Math 115 HW #5 Solutions

From §12.9

4. Find the power series representation for the function

$$f(x) = \frac{3}{1-x^4}$$

and determine the interval of convergence.

Answer: Using the geometric series formula,

$$\frac{3}{1-x^4} = \sum_{n=0}^{\infty} 3(x^4)^n = \sum_{n=0}^{\infty} 3x^{4n}.$$

Using the Ratio Test,

$$\lim_{n\to\infty} \left| \frac{3x^{4(n+1)}}{3x^{4n}} \right| = \lim_{n\to\infty} |x|^4,$$

which is less than 1 when $|x| < 1$. Checking the endpoints, we see that when $x = 1$, the series is

$$\sum_{n=0}^{\infty} 3\,(1)^{4n} = \sum_{n=0}^{\infty} 3,$$

which diverges. When $x = -1$, the series is

$$\sum_{n=0}^{\infty} 3\,(-1)^{4n} = \sum_{n=0}^{\infty} 3,$$

which diverges. Therefore, the interval of convergence is

$$(-1, 1)\,.$$

10. Find a power series representation for the function

$$f(x) = \frac{x^2}{a^3 - x^3}$$

and determine the interval of convergence.

Answer: Re-writing f as

$$f(x) = x^2 \left(\frac{1}{a^3 - x^3} \right) = \frac{x^2}{a^3} \left(\frac{1}{1 - \frac{x^3}{a^3}} \right),$$

we can use the geometric series to see that

$$f(x) = \frac{x^2}{a^3} \sum_{n=0}^{\infty} \left(\frac{x^3}{a^3} \right)^n = \frac{x^2}{a^3} \sum_{n=0}^{\infty} \frac{x^{3n}}{a^{3n}} = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{a^{3n+3}}.$$

Using the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{\frac{x^{3(n+1)+2}}{a^{3(n+1)+3}}}{\frac{x^{3n+2}}{a^{3n+3}}}\right| = \frac{|x|^3}{a^3},$$

which is less than 1 when $|x| < a$. Checking the endpoints, when $x = a$, the series is

$$\sum_{n=0}^{\infty}\frac{a^{3n+2}}{a^{3n+3}} = \sum_{n=0}^{\infty}\frac{1}{a},$$

which diverges. When $x = -a$, the series is

$$\sum_{n=0}^{\infty}\frac{(-a)^{3n+2}}{a^{3n+3}} = \sum_{n=0}^{\infty}(-1)^{3n+2}\frac{1}{a},$$

which also diverges. Therefore, the interval of convergence is

$$(-a, a).$$

16. Find a power series representation for the function

$$f(x) = \frac{x^2}{(1-2x)^2}$$

and determine the radius of convergence.

Answer: Write $f(x)$ as

$$f(x) = x^2\left(\frac{1}{(1-2x)^2}\right).$$

Therefore,

$$f(x) = x^2\sum_{n=1}^{\infty}n(2x)^{n-1} = x^2\sum_{n=1}^{\infty}2^{n-1}nx^{n-1} = \sum_{n=1}^{\infty}2^{n-1}nx^{n+1}.$$

Using the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{2^n(n+1)x^{n+2}}{2^{n-1}nx^{n+1}}\right| = \lim_{n\to\infty}2|x|\frac{n+1}{n} = 2|x|\lim_{n\to\infty}\frac{n+1}{n} = 2|x|,$$

which is less than 1 when $|x| < 1/2$. Therefore, the radius of convergence is $1/2$.

24. Evaluate the indefinite integral

$$\int\frac{\ln(1-t)}{t}dt$$

as a power series. What is the radius of convergence?

Answer: From Example 6, we know that the power series for $\ln(1-t)$ is

$$-t - \frac{t^2}{2} - \frac{t^3}{3} - \ldots = -\sum_{n=1}^{\infty}\frac{t^n}{n}.$$

Therefore, the series for $\frac{\ln(1-t)}{t}$ is

$$\frac{1}{t}\left(-t-\frac{t^2}{2}-\frac{t^3}{3}-\ldots\right)=-1-\frac{t}{2}-\frac{t^2}{3}-\ldots=-\sum_{n=0}^{\infty}\frac{t^n}{n+1}.$$

Therefore,

$$\int\frac{\ln(1-t)}{t}dt=\int\left(-1-\frac{t}{2}-\frac{t^2}{3}-\ldots\right)dt=C-t-\frac{t^2}{4}-\frac{t^3}{9}-\ldots=C-\sum_{n=1}^{\infty}\frac{t^n}{n^2}.$$

Using the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{\frac{t^{n+1}}{(n+1)^2}}{\frac{t^n}{n^2}}\right|=|t|\lim_{n\to\infty}\frac{n^2}{n^2+2n+1}=|t|,$$

so the radius of convergence is 1.

