to explain what they did, but why their process makes sense. In the case of dot card number talks, this involves where they “saw” the numbers they used. In the case of arithmetic operations, it involves understanding the mathematics that underlies any procedure that they use.
- The teacher’s job is to ask questions that clarify what the students see rather than how they “should” see.

These are not activities to do every day, nor are they activities to occupy an entire class period. Rather, these types of thinking, visualizing, and reasoning activities can be a great way to start or end class perhaps once a week during this session. They tie very nicely to the “dots and boxes” representations of numbers.

You probably don’t want to do too many of these number talks (you may use them more in the nedt session to practice computation). But here are some other designs to try:







2.3. **Pitfalls and Additional Practice.** Most likely, your students will be fine with the mechanics of base 10 place value: writing numbers in expanded form, naming the place value for specific digits, saying what a digit is "worth," and translating from written words to numerals. If you find some or all of your students need to brush up on these skills, you can point them towards an online resource such as Kahn Academy[8] or IXL[9]. If you want to require them to complete such practice work, you can insist they email you a screen shot of completed worksheets or some other proof of completion.

More likely, students will make mistakes like mixing up "power," "multiple," and "factor" (saying, for example, that in base 5 the positions are "multiples of five"). Correcting these mistakes requires reinforcement in class, gentle insistence or proper terminology in presentations and write-ups, and using problems targeting these issues. (Some appear in the problem bank, and it is a simple matter to generate additional problems with a similar flavor.)

2.4. **Quizzes.** Here are some sample quiz questions that can be used with this chapter:

(1) For each problem, say if the equation is true or false. Carefully justify your answer.
  (a) $8_{\text{nine}} = 8_{\text{eleven}}$.
  (b) $80_{\text{nine}} = 80_{\text{eleven}}$.
(2) What number comes after 233_{four}? (Write your answer in base four.)
(3) Consider the number 122_b, for some base b that's bigger than 2. Is that number even, no matter what base b is? Or is it possible for that number to be odd? Justify your answer.
(4) Consider the number 222_b, for some base b that's bigger than 2. Is that number even, no matter what base b is? Or is it possible for that number to be odd? Justify your answer.

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[8] `https://www.khanacademy.org/math/arithmetic/multiplication-division/place_value/v/place-value-1`

[9] `http://www.ixl.com/math/place-values`

3. Chapter 3: Numbers and Operations

The focus of this chapter is on encouraging students to think more deeply about what they "know" about addition, subtraction, multiplication, and division of whole numbers. We use two models for thinking about the operations:

- We continue with James Tanton's "Dots and Boxes" model, which will allow us to explore the operations in bases other than ten.
- We introduce a measurement model, with a focus on the number line.

Note: The "Dots and Boxes" approach is essentially a set-theoretic development of the operations, without the formalism of sets, unions, intersections, and so on. Some instructors like to introduce the formality of sets, explore these other binary operations, and include activities using Venn diagrams. Such activities are not included in this chapter, but of course if you find these ideas appealing and engaging, you should take some class time out to explore them with your students.

One goal of this chapter is to get students in the mindset that elementary mathematics makes sense, and is not simply a set of rules to memorize. We want to connect fundamental understanding of the operations — addition is combining, subtraction is taking away, multiplication is repeated addition, and division is forming equal-sized groups — with our models. We then connect the models with the standard algorithms. Finally, we use the models to explain some familiar facts like:

- addition and multiplication are commutative and associative,
- multiplication distributes over addition, and
- division by 0 is undefined.

Often, our students know these facts procedurally — they can use them appropriately in computations and perhaps even give the correct names. However, they will say things like, "Addition is commutative because $a + b = b + a$." They have no real understanding *why* these arithmetic facts are true, or even that there is (and should be) an explanation for "why."

The most important thing that students should get from the chapter is a clear understanding that they can always figure out "why" for some arithmetic fact. They should also know the difference between the definition of a property, some examples showing the property holds, and an actual explanation for why it is always true. Explanations for only a couple of these arithmetic facts are actually presented in the student text, and students are asked to generate the others in the problems. This is intentional. These materials provide a model for how to think about mathematics and how to figure things out, but are not meant to be an encyclopedic reference for elementary mathematics.

