Math 128A: Homework 8 Solutions

Due: August 8

1. Show for any induced matrix norm that

   (a) if I is the identity matrix, then $\|I\| = 1$.

   By definition, we have

   $$\|I\| = \max_{\|\mathbf{x}\|=1} \|I\mathbf{x}\| = \max_{\|\mathbf{x}\|=1} \|\mathbf{x}\| = 1.$$

   (b) if A is invertible, then $\|A^{-1}\| \geq \|A\|^{-1}$

   From (a), we have

   $$AA^{-1} = I \Rightarrow \|AA^{-1}\| = \|I\| \Rightarrow \|A\|\|A^{-1}\| \geq 1$$

   whence it follows that $\|A^{-1}\| \geq \|A\|^{-1}$.

   (c) if $\|A - I\| < 1$, then A is invertible.

   Assume to the contrary that A is not invertible. Then, there exists a non-zero $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = \mathbf{0}$. Note then that

   $$(A - I)\mathbf{x} = A\mathbf{x} - \mathbf{x} = -\mathbf{x} \Rightarrow \|(A_I)\mathbf{x}\| = \|\mathbf{x}\| \Rightarrow \frac{\|(A - I)\mathbf{x}\|}{\|\mathbf{x}\|} = 1.$$

   It follows from the definition of an induced norm that $\|A - I\| \geq 1$, a contradiction. We conclude that A must be invertible.

2. The Frobenius norm is defined for an $n \times n$ matrix A by

   $$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}.$$

   (a) Show that $\|A\mathbf{x}\|_2 \leq \|A\|_F \|\mathbf{x}\|_2$ for any $\mathbf{x} \in \mathbb{R}^n$.

We have

$$\begin{aligned}
\|A\mathbf{x}\|_2^2 &= \sum_{i=1}^{n} |(A\mathbf{x})_i|^2 \\
&= \sum_{i=1}^{n} \left| \sum_{j=1}^{n} A_{ij} x_j \right|^2 \\
&\leq \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |A_{ij}|^2 \right) \left(\sum_{k=1}^{n} |x_k|^2 \right) \qquad (\because \text{Cauchy-Schwarz Ineq.}) \\
&= \|A\|_F^2 \|\mathbf{x}\|_2^2
\end{aligned}$$

so that $\|A\mathbf{x}\|_2 \leq \|A\|_F \|\mathbf{x}\|_2$.

(b) Show that $\|AB\|_F \leq \|A\|_F \|B\|_F$ for any two $n \times n$ matrices A and B.

We have

$$\begin{aligned}
\|AB\|_F^2 &= \sum_{i=1}^{n} \sum_{j=1}^{n} |(AB)_{ij}|^2 \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left| \sum_{k=1}^{n} A_{ik} B_{kj} \right|^2 \\
&\leq \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\sum_{k=1}^{n} |A_{ik}|^2 \right) \left(\sum_{l=1}^{n} |B_{lj}|^2 \right) \qquad (\because \text{Cauchy-Schwarz Ineq.}) \\
&= \left(\sum_{i=1}^{n} \sum_{k=1}^{n} |A_{ik}|^2 \right) \left(\sum_{j=1}^{n} \sum_{l=1}^{n} |B_{lj}|^2 \right) \\
&= \|A\|_F^2 \|B\|_F^2
\end{aligned}$$

whence we conclude that $\|AB\|_F \leq \|A\|_F \|B\|_F$.

This shows that even though the Frobenius norm isn't an induced norm, it still has many of the nice properties that induced norms possess.

3. Show that if $\|\cdot\|$ is a norm on $\mathbb{R}^n$ and A is an invertible matrix, then $x \mapsto \|Ax\|$ is also a norm on $\mathbb{R}^n$.

Observe that

(i) As $\|\cdot\|$ is a norm, we have $\|A\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

(ii) Suppose that $\|A\mathbf{x}\| = 0$ for some $\mathbf{x} \in \mathbb{R}^n$. As $\|\cdot\|$ is a norm, it follows that $A\mathbf{x} = \mathbf{0}$. As A is invertible, we must have $\mathbf{x} = \mathbf{0}$.
Conversely, note that $\|A\mathbf{0}\| = \|\mathbf{0}\| = 0$.

(iii) Let $a \in \mathbb{R}$. We then have

$$\|A(a\mathbf{x})\| = \|a(A\mathbf{x})\| = |a|\|A\mathbf{x}\|$$

as $\|\cdot\|$ is a norm.