From §12.10

8. Find the Maclaurin series for

$$f(x)=\cos 3x$$

using the definition of a Maclaurin series. Also find the associated radius of convergence.

Answer: We compute the first few derivatives:

$$\begin{aligned}f'(x)&=-3\sin 3x\\ f''(x)&=-9\cos 3x\\ f'''(x)&=27\sin 3x\\ f^{(4)}(x)&=81\cos 3x\\ &\vdots\end{aligned}$$

Therefore,

$$\begin{aligned}f(0)&=1\\ f'(0)&=0\\ f''(0)&=-9\\ f'''(0)&=0\\ f^{(4)}(0)&=81\\ &\vdots\end{aligned}$$

So, by the definition of the Maclaurin series,

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\ldots=1-\frac{9}{2}x^2+\frac{81}{4}x^4-\ldots=\sum_{n=0}^{\infty}(-1)^n\frac{3^{2n}}{(2n)!}x^{2n}.$$

Using the Ratio Test,

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{3^{2n+2}}{(2n+2)!}x^{2n+2}}{(-1)^n\frac{3^{2n}}{(2n)!}x^{2n}}\right| = \lim_{n\to\infty}\frac{3^2}{(2n+2)(2n+1)}|x|^2 = |x|^2\lim_{n\to\infty}\frac{9}{4n^2+6n+2} = 0,$$

so this series always converges. Therefore, the radius of convergence is ∞.

16. Find the Taylor series for

$$f(x) = \frac{1}{x}$$

centered at $a = -3$.

Answer: Note that

$$\begin{aligned} f'(x) &= -\frac{1}{x^2} \\ f''(x) &= \frac{2}{x^3} \\ f'''(x) &= -\frac{6}{x^4} \\ f^{(4)}(x) &= \frac{24}{x^5} \\ &\vdots \\ f^{(n)}(x) &= (-1)^n\frac{n!}{x^{n+1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} f(-3) &= -\frac{1}{3} \\ f'(-3) &= -\frac{1}{9} \\ f''(-3) &= -\frac{2}{27} \\ f'''(-3) &= -\frac{6}{81} \\ f^{(4)}(-3) &= -\frac{24}{243} \\ &\vdots \\ f^{(n)}(3) &= -\frac{n!}{3^{n+1}}. \end{aligned}$$

Hence, by the definition of the Taylor series,

$$f(x) = -\sum_{n=0}^{\infty}\frac{(x+3)^n}{3^{n+1}}$$

is the Taylor series for $f(x) = \frac{1}{x}$ centered at 3.

34. Use a Maclaurin series in Table 1 to obtain the Maclaurin series for the function

$$f(x) = x^2 \tan^{-1}\left(x^3\right)$$

Answer: From Table 1, we know that the Maclaurin series for $\tan^{-1} x$ is

$$\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

Therefore, the series for $\tan^{-1}(x^3)$ is

$$\tan^{-1}(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{2n+1} = x^3 - \frac{x^9}{3} + \frac{x^{15}}{5} - \frac{x^{21}}{7} + \dots$$

In turn, this means that the series for f is

$$x^2 \tan^{-1}(x^3) = x^2 \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+5}}{2n+1} = x^5 - \frac{x^{11}}{3} + \frac{x^{17}}{5} - \frac{x^{23}}{7} + \dots$$

48. Evaluate the indefinite integral

$$\int \frac{e^x - 1}{x} dx$$

as an infinite series.

Answer: The Maclaurin series for e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots,$$

so the series for the numerator is

$$e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

In turn, this means the series for the integrand is

$$\frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} = 1 + \frac{x}{2!} + \frac{x^2}{3!} + \dots$$

Therefore, we can integrate term-by-term to get

$$\int \frac{e^x - 1}{x} dx = C + x + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{3 \cdot 3!} + \dots = C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$

56. Use series to evaluate the limit

$$\lim_{x \to 0} \frac{1 - \cos x}{1 + x - e^x}.$$

Answer: Using the Maclaurin series for $\cos x$ we can write the numerator as the series

$$1-\cos x = 1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\dots\right)=\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\dots.$$

Using the Maclaurin series for e^x, we can write the denominator as

$$1+x-e^x = 1+x-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right)=-\frac{x^2}{2!}-\frac{x^3}{3!}-\dots$$

Therefore,

$$\lim_{x\to 0}\frac{1-\cos x}{1+x-e^x}=\lim_{x\to 0}\frac{\frac{x^2}{2!}-\frac{x^4}{4!}+\dots}{-\frac{x^2}{2!}-\frac{x^3}{3!}-\dots}$$