It is not essential that *every* student explains *each* of the properties. You may want to distribute the work amongst group, and perhaps take this opportunity to reprise activities where students look critically at each others' work after working on different problems (see the "Writing Activity" described in the Notes for Chapter 1).

The chapter consists of four sections and a "Problem Bank."

Model 1: Dots and Boxes: This section develops the four operations — addition as combining, subtraction as taking away, multiplication as repeated addition, and division as forming equal groups — all in the context of the base 10 dots-and-boxes model. (Some problems ask students to use the techniques in other bases, which will demonstrate a deeper understanding of the ideas, of course.) Rather than jumping right into these activities, you may want to ask students to take their concepts of the operations from the chapter-launching Think/Pair/Share activity, and demonstrate what it looks like in the dots-and-boxes context, giving specific problems for them to demonstrate. Students will probably easily demonstrate addition, but they often have trouble with subtraction. They may line up the two numbers as they would for addition, and are then unsure what to do from there. If some students work out a reasonable approach to subtraction, you can have them demonstrate it for the whole class, explicitly making the connection between subtraction and the action of taking away. Alternately, students can read the materials to see a more coherent development and to get common language moving forward. Note that this section connects standard algorithms for addition, subtraction, and division to the model. We work with the standard algorithm for multiplication in the next section.

Model 2: Measurement: This section introduces the idea of a basic unit and a measurement model for the operations based on assigning that unit the value one. This idea comes back in the next chapter ("Fractions"). That is previewed a bit in the launch activity, but you don't need spend a lot of time on fractions at this point. Most of the section takes place on a number line. We also introduce an area model for multiplication and connect it to the standard algorithm. The distributive law is, of course, hiding just beneath the surface here. Someone may mention this in class, but if not do not force it. The formal properties of operations are discussed explicitly in the next section. Two optional problems presenting non-standard multiplication algorithms close out the section. These could be optional extra credit or the basis for an activity where different groups make sense of an algorithm and then teach the method to the class (including demonstrating examples and explaining why it works).

Operations: This rather lengthy section explores both the connections between operations — subtraction as "missing addend" addition problems and division as "missing factor" multiplication problems — and standard properties of the operations. The emphasis throughout should be on finding ways to *explain* these properties using our models. In an IBL class, students will come up with their own explanations, which will be refined and polished through questioning by you and by the rest of the class. (This may be a good time to divvy up work, asking different groups to focus on different properties and then presenting their work to the class.) Alternately, you may want to present these explanations as a short lecture, or you may ask students to read the examples in the book. Division is always the most difficult of the operations for elementary students (and teachers!) to master. The section ends with a more lengthy discussion of the operation. If you have all students work through all of the problems, this section may start to feel repetitive, and it may be a bit of a slog for some classes. You may want to break it up with a day of problem solving using problems from Chapter 1.

Division Explorations: This optional section extends the dots-and-boxes division method to both base 5 and base x. This gives students a chance to connect long division with the polynomial long division they likely learned in high school, perhaps making more sense of the latter. Additionally, the problems suggest the idea of algebra as abstracting properties of *operations*, and that true algebraic statements are true for any substitution of the variables. Again, this section is optional, but we encourage you to give your students the opportunity to dig into one of these optional sections at some point during the course.

Problem Bank: As usual, the problem bank contains potential homework and extra problems for use in class. Pick and choose those that appeal to you.

3.1. **In Class Activities.** There is not much in the way of "practice of basic facts" in the student chapter. Remember that we assume students have some basic facility with computation, and we are trying to get them to think more deeply about the ideas in elementary mathematics.

However, one of the main tasks for elementary teachers is develop number sense and flexible ideas about computation in their students. Recent research by Eddie Gray and David Tall[10] shows that students identified as high achieving by their teachers use more number sense in performing computational tasks, as opposed to memorization of facts or counting strategies. In order to develop this number sense in their students, teachers certainly must have it themselves.