(iv) Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. We then have

$$\|A(\mathbf{x} + \mathbf{y})\| = \|A\mathbf{x} + A\mathbf{y}\| \leq \|A\mathbf{x}\| + \|A\mathbf{y}\|$$

where we used the fact that $\|\cdot\|$ satisfies the triangle inequality as it is a norm.

As all the properties hold, we conclude that $x \mapsto \|Ax\|$ is also a norm on $\mathbb{R}^n$.

4. A square matrix A is said to be *orthogonal* if $A^T A = I$. Show that if a matrix is orthogonal and triangular, it must be diagonal. What are the diagonal entries?

   It follows from the definition of orthogonality that $A^{-1} = A^T$. Recall that the inverse of a lower triangular matrix must also be lower triangular. Thus, A^T is both lower and upper (as it is the transpose of a lower triangular matrix) triangular; we conclude that A^T, and hence A, is diagonal.

   Let d_i be the ith diagonal entry of A. Then, $A^T = A$ so the ith diagonal entry of $A^T A$ is d_i^2. As $A^T A$ also equals the identity matrix, we have $d_i^2 = 1 \Rightarrow \boxed{d_i = \pm 1}$.

5. A matrix is strictly upper triangular if it is upper triangular with zero diagonal elements. Show that if A is an $n \times n$ strictly upper triangular matrix, then $A^n = 0$.

   We prove that for $1 \leq k \leq n$, the diagonal and the first $(k-1)$ super-diagonals of A^k have only zero elements, i.e., $(A^k)_{ij} = 0$ for $j - i \leq k - 1$. Note that the statement holds for $k = 1$: we are given that the diagonal of A is full of zeros.

   Assuming now that it holds for some $k = k_0$, we have

$$A^{k_0+1} = AA^{k_0} \Rightarrow (A^{k_0+1})_{ij} = \sum_{l=1}^{n} A_{il}(A^{k_0})_{lj}. \tag{1}$$

   Observe that

   (i) $A_{il} \neq 0$ only if $l - i \geq 1 \Rightarrow l \geq i + 1$;

   (ii) $(A^{k_0})_{lj} \neq 0$ only if $j - l \geq k_0 \Rightarrow l \leq j - k_0$.

   It follows that in (1), the entry $(A^{k_0+1})_{ij}$ is non-zero only if

$$i + 1 \leq j - k_0 \Rightarrow j - i \geq k_0 + 1.$$

   In other words, $(A^{k_0+1})_{ij} = 0$ if $j - i \leq k_0$. This shows that the claim holds for $k = k_0 + 1$ if it holds for $k = k_0$.

   It follows from the principle of induction that $(A^k)_{ij} = 0$ for $j - i \leq k - 1$. In particular, for $k = n$, we have $(A^n)_{ij} = 0$ if $j - i \leq n - 1$; as this inequality is satisfied by all $i, j \in \{1, 2, \ldots, n\}$, we conclude that $A^n = 0$.

6. For the following pairs of A and $\mathbf{b}$, find (i) the LU factorization of PA where P is an appropriate permutation matrix; (ii) the determinant of A; (iii) the solution to $A\mathbf{x} = \mathbf{b}$.

(a) $A = \begin{pmatrix} 2 & -1 & 2 \\ -6 & 3 & 0 \\ 1 & 5 & -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 10 \\ 0 \\ 1 \end{pmatrix}$.

(i) We have

$$\begin{matrix} (R_2 - (-3)R_1 \to R_2) \\ (R_3 - (1/2)R_1 \to R_3) \end{matrix} \longrightarrow \begin{pmatrix} 2 & -1 & 2 \\ 0 & 0 & 6 \\ 0 & 11/2 & -2 \end{pmatrix}$$

$$(R_2 \leftrightarrow R_3) \longrightarrow \begin{pmatrix} 2 & -1 & 2 \\ 0 & 11/2 & -2 \\ 0 & 0 & 6 \end{pmatrix}$$

We conclude that $PA = LU$ where $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$ and $U = \begin{pmatrix} 2 & -1 & 2 \\ 0 & 11/2 & -2 \\ 0 & 0 & 6 \end{pmatrix}$.

