

Math 139: Plane Analytic  
Geometry  
Notes and Problems

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## Introduction

In this course you will learn about geometry by solving a carefully designed sequence of problems. It is important that you understand every problem. As hard as it is to imagine, you will occasionally want to have more questions to work on in order to fully understand the ideas in this class. Feel free to ask me about additional problems to consider when you are stuck. The notes and problems will refer back to previous parts, so I suggest you keep a binder with the notes and your work together and bring all of these materials to class and any office hours.

Unlike mathematics courses you have had in the past, solving a problem in this course will **always** have two components:

- Find a solution
- Explain how you know that your solution is correct.

This will help prepare you to use mathematics in the future, when there will not be someone to tell you if your solution is correct or not. That is also why it will be primarily up to you (the students) to assess the correctness of work done and presented in class. This means that using the proper language (both written and spoken) and specificity are keys to effective communication. This class will teach you about the specificity and precision of mathematical language, so it is important that you practice and work on this. In order for you to understand the ideas in this class you will need to evaluate other people's ideas as well as your own for correctness.

The work in this class is *not about getting an answer* but rather **making sense** of the process and ideas involved in the problems. For this reason, justification in your work and ideas is very important. Why you did or tried something is just as important as what you did or what result you got. In fact, clearly articulating your thought process will make you a more efficient thinker.

You are not alone in this course. The role of the instructor is to guide the discussion and make sure you have the resources to be successful. While this new learning environment can be a bit unsettling for students at first,

you will get comfortable as you do more problems and get feedback from other students and the instructor. I am also here to help you outside of class time and expect you to find a way to get the help you need, whether face to face or over email. You will find that once you have thought about a problem for a little while, it will only take a little push to get unstuck.

Some Notation:

- $\mathbb{N} = \{0, 1, 2, 3, \dots\}$  is the set of natural numbers.
- $\mathbb{Z} = \{\dots - 3, -2, -1, 0, 1, 2, 3, \dots\}$  is the set of integers.
- $\mathbb{R}$  is the set of real numbers.
- $\mathbb{R}^2$  is the cartesian plane.
- $\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$  is the set of complex numbers.
- DNMS is an acronym for 'Does not make sense'.
- DNE is an acronym for 'Does not exist'.
- #pf describes productive failure

Definitions will be bolded for some terms and others will have their own heading and number.

Many students use Desmos as a graphing tool and we will occasionally use Desmos in class. You can set up an account and download Desmos at <https://www.desmos.com/> .

# Chapter 1

## Algebra and Geometry

### 1.1 Algebra Problems

You have probably had several algebra classes (for some you, it was last semester) where you learned a bunch of rules about how you were allowed to move symbols on a page. Algebra is a whole lot more useful than just memorizing how to do a bunch a problems. One of the most useful ways we will use algebra in this course is to trade the problem we have for one that is more familiar or similar to a problem we have already solved. In this way, it is as important to know what you are allowed to do in a symbolic setting as it is to understand how your problem relates to the algebra. Something we will emphasize this semester is that it should make sense why you are doing the algebra. In particular, you need to have a goal in mind before you start doing algebra. Sometimes we will be using notation that is familiar to you and other times we will be using notation that you have not seen before. Since this varies greatly from student to student, it is important for **you** to make sure you know what the symbols on the page mean. In the same way you wouldn't want to try to understand the plot of a story written in French until you could read French, you don't want to try to understand why we are doing some math without knowing what the symbols on the page mean. We will talk more about each of these ideas throughout the semester.

For our first set of problems, we will look at some basic ideas of how to transform English sentence descriptions of math into an equation and viceversa. We say that an equation makes a true statement if the right-hand side of the equation is equal to the left-hand side of the equation. For instance, the equation  $2=1+1$  is true and the equation  $2=0$  is false.

**Question 1.** *Write an equation that describes the following sentence (Make sure to consider which side of the equation each expression should go on):*

*The sum of the squares of two numbers is the square of the sum of the two numbers.*

**Question 2.** a) Give values for two numbers that shows an example of when the statement in Question 1 is false.

b) Give values for two numbers that shows an example of when the statement in Question 1 is true.