We provide here some in-class activities that may help students develop and reinforce their number sense. Depending on the needs of your students, you may choose to do activities like these more or less often.

3.1.1. *Math talks.* Regular "Number Math Talks" provide one strategy for bringing number sense and computational practice into the classroom without making the chapter *about* computation facility. (See the notes above on Chapter 2 for a similar version of this activity.) Activities like this allow students to practice both computing and articulating their strategies. Having students share their strategies will allow everyone to grow their bank of strategies and their number sense. But the focus in this case is squarely on the number sense and reasoning, rather than on speed and accuracy in computation.

A Number Math Talk[11] proceeds in this way:

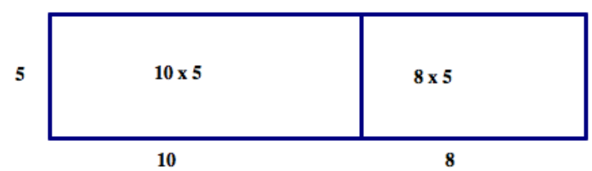
- Give students directions: They are to solve the arithmetic problem without writing anything down, and without using any standard algorithms. When they have the answer, they are to simply give you a subtle "thumbs up" signal. (Don't ask them to raise their hands. This can be intimidating for students who work more slowly, if they notice lots of hands are up around them. They can simply hold a thumb up close to their chest.)
- Pose an arithmetic question like 18×5, and wait until all students are giving you the thumbs up.
- Call on a student (this is a good time to randomly pull a name from the class list) to explain their answer. Even if the answer is wrong, your response should be "tell us how you got that,"

[10] `http://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot1991h-gray-procept-pme.pdf`

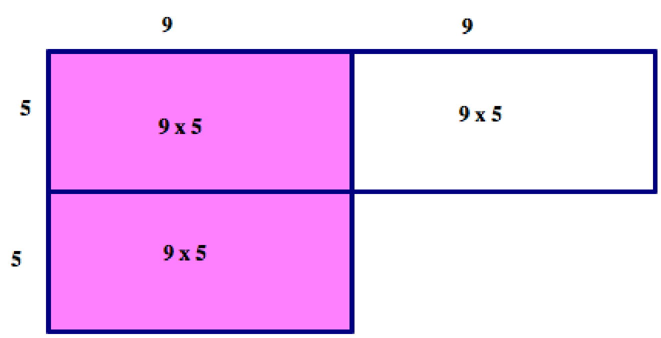
[11] For more on number talks, see `http://www.mathperspectives.com/num_talks.html`.

and ask the class to critique the method. Usually students will self-correct as they explain, or someone in class can help them see an error.

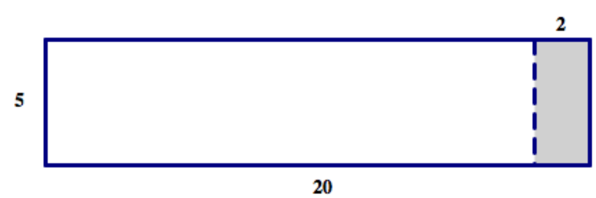
- Ask for a show of hands for students who thought about the problem in exactly the same way. Then ask if anyone else thought about it differently, and have one of those students share a different strategy. Keep going until you have several different strategies to talk about.
- Give a visual representation for each strategy presented. For example, in 18×5, some students might compute 10×5 and 8×5 and then add. So you might draw this picture:



Other students might use a double-and-half strategy like this: $18 \times 5 = 9 \times 10$. So you might draw this picture.



Another student might compute 20×5 and subtract 2×5, leading to a picture like this one.



- You can use the strategies and pictures to talk about what properties of arithmetic are being used: associativity? distributive law? commutativity? others?
- If you'd like to spend more time on these strategies, you can present follow-up computations, even asking students to use

a particular strategy this time. ("See if you can use Tania's method on this computation.")