(ii) We have $\det(PA) = \det(LU)$ so that

$$\det(P)\det(A) = \det(L)\det(U) \Rightarrow -\det(A) = (1)(66) \Rightarrow \boxed{\det(A) = -66}.$$

(iii) Note that $A\mathbf{x} = \mathbf{b} \Rightarrow PA\mathbf{x} = P\mathbf{b} \Rightarrow LU\mathbf{x} = P\mathbf{b}$. Let $\mathbf{y} = U\mathbf{x}$; we then have $L\mathbf{y} = P\mathbf{b}$, i.e.,

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 10 \\ 1 \\ 0 \end{pmatrix}.$$

Forward Substitution then yields $y_1 = 10 \Rightarrow y_2 = -4$ and $y_3 = 30$. We next solve $U\mathbf{x} = \mathbf{y}$:

$$\begin{pmatrix} 2 & -1 & 2 \\ 0 & 11/2 & -2 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 10 \\ -4 \\ 30 \end{pmatrix}.$$

Back Substitution gives $\boxed{x_3 = 5}$, $\boxed{x_2 = \frac{12}{11}}$ and $\boxed{x_1 = \frac{6}{11}}$.

(b) $A = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 2 & 4 & 5 \\ 1 & -1 & 1 & 7 \\ 2 & 3 & 4 & 6 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} -2 \\ 13 \\ -3 \\ 13 \end{pmatrix}$.

(i) We have

$$\begin{array}{c}(R_2-(2)R_1\to R_2)\\(R_3-(1)R_1\to R_3)\\(R_4-(2)R_1\to R_4)\end{array}\longrightarrow\begin{pmatrix}1&1&-1&2\\0&0&6&1\\0&-2&2&5\\0&1&6&2\end{pmatrix}$$

$$(R_2\leftrightarrow R_4)\longrightarrow\begin{pmatrix}1&1&-1&2\\0&1&6&2\\0&-2&2&5\\0&0&6&1\end{pmatrix}$$

$$(R_3-(-2)R_2\to R_3)\longrightarrow\begin{pmatrix}1&1&-1&2\\0&1&6&2\\0&0&14&9\\0&0&6&1\end{pmatrix}$$

$$(R_4-(3/7)R_3\to R_4)\longrightarrow\begin{pmatrix}1&1&-1&2\\0&1&6&2\\0&0&14&9\\0&0&0&-20/7\end{pmatrix}$$

We conclude that $PA=LU$ where $P=\begin{pmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{pmatrix}$, $L=\begin{pmatrix}1&0&0&0\\2&1&0&0\\1&-2&1&0\\2&0&3/7&1\end{pmatrix}$

and $U=\begin{pmatrix}1&1&-1&2\\0&1&6&2\\0&0&14&9\\0&0&0&-20/7\end{pmatrix}$.

(ii) We have $\det(PA)=\det(LU)$ so that

$$\det(P)\det(A)=\det(L)\det(U)\Rightarrow-\det(A)=(1)(-40)\Rightarrow\boxed{\det(A)=40}.$$

(iii) Note that $A\mathbf{x}=\mathbf{b}\Rightarrow PA\mathbf{x}=P\mathbf{b}\Rightarrow LU\mathbf{x}=P\mathbf{b}$. Let $\mathbf{y}=U\mathbf{x}$; we then have $L\mathbf{y}=P\mathbf{b}$, i.e.,

$$\begin{pmatrix}1&0&0&0\\2&1&0&0\\1&-2&1&0\\2&0&3/7&1\end{pmatrix}\begin{pmatrix}y_1\\y_2\\y_3\\y_4\end{pmatrix}=\begin{pmatrix}-2\\13\\-3\\13\end{pmatrix}.$$

Forward Substitution then yields $y_1=-2\Rightarrow y_2=17$, $y_3=33$ and $y_4=20/7$. We next solve $U\mathbf{x}=\mathbf{y}$:

$$\begin{pmatrix}1&1&-1&2\\0&1&6&2\\0&0&14&9\\0&0&0&-20/7\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}=\begin{pmatrix}-2\\17\\33\\20/7\end{pmatrix}.$$

Back Substitution gives $\boxed{x_4=-1}$, $\boxed{x_3=3}$, $\boxed{x_2=1}$ and $\boxed{x_1=2}$.