Dividing both numerator and denominator by x^2, this is equal to

$$\lim_{x\to 0}\frac{\frac{1}{2!}-\frac{x^2}{4!}+\dots}{-\frac{1}{2!}-\frac{x}{3!}-\dots}=\frac{\frac{1}{2!}}{-\frac{1}{2!}}=-1$$

68. Find the sum of the series

$$1-\ln 2+\frac{(\ln 2)^2}{2!}-\frac{(\ln 2)^3}{3!}+\dots$$

Answer: Notice that

$$e^{-\ln 2}=1-\ln 2+\frac{(\ln 2)^2}{2!}-\frac{(\ln 2)^3}{3!}+\dots$$

is just the given series, so the sum of the series is

$$e^{-\ln 2}=\frac{1}{e^{\ln 2}}=\frac{1}{2}.$$

From §12.11

16. (a) Approximate

$$f(x)=\sin x$$

by a Taylor polynomial with degree 4 at the number $\pi/6$

Answer: The first four derivatives of f are

$$f'(x)=\cos x$$
$$f''(x)=-\sin x$$
$$f'''(x)=-\cos x$$
$$f^{(4)}(x)=\sin x$$

so we have

$$\begin{aligned} f(\pi/6) &= 1/2 \\ f'(\pi/6) &= \sqrt{3}/2 \\ f''(\pi/6) &= -1/2 \\ f'''(\pi/6) &= -\sqrt{3}/2 \\ f^{(4)}(\pi/6) &= 1/2. \end{aligned}$$

Therefore, the degree 4 Taylor polynomial for f at $\pi/6$ is

$$\frac{1}{2} + \frac{\sqrt{3}}{2}(x - \pi/6) - \frac{1}{2 \cdot 2!}(x - \pi/6)^2 - \frac{\sqrt{3}}{2 \cdot 3!}(x - \pi/6)^3 + \frac{1}{2 \cdot 4!}(x - \pi/6)^4.$$

(b) Use Taylor's Inequality to estimate the accuracy of the approximation $f(x) = T_4(x)$ when x lies in the interval $0 \leq x \leq \pi/3$.

Answer: When $0 \leq x \leq \pi/3$, Taylor's Inequality says that the remainder $R_4(x)$ is bounded by

$$|R_4(x)| \leq \frac{M}{5!}|x - \pi/6|^5$$

where M is an upper bound on $f^{(5)}$ in this interval. Since

$$f^{(5)}(x) = \cos x$$

and since

$$\frac{1}{2} \leq \cos x \leq 1$$

for all x between 0 and $\pi/3$, we can pick $M = 1$.

Therefore,

$$|R_4(x)| \leq \frac{1}{5!}|x - \pi/6|^5 = \frac{1}{120}|x - \pi/6|^5.$$

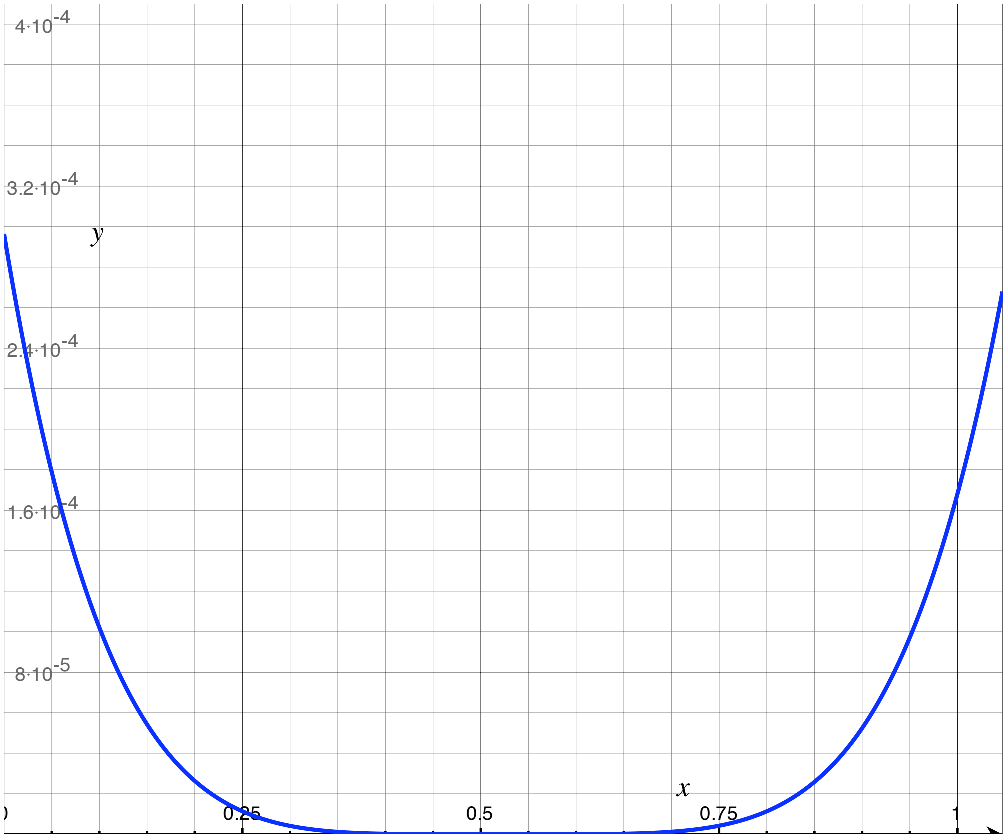
When $0 \leq x \leq \pi/3$, the quantity $|x - \pi/6| \leq \pi/6$, so