You may wish to start one class per week with a Math Talk, or you may want to do them every day if you find your students need additional practice. You may want to start with one digit and two digit examples, then expand the activity to some well-chosen larger numbers. Here are some suggested problems for each of the operations, with varying levels of difficulty. You can easily come up with many computations that lend themselves to this activity, using any of the four basic operations.

addition	subtraction	multiplication	division
$7 + 18$	$16 - 13$	$8 \times 9 \times 3$	$252 \div 4$
$27 + 52$	$56 - 29$	4×37	$1710 \div 9$
$342 + 561 + 52$	$751 - 647$	37×98	$1225 \div 25$
$198 + 387$	$83 - 37$	$4 \times 13 \times 25$	$72 \div 12$
$32 + 29 + 56$	$214 - 86$	$26 \times 24 - 21 \times 24$	$120 \div 15$
$54 + 28 + 67$	$52 - 35$	84×5	$145 \div 5$
$49 + 252$	$173 - 96$	37×99	

3.1.2. *Four 4's.* The four 4's game works this way: Challenge students (individually or in groups) to make every number from 1–20 using just four 4's (as digits) and basic operations. For example, you can make the number 1 in several ways: $1 = 44 \div 44$, $1 = (4 \div 4)^{44}$, $1 = (4 \div 4) + (4 - 4)$, and so on.

To complete some of the challenges, students may need to go beyond the four operations in this chapter, using square roots, exponents, and maybe even factorials. You can take the opportunity of having them share their answers to discuss the meanings of these functions.

The activity may also lead to a conversation about order of operations, though it is not an explicit part of these materials since it is not in the K–5 for the Common Core State Standards. Like the other functions mentioned, order of operations should be familiar to students, and it's fine to take some time to discuss it when it comes up naturally in an activity.

You can play the same game with other digits. For example, you might use the digits of a year. On President Obama's birthday, you can play the activity with his birth year of 1961.

3.1.3. *24.* This is a card game similar to the "Four 4's" game above. In groups of three or four, each group is given a deck of cards. (You may choose to remove face cards, assign them values, treat them all as ten, or whatever.) Someone deals out four cards face up. The goal is to use the four cards and any arithmetic operations / functions to create the number 24. For example, if the group is dealt a 3, 4, 5, and 10 you could write: $10 \cdot (5-3)+4 = 24$.

Whoever can form 24 first gets a point, and when all the cards have been used the person with the most points wins. If a group thinks making 24 is impossible with the four cards that have been dealt, they can agree to re-shuffle and deal four new cards. Or you may ask that they check with you, and if you see a way to make 24, encourage them to keep trying because it is possible.

3.2. **Materials.** At the end of Section 1, if you have base-10 blocks available, you may want to ask students to demonstrate the same calculations they have done with dots-and-boxes using those materials and physically regrouping. Make note for them that the base 10 materials (common in elementary schools) allow them to get at many of the same ideas about standard algorithms. However, they force you to work in base 10 whereas the dots and boxes approach is more flexible.

In the introduction to Section 2, if you have access to the materials to do so, you might wish to include a short activity on using volume to model operations in a measurement model as well. This would require some containers with various sizes / shapes or with markings at different levels so that one of the containers (or markings) can be measured off in terms of the other one. (Colored water is particularly nice to use for these demonstrations, since it's more visible.)

Ask students to demonstrate adding and subtracting using the containers. For multiplying, note that you can multiply some volume by a number, but you can't multiply two volumes. You can, however, divide one volume by another. Ask students to demonstrate what that would mean.

Also in Section 2, we encourage you to actually get students up and moving in this activity. If you can, take them outside and draw number lines in sidewalk chalk. Have them physically solve the problems with a partner, taking turns pacing out the answers. They need to internalize "subtraction means walk backwards," as well as how to do the division algorithm by keeping track of how many times they move forward.

If you opt to discuss negative numbers with your class, extending this model allows students to explain, for example, why subtracting a negative number and adding a positive number have the same effect: face negative and walk backwards (subtract a negative) is the same as face positive and walk forwards (add a positive). Similar explanations allow students to explain familiar rules like "negative times negative equals positive."