7. For the following numerical schemes, find the amplification factor $R(z)$ and determine if they are A-stable.

(a) The three stage Runge-Kutta method

$$\begin{array}{c|ccc} 0 & 0 & & \\ 1/2 & 1/2 & & \\ 1 & -1 & 2 & \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

We apply this method to the problem $y'(t) = \lambda y$. We have

$$\begin{aligned} k_1 &= \lambda u_i \\ k_2 &= \lambda \left(u_i + \frac{h}{2} k_2 \right) = \lambda u_i \left(1 + \frac{h\lambda}{2} \right) \\ k_3 &= \lambda \left(u_i - hk_1 + 2hk_2 \right) = \lambda u_i \left(1 - h\lambda + 2h\lambda \left(1 + \frac{h\lambda}{2} \right) \right) \\ u_{i+1} &= u_i + \frac{h}{6}(k_1 + 4k_2 + k_3) \end{aligned}$$

Note that

$$k_1 + 4k_2 + k_3 = \lambda u_i \left(6 + 3h\lambda + (h\lambda)^2 \right)$$

so that

$$u_{i+1} = u_i + u_i \frac{h\lambda}{6} \left(6 + 3h\lambda + (h\lambda)^2 \right) = u_i \left(1 + h\lambda + \frac{(h\lambda)^2}{2} + \frac{(h\lambda)^3}{6} \right).$$

We conclude that the amplification factor is

$$\boxed{R(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6}}.$$

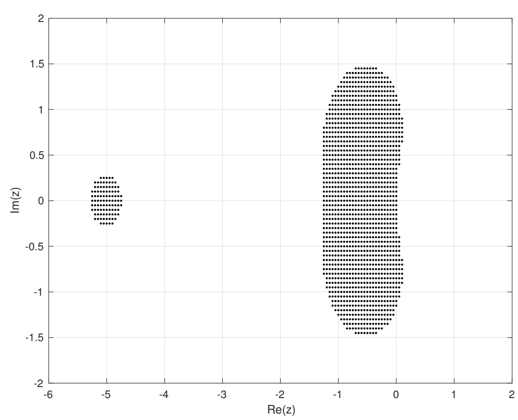

Figure 1: The region of absolute stability for 7(a)

This method would be A-stable if $|R(z)| \leq 1$ whenever z is a complex number with a negative real part. Note however that $|R(-3)| = 2$ so this method fails to be A-stable. See Figure 1 for the region of absolute stability

(b) The implicit method

$$u_{i+1} = u_i + \frac{h}{4}(f(t_i, u_i) + 3f(t_{i+1}, u_{i+1}))$$

We apply this method to the problem $y'(t) = \lambda y$. We have

$$\begin{aligned} u_{i+1} &= u_i + \frac{h}{4}(\lambda u_i + 3\lambda u_{i+1}) \\ 4u_{i+1} &= 4u_i + h\lambda(u_i + 3u_{i+1}) \\ (4 - 3\lambda h)u_{i+1} &= (4 + \lambda h)u_i \\ u_{i+1} &= \left(\frac{4 + \lambda h}{4 - 3\lambda h}\right) u_i. \end{aligned}$$

We conclude that the amplification factor is $\boxed{R(z) = \frac{4+z}{4-3z}}$.

The region of absolute stability for this method is shown in Figure 2.

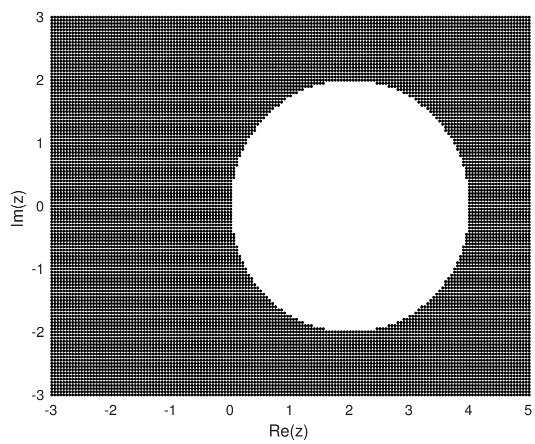

Figure 2: The region of absolute stability for 7(b)

The diagram suggests that this method is A-stable. For a definitive proof, note that

(i) the only pole of $R(z)$ is at $z = 4/3$ which does not lie in the left half plane.

(ii) for any $b \in \mathbb{R}$, we have

$$|R(ib)| = \frac{|4 + ib|}{|4 - 3bi|} = \sqrt{\frac{4 + b^2}{4 + 9b^2}} \leq 1.$$

We conclude that $|R(z)| \leq 1$ for all $z \in \mathbb{C}^-$, i.e., the method is indeed A-stable.