**Question 3.** Write an equation that describes the following sentence (Make sure to consider which side of the equation each expression should go on):

*The difference of the squares of two numbers is the square of the differences of the two numbers.*

**Question 4.** a) Give values for two numbers that shows an example of when the statement in Question 3 is false.

b) Give values for two numbers that shows an example of when the statement in Question 3 is true.

Part of what make equations and algebra useful is that they can help us understand under what conditions a statement is true and when that statement is false. The previous questions shows how for different values of variables in our equations, the equation may be true or false. Equations that are true no matter what values are used are called identities (remember things like  $2a = a + a$  or  $\sin(\theta)^2 + \cos(\theta)^2 = 1$ ). While identities can be very useful, you spent a lot more of your mathematical career dealing with equations that are only true for some variable values. A collection of variable values that makes an equation true is called a **solution** to the equation.

**Question 5.** For each of the following equations, state how many variable values are needed to give a solution (you do not need to give a solution):

a)  $3x + 1 = 0$

b)  $3x + 1 = y$

c)  $x^2 + y^2 = 1$

d)  $\clubsuit - 2\diamond - 4 = 3\heartsuit + \spadesuit$

For the next several problems, we will be looking at equations that are not identities (they are not always true). In fact, the following equations are based off of common algebraic errors.

**Question 6.** For each of the following equations, give a set of variable values that is not a solution, i.e. give values for  $a$ ,  $b$ ,  $c$ , and,  $d$  such that each of the following statements is false. Your choices may be different for each part and you may want to do some algebra to simplify the expressions to make it easier to choose your variable values.

a)  $a(b + c) = ab + c$

b)  $a(b + c)^2 = (ab + ac)^2$

c)  $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$

d)  $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

e)  $\frac{a+b}{c+b} = \frac{a}{c}$

f)  $\sqrt{a^2 + b^2} = a + b$

g)  $c\sqrt{a+b} = \sqrt{ca+cb}$

**Question 7.** For each of the following equations, give a set of variable values that is a solution, i.e. give values for  $a$ ,  $b$ ,  $c$ , and,  $d$  such that each of the following statements is true.

a)  $a(b + c) = ab + c$

b)  $a(b + c)^2 = (ab + ac)^2$

c)  $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$

d)  $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$

e)  $\frac{a+b}{c+b} = \frac{a}{c}$

f)  $\sqrt{a^2 + b^2} = a + b$

g)  $c\sqrt{a+b} = \sqrt{ca+cb}$

**Information 8.** Sets are collections of objects which you can think of as a way to list what is in the collection and what is not in the collection. You have probably seen sets before in math classes but we will go over a couple of ways to write them. You know you are working with a set when you see the brace that looks like  $\{$  or  $\}$ . The natural numbers can be written as  $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ . Notice that you don't write out all of the numbers in  $\mathbb{N}$  since that would take too long. Similarly, you may have seen the integers written as  $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ . Basically you are writing out enough of the set in a patterned way that someone else would be able to

say what is in the set and what is not. The symbol  $\in$  stands for "is an element of the set". For instance,  $2 \in \mathbb{N}$  would be read as "the number two is an element of the set of natural numbers" or more simply "two is in the natural numbers."

**Question 9.** State whether each statement is true or false:

a)  $0 \in \mathbb{Z}$

b)  $-2 \in \mathbb{N}$

c)  $-9 \in \mathbb{Z}$

d)  $4 - 3 \in \mathbb{N}$

e)  $\frac{1}{2} \in \mathbb{Z}$

f)  $\sqrt{4} \in \mathbb{N}$

**Information 10.** *Set builder notation* is a very convenient way to describe sets since it does not rely on listing out all of the elements of the set, which usually can't be done. In general set builder notation looks like:

$\{ \text{the kinds of objects that are in the set} \mid \text{restrictions on the description} \}$

For instance,  $A = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$  would be read as "A is the set of ordered pairs of real numbers  $(x, y)$  such that  $x$  is equal to  $y$ ". The set  $A$  can be written much more simply as  $\{(a, a) \mid a \in \mathbb{R}\}$  and can be described as "the set of points in the plane that have the same horizontal and vertical coordinate." You can already see that there is not just one way to write sets but they should all describe the set collection. Note that the first part of set builder notation tells what kind of objects are in the set and the second part tells us any restrictions. A couple of other common sets we will deal with are the rational numbers ( $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z} \& n \neq 0\}$ ) and the real numbers ( $\mathbb{R}$  is the set of numbers that have a decimal expansion).