3.3. **Quizzes.** Some options for quizzes:

- Ask students to use both models to demonstrate a simple calculation.
- After an arithmetic fact has been discussed in class, ask students to explain why it is true on a quiz.
- As an exit ticket, ask students to generate a good quiz question. Then pick one of two of these student-generated questions to use on a quiz during the next class.
- Some of the "Problem Bank" problems are reasonable quiz questions.

4. Chapter 4: Fractions

The topic of fractions is one of the hardest to teach and learn in elementary school, and it is the one area where Math 111 and 112 students are consistently weak in both skills and conceptual understanding.

We address this issue directly, with three sections on the subject of "What is a fraction?" in which we unpack some of the difficulty that K–5 students have with these ideas, and where that difficulty comes from.

As in the "Number and Operations" chapter, we focus on understanding different models for fractions, using those models to make sense of operations, and focusing on why the algorithms for computing with fractions make sense.

We also work on developing some fraction number sense. Throughout this chapter, you should discourage students from using decimal equivalents. Remind them that elementary students learn about fractions first, and they will need to teach fraction ideas without resorting to decimal representations. You should discourage the use of calculators. If students offer problem solutions or justifications based on decimal representations, ask students if they can also work it out another way.

Much of the material in this chapter was adapted from James Tanton's "Guide to Everything Fractions," which you can find in its entirety on his website.[12]

The chapter consists of nine sections, a Problem Bank, and two optional extensions. This chapter, in particular, lends itself to many in-class activities that are not written up in the student text. See later in these notes for ideas for these activities.

What is a Fraction?: We introduce our first model for fractions: a fraction is the answer to a division problem. We call this the "Pies Per Boy" model. That is, the fraction $\frac{a}{b}$ represents the answer to the division problem $a \div b$, and the answer to the question "If a boys share b pies, and everyone gets the same amount, how much pie does each individual boy get?" Students are asked to explore the model and think about questions like what $\frac{a}{a}$ means and what it might mean to divide by $\frac{1}{2}$. All of these topics will be revisited later, so there's no need to delve into them too deeply at this point.

The Key Fraction Rule: This section tackles equivalent fractions and uses the "Pies Per Boy" model to explain why it is true that $\frac{xa}{xb} = \frac{a}{b}$, at least for positive whole numbers x. Right

[12] `http://www.jamestanton.com/wp-content/uploads/2009/07/fractions-guide.pdf`

now, we are looking for intuitive understanding. We will provide a more rigorous proof of the equivalence in a later section.

Adding and Subtracting Fractions: This section develops the algorithms for adding and subtracting fractions with like and then unlike denominators. The method presented is via equivalent fractions, in order to reinforce our "key fraction rule." Some students may recall that they "should" use the LCM of the denominators (or they may incorrectly remember that they "should" use the GCF of the denominators) when adding and subtracting. Since LCM and GCF are not part of the K–5 curriculum in the Common Core State Standards, these topics are not covered in depth in our Math 111 and 112 courses. As the instructor, you should decide how much to pursue these ideas should they arise in your class. It is important to keep the focus on sense-making: In fact, it doesn't matter *which* common denominator we use. Any one will do! We could just as easily say that we "should" use the product of the denominators since it always works and doesn't require the additional LCM computation. If these ideas come up in class, you might pose a question to students: why do they think elementary students are taught to use the LCM as a common denominator? What are the benefits of that choice versus another choice you might make?

What is a Fraction? Revisited: This section expands our fraction model to be simply "parts of some whole," discussing units and unitizing. The motivation here is that if we wish to move on to multiplying and dividing fractions (and certainly we do!), then the "Pies Per Boy" model is no longer adequate because we don't have a way of making sense of multiplication in that model. This section also discusses ordering fractions, and arithmetic sequences are introduced. Students are challenged to form arithmetic sequences of fractions with given starting and ending values. These problems can be challenging, but we encourage you to let students struggle and make sense of them rather than providing canned methods for solving them.

Multiplying Fractions: Given the number line model for fractions, we are able to return to the area model for multiplication used in the "Number and Operations" chapter. Using this model, we develop the standard "multiply the numerators and multiply the denominators" procedure for fraction multiplication. Tying that procedure back to the picture lets students understand why it makes sense in terms of "parts of a whole."