$$\frac{1}{240}|x - \pi/6|^5 \leq \frac{1}{120}(\pi/6)^5 = \frac{\pi^5}{120 \cdot 7776} \approx 0.000328$$

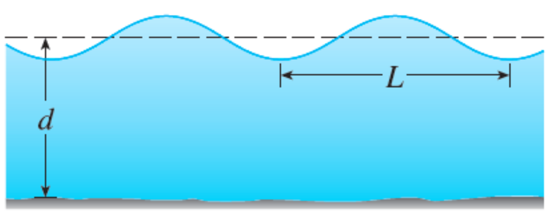
is the worst the error could possibly be on this interval.

(c) Check your result in (b) by graphing $|R_4(x)|$.

Answer:



35.



If a water wave with length L moves with velocity v across a body of water with depth d, as in the figure, then

$$v^2 = \frac{gL}{2\pi} \tanh \frac{2\pi d}{L}$$

(a) If the water is deep, show that $v \approx \sqrt{gL/(2\pi)}$.

Answer: As $u \to \infty$, $\tanh u \to 1$, so, when the water is very deep,

$$v^2 \approx \frac{gL}{2\pi},$$

meaning that

$$v \approx \sqrt{\frac{gL}{2\pi}},$$

as desired.

(b) If the water is shallow, use the Maclaurin series for tanh to show that $v \approx \sqrt{gd}$. (Thus in shallow ater the velocity of a wave tends to be independent of the length of the wave.)

Answer: The first few derivatives of $f(x) = \tanh x$ are

$$\begin{aligned} f'(x) &= \operatorname{sech}^2 x \\ f''(x) &= -2 \operatorname{sech} x \operatorname{sech} x \tanh x = -2 \operatorname{sech}^2 x \tanh x \\ f'''(x) &= -4 \operatorname{sech} x \operatorname{sech} x \tanh x \tanh x - 2 \operatorname{sech}^2 x \operatorname{sech}^2 x = -4 \operatorname{sech}^2 x \tanh^2 x - 2 \operatorname{sech}^4 x \end{aligned}$$

Therefore,

$$\begin{aligned} f(0) &= \tanh 0 = 0 \\ f'(0) &= \operatorname{sech}^2 0 = 1 \\ f''(0) &= -2 \operatorname{sech}^2 0 \tanh 0 = 0 \\ f'''(0) &= -4 \operatorname{sech}^2 0 \tanh^2 0 - 2 \operatorname{sech}^4 0 = -2 \end{aligned}$$

Hence,

$$\tanh x = x - \frac{x^3}{3} + \dots$$

This gives a Maclaurin series for v^2:

$$\begin{aligned} v^2 &= \frac{gL}{2\pi} \left(\frac{2\pi d}{L} - \frac{\left(\frac{2\pi d}{L}\right)^3}{3} + \dots \right) \\ &= gd - \frac{gL}{6\pi} \left(\frac{2\pi d}{L} \right)^3 + \dots \end{aligned}$$

When d is small, $\frac{2\pi d}{L}$ is also small and higher powers of it are even smaller. Therefore, the first term above gives a good approximation for v^2 when d is small. Thus,

$$v \approx \sqrt{gd}.$$

(c) Use the Alternating Series Estimation Theorem to show that if $L > 10d$, then the estimate $v^2 \approx gd$ is accurate to within $0.014gL$.

Answer: The error is no bigger than the first unused term in the series:

$$|\text{error}| \leq \frac{gL}{6\pi} \left(\frac{2\pi d}{L} \right)^3.$$

When $L > 10d$, the right hand side (and, thus, the error) is smaller than

$$\frac{gL}{6\pi} \left(\frac{2\pi d}{10d} \right)^3 = \frac{gL}{6\pi} \left(\frac{\pi}{5} \right)^3 = \frac{\pi^3}{6\pi \cdot 5^3} gL = \frac{\pi^2}{750} gL \approx 0.013gL$$