The set of numbers in a certain range comes up so often that we have a specific notation for them. **Intervals** can be written as  $(a, b)$ ,  $(a, b]$ ,  $[a, b)$ , or  $[a, b]$  depending on whether the end points are included. For instance, the interval  $[-1, 3)$  is the set of real numbers between  $-1$  and  $3$  and includes  $-1$ , but does not include  $3$ . Unfortunately, the interval notation can cause confusion with the notation for things like points or vectors, so you usually have to use the context of the problem to know the difference between points and interval notation.

**Question 11.** a) If  $A = \{x \in \mathbb{R} \mid x \leq 1.23\}$ , is  $3 \in A$ ?

b) If  $A = \{x \in \mathbb{R} \mid x \leq 1.23\}$ , is  $0 \in A$ ?

- c) If  $A = \{x \in \mathbb{R} \mid x \leq 1.23\}$ , is  $1.23 \in A$ ?
- d) Write the interval  $(-2.1, 4]$  as a set using set builder notation.
- e) Write the set of points in the plane where the  $x$ -coordinate is four times larger than the  $y$ -coordinate using set builder notation.
- f) Write the set of points in the plane that are on the graph of  $y = 3x^2 + 1$  using set builder notation.
- g) If the set  $B$  is the interval  $[2.1, 5)$ , write the set of real numbers that are **not** in  $B$  using set builder notation.

**Question 12. Bonus:** Use set builder notation to give the solution sets for each of the following equations. In other words, for what values of  $a$ ,  $b$ ,  $c$ , and  $d$  are each of the following statements true. Be sure to give **all** possible values. It will be very helpful to simplify these equations with valid algebraic steps.

- a)  $a(b + c) = ab + c$
- b)  $a(b + c)^2 = (ab + ac)^2$
- c)  $\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$
- d)  $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$
- e)  $\frac{a+b}{c+b} = \frac{a}{c}$
- f)  $\sqrt{a^2 + b^2} = a + b$
- g)  $c\sqrt{a+b} = \sqrt{ca+cb}$
- h)  $a^2 + b^2 = (a+b)^2$
- i)  $(a-b)^2 = a^2 - b^2$

**Information 13.** As we said earlier, algebra is extremely useful in transforming problems into a more familiar setting while keeping the same solution sets or obtaining a more familiar form. For instance, I wouldn't expect you to know what the graph of

$$\sqrt{(x-1)^2 + (y-2)^2} = \sqrt{(x+1)^2 + (y-0)^2}$$

looks like but you could algebraically simplify to the more familiar

$$y = -x + 1$$

You have a lot of training in how to do this but we will practice this throughout this course. We will start looking at how to make our expression fit a particular form, a skill that will be immensely helpful in this course.



**Question 14.** a) For what values of  $A$  and  $q$  will the expression  $3x + 2$  be of the form  $A(x + q)$ ?

b) For what values of  $A$  and  $q$  will the expression  $3 + 2x$  be of the form  $A(x - q)$ ?

**Question 15.** a) Expand  $(x - a)^2$

b) What value should  $\star$  be so that  $x^2 - 4x + \star$  is of the form  $(x - a)^2$ ?  
What is  $a$  for your expression?

c) What about for  $x^2 + 9x + \star$ ?

d) Or  $2x^2 - \frac{2}{3}x + \star$ ?

e) For what value of  $\star$  will  $2x^2 - \frac{2}{3}x + \star$  be of the form  $B(x - a)^2$ ?