The complication here is that the factors are parts of *segments* and the product is part of a *square*. This shift in dimension may be confusing to students, but point out that it is no different than the whole number case. Even there, the sides are numbers of unit lengths, and the product is a number of unit squares. The section ends with a careful justification of our "key fraction rule" based on thinking about multiplication by 1.

Dividing Fractions: Meaning: This section revisits the partitive, quotative, and missing factor models of division where the numbers in question are fractions. The focus is on making sense of the problems and drawing pictures to represent the situations. Computational methods are discussed in the next section.

Dividing Fractions: Computations: This section provides several ways to think about computing division of fractions problems. We start with "if the fractions have the same denominator, then just divide the numerators," and use that to develop a common denominator method for division. We also use a missing factor approach, but it is clearly limited in its utility based on the fractions in the problem. This method is particularly satisfying because it makes sense to students. Our goal is to move away from the mindless, "ours is not to reason why... just invert and multiply" philosophy that most students have about these problems. We then introduce division of fractions by "simplifying an ugly fraction" (one where the numerator and denominators are themselves fractions or mixed numbers). This leads finally to a justification for the "invert the second fraction and multiply" method. Finally, we ask students to compare the four methods for dividing fractions and discuss the pros and cons of each.

Fraction Sense: In this section, students think about how the size of (positive) numbers change when the four basic operation are performed. Because of the symbols, the opening problem may look quite intimidating to students. You might want to pose just the first line (addition) and see what they can do with it. The goal is to get students to invent lots of examples to test out in order to make a decision, and then to use those examples to help them craft an explanation. The section ends by revisiting the question of division by zero, this time in the context of fractions $\frac{0}{a}$ and $\frac{b}{0}$.

Problem Bank: There are lots of problems here, again too many to assign or use all of them. In addition, if you spend time in class on some of the activities described below, you may assign more of the readings, exercises, and problems from the other sections for homework. Draw on these problems as you see fit.

Egyptian Fractions: This optional section explores the idea of Egyptian fractions. You may choose to spend time in class on the subject (providing extra practice with addition, subtraction, and problem solving with fractions). Alternately, you may want to offer this section as an optional extra credit activity for interested students.

Algebra Connections: For students with a stronger background, this section gets them to simplify seemingly complicated algebraic expressions using the techniques of this chapter. The goal is to reinforce the idea of algebra as "generalized arithmetic".

What is a Fraction? Part 3: This final section revisits for a third time the reason that fractions can be so difficult a topic for elementary teachers and students. We present a bit of the mathematical formalism of fractions as equivalence classes of whole numbers. Finally, we ask students to think about what they've learned and discuss how they like to think about fractions, now that the chapter is complete.

In Class Activities. This chapter, perhaps more than the others, lends itself to many in-class activities that don't necessarily translate well to a student text. We describe several such activities here. These are all, of course, optional. The idea is to provide lots of examples of the kinds of activities that might engage students and reinforce the ideas in the chapter.

Math talks. You may wish to adapt the "Math Talks" described in Chapters 2 and 3 to include simple fraction computations, especially if you feel your students need additional computation practice. The talks can be adapted to include questions like "Is this sum more or less than $\frac{1}{2}$?: $\frac{1}{3}+\frac{1}{4}$" or "Is the answer to $1 \div \frac{2}{3}$ more or less than 1? (Answer without computing.)" or "Estimate $\frac{7}{8}+\frac{11}{13}$ without computing it. Which number is it closest to and why? 1, 2, 19, or 20?"

Fracitons in-between. Draw a long number line on the board, with zero on one end and one on the other. Ask someone: put a fraction about where it belongs on the board. Continue until everyone has placed a fraction. If necessary, repeat with rule that students have to put a fraction between two that were placed by other students (to avoid sequences like $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ...

You can debrief the activity with a discussion: How many fractions are there between 0 and 1? Why? How many fractions between any two fractions? How to find a fraction between any two that are given?