Check on [d2l.sfasu.edu](http://d2l.sfasu.edu) for your first Individual Investigation (under the Quizzes tab) which will be due soon.

## 1.2 Coordinates and Points

In this section, we will look at the ideas of points, coordinates and their uses in both one-dimension (the number line) and two-dimensions (the Cartesian Plane).

**Question 16.** Draw and label a number line in the space below.

Describe the process you did to draw your number line and put a coordinate system to it. There should be three steps:

a)

b)

c)

**Information 17.** A **point** is a location, usually denoted by a capital letter like  $P$  or  $Q$ . A point is usually described by some measurement called a coordinate (which depends on a coordinate system). We will talk about different ways to give coordinates several times in this course but for now we will use the typical (rectangular) coordinate ideas. In other words, the way we will describe a point (a location) is by giving its signed (positive or negative) distance from the origin of our coordinate system.

It would not make sense to say that my house is 1.8 miles. It would make sense to say that my house is 1.8 miles away from my office or that my house is .4 miles away from the high school. While the location of my house does not change, the way we describe the location is relative to where we begin measuring.

**Question 18.** Draw and label a number line. Label the following points (we will use these points through Question 21):

$$P_1 = (2)$$

$$P_2 = (-3)$$

$$P_3 = (4)$$

Why are the parentheses needed?

**Question 19.** a) How far is  $P_1$  from  $P_2$ ?

b) How far is  $P_2$  from  $P_1$ ?

c) What about  $P_3$  and  $P_1$ ?

d) In general, if a point  $P$  has coordinate  $a$  (written as  $P = (a)$ ) and point  $Q$  has coordinate  $b$  (written as  $Q = (b)$ ), how far is  $P$  from  $Q$ ? Check your answer using the values from previous three parts.

**Question 20.** When describing a change, be sure to tell how much of a change occurred and what direction the change was in.

a) Use a sentence to describe the change that occurs going from  $P_1$  to  $P_2$ ?

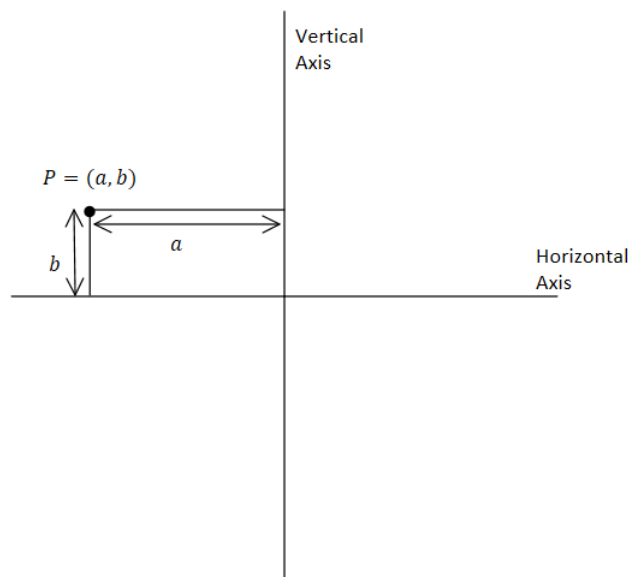
b) Use a sentence to describe the change that occurs going from  $P_1$  to  $P_3$ ?

c) Use a sentence to describe the change that occurs going from  $P_2$  to  $P_1$ ?

**Question 21.** We said earlier that the coordinate of a point can change depending on the coordinate system used. Remember that a point (a location) does not change if we switch coordinate systems but the way we describe the location (called the coordinate or coordinates) will change.

- a) If you put your origin at  $P_1$ , how could you write the new coordinate, which we will call  $\hat{x}$ , in terms of the old coordinate  $x$ ?
- b) Using your answer, give the coordinates of  $P_1$ ,  $P_2$ , and  $P_3$  in both coordinate systems. Hint: draw two overlapping coordinate systems for both  $x$  and  $\hat{x}$ .