Justification of "sometimes true". We find that "always / sometimes / never true?" questions are useful for quick assessments at the end of class. For example is the following statement always true, sometimes true or never true?

The greater the numerator, the greater the fraction.

Students often have a difficult time justifying that statements are "sometimes true." Some students may correctly answer "sometimes true," but may not realize that to justify that answer they must show *both* an example where it is true and an example where it is false. You can pose the question to students, ask them to come up with a good justification, and then have several groups share their justifications.

Multiplication and Division. For each of the following expressions, ask students to write a word problem. Then they should solve their own problems (or exchange problems with a partner or another group and solve each others' problems).

$3\times\frac{1}{2}$ $\quad$ $\div\frac{1}{2}$ $\quad$ $\frac{1}{3}\times 4$ $\quad$ $\frac{1}{3}\div 4$ $\quad$ $\frac{5}{8}\times\frac{1}{4}$ $\quad$ $\frac{5}{8}\div\frac{1}{4}$

You can, of course, collect these problems and pull a few of them to use as questions on a future quiz.

Rational tangles. Conway's rational tangles activity is described in detail here: `http://www.geometer.org/mathcircles/tangle.pdf`[13]. It's an engaging class that deals with creating a mathematical model of a fun situation along with practice computing with fractions. It's quite satisfying to students when they compute the answer and then it works out, the knot actually unties!

Fraction war. Students can play fraction war with cards. You might want to start by having someone explain the rules to the card game "War." Then describe this modified version:

In pairs or threess: Shuffle cards well, then split the deck of cards evenly between the players. (They will need to leave out a single card if there are three players. You can designate a card to be left out, like the ace of spades.)

Each person turns over the top two cards in their hand and forms a fraction by putting low number / high number. The person with the

[13] You can view a (long) video of the activity run with teachers here: `http://www.youtube.com/watch?v=iE38AXV_dHc`.

largest fraction wins. (Students need to agree on which is bigger... no calculators!) The winner takes all four (or six) cards and puts them face up below his cards.

As with War, if there's a tie the cards stay out and there is another round of play. The winner of the second round takes all cards from both rounds. (Repeat as many times as necessary until there is no longer a tie.)

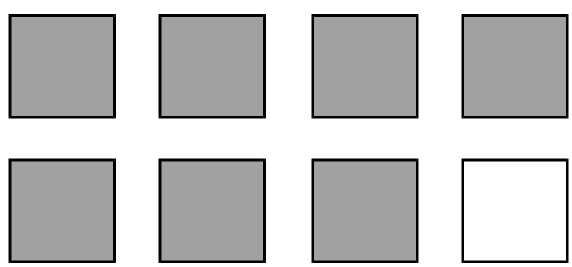
When all cards have been played (players get to face up cards), the winner is the one with the most cards in her deck.

Variations: Smallest fraction wins. Put high number / low number and smaller (or larger) wins. Put low / high and fraction closest to $\frac{1}{2}$ wins.

Quiz and test questions. You can, of course, draw problems from the "Problem Bank" for quizzes, or use questions that students generate in class activities like those described above. Here are some other questions you might want to use.

- Your student asks if you can draw a picture to explain what $\dfrac{3}{4}$ of $\dfrac{5}{7}$ means. What would you draw?
- Explain how each of the following fractions could be represented by the drawing below.

  $\dfrac{7}{8}$ $\quad$ $3\dfrac{1}{2}$ $\quad$ $1\dfrac{3}{4}$

- Lynne calculates
  $$\frac{3}{4} \div \frac{1}{4} = 3.$$
  Then she says, "I don't get it! The answer is bigger than both $\dfrac{3}{4}$ and $\dfrac{1}{4}$. How can that be?"
  (a) Did Lynne calculate correctly, or did she make a mistake? Explain how you know.
  (b) How would you help answer Lynne's question?

- Here is an unordered list of fractions. Answer the following questions using benchmarks and other intuitive methods. **Do not use decimals, and do not use a common denominator.**

$$\frac{3}{2}, \quad \frac{7}{8}, \quad \frac{4}{5}, \quad \frac{1}{6}, \quad \frac{3}{4}, \quad \frac{1}{3}, \quad \frac{3}{5}, \quad \frac{5}{11}, \quad \frac{9}{8}, \quad \frac{1}{4}, \quad \frac{1}{10}.$$

  (a) List all of the fractions given above that are greater than 1. Explain how you know the fractions are greater than 1.
  (b) List all of the fractions given above that are greater than $\frac{1}{2}$ but less than 1. Explain your answer.
  (c) List all of the fractions given above that are less than $\frac{1}{2}$. Explain how you know the fractions are less than $\frac{1}{2}$.
  (d) Put the whole list of fractions in order, from smallest to largest. You do not have to explain your work.

- For each pair of fractions, decide which is larger. Circle your choice and **justify** it.

$$\frac{997}{999} \quad \text{or} \quad \frac{99997}{99999}$$

$$\frac{15}{337} \quad \text{or} \quad \frac{15}{773}$$

- Write the missing-factor multiplication problem for this division problem. Then use what you know about multiplication of fractions to solve the problem. Show your work clearly. (Do not draw an area model!)

$$\frac{5}{12} \div \frac{1}{3} = \underline{\qquad\quad}.$$

- Solve this division problem with the common denominator method. Show your work clearly.

$$\frac{3}{4} \div \frac{1}{12} = \underline{\qquad\quad}.$$

- Draw an area model to show $\frac{2}{3} \times \frac{3}{5}$ and then find the answer.
- For each problem circle either TRUE or FALSE. Provide a clear justification of your choice. (Don't just state a "rule'," but explain *why* your choice is correct.)

(a) $\frac{5}{7} + \frac{3}{8} = \frac{8}{15}$.

TRUE FALSE

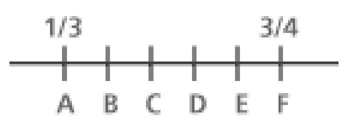
(b) The quotient $\frac{6}{11} \div \frac{7}{13}$ is the same as the product $\frac{11}{6} \cdot \frac{7}{13}$.

TRUE FALSE

(c) There is a fraction between $\frac{3}{29}$ and $\frac{4}{29}$.

TRUE FALSE

- Solve each of these problems. Think carefully about what answer makes sense. Drawing a picture might help!
  (a) Inga was making a cake that called for $2\frac{2}{3}$ cups of flour. But the only clean measuring cup was $\frac{1}{3}$ cup. How many scoops of flour with the $\frac{1}{3}$ cup will she need to make the recipe?
  (b) After his birthday party, John had $2\frac{2}{3}$ pizzas left over. John ate $\frac{1}{3}$ of the leftover pizza. How much pizza did John eat?
- Determine which of the following are correct or incorrect. **Justify your answer.**
  (1) $\frac{a \cdot \cancel{b} + c \cdot \cancel{b}}{\cancel{b}} = a + c$.
  (2) $\frac{\cancel{a} + b}{\cancel{a} + c} = \frac{b}{c}$.
- Jim gave a fourth of the money in his pocket to Tom. Then, he gave a third of what was left to Zoe. Then he split the remainder with you. If Jim gave you \$20, how much money did he have before he started giving it away?
- The points are evenly spaced on the number line. What fraction corresponds to point E?



- Draw an area model to show $\frac{3}{4} \times \frac{2}{5}$ and then find the answer.
- Draw a picture to illustrate this situation; use your picture to answer the questions: There was $\frac{2}{3}$ of a pie leftover in the fridge. John ate $\frac{2}{3}$ of that leftover pie.
  (1) How much of the pie did John eat? Justify your answer.
  (2) How much of the pie was left when he was done? Justify your answer.
- Draw a picture of the following problem, and then use your picture to solve it.

Jeanette wants to tie pieces of ribbon around bags of cookies that she'll sell at a fundraiser. She wants to cut each piece of ribbon to be exactly $\frac{2}{3}$ *of an inch long. Her original ribbon is* 6 *inches long. How many pieces can she cut?*