**Information 22.** In two dimensions, we have two axes since we measure locations with two measurements. We usually measure the two coordinates as signed distances from the vertical axis (called the horizontal coordinate) and the horizontal axis (called the vertical coordinate). Coordinates are usually given as an ordered pair of the form (horizontal coordinate, vertical coordinate).



**Question 23.** Draw a two dimensional coordinate system. (Did you have the same three steps as in Question 16? Make sure you draw large graphs (about a quarter page) so that you (and I) will be see and label all of your work.

Draw and label the following points or collections of points with the corresponding letter:

- a) the origin
- b) the point that is 5 units above the horizontal axis and 3 units to the left of the vertical axis

- c) *the set of points with horizontal coordinate 1. Be sure to plot and label all points that satisfy this description.*
- d) *the set of points with the same value for the horizontal and vertical coordinate. Be sure to plot and label all points that satisfy this description.*
- e) *the set of points with horizontal coordinate equal to the square of the vertical coordinate. Be sure to plot and label all points that satisfy this description.*

- Question 24.** a) *Describe what the coordinates of points in the first quadrant have in common.*
- b) *What about the second quadrant?*
  - c) *What about the vertical axis?*
  - d) *What about the fourth quadrant?*

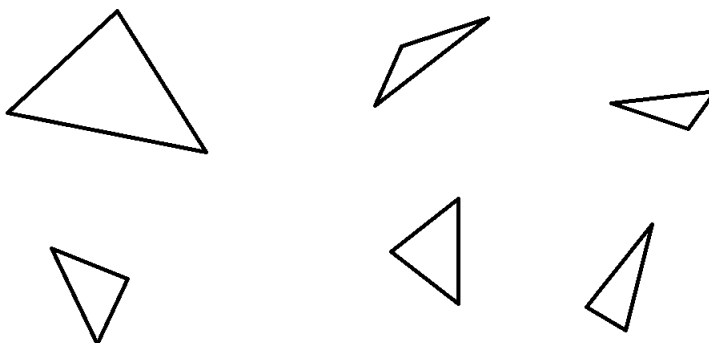
### 1.2.1 Applications of Coordinates

- Question 25.** a) *Look at a map of Manhattan (NY not KS). Where should the axes be placed so that the street numbers make sense as coordinates? In other words, where should the axes be placed so that the points on 5th Street or 5th Avenue have a coordinate of 5. (The coordinate axes do not need to go on a street and some of the distances may not be exactly the same but do the best you can.)*
- b) *What is at the origin?*
  - c) *What are the coordinates of Carnegie Deli?*

- Question 26.** a) *Look at a map of Washington DC. Where are the axes placed so that the street numbers (and letters) make sense as coordinates?*
- b) *What are the coordinates of the International Spy Museum?*

- Question 27.** a) *Look at a map of Lincoln NE. Where are the axes placed so that the street numbers (and letters) make sense as coordinates?*
- b) *What are the coordinates of the Nebraska State Capitol Building?*

- Question 28.** *Draw a set of axes on each triangle such that each of the vertices is on an axis. Hint: All the sides of the triangles do not need to be on the axes.*



**Question 29.** *Given any triangle, can you always draw axes such that each triangle has its vertices on the axes? If so, describe how to draw the axes. If not, give an example of a triangle which cannot have its vertices on some set of axes.*

**Information 30.** *The line segment between points  $P$  and  $Q$  is denoted  $\overline{PQ}$ . Remember that when describing change, you need to give both the amount of change and the direction.*

**Question 31.** *Let  $P_1 = (3, -1)$  and  $P_2 = (-2, 1)$ .*

- What is the horizontal change (sometimes called  $\Delta x$ ) that occurs in going from  $P_1$  to  $P_2$ ?*
- What is the vertical change (sometimes called  $\Delta y$ ) that occurs in going from  $P_1$  to  $P_2$ ?*
- Draw a graph with  $P_1$ ,  $P_2$ ,  $\overline{P_1P_2}$ , and label  $\Delta x$  and  $\Delta y$  on your graph.*
- Using your picture, how long is  $\overline{P_1P_2}$  in terms of  $\Delta x$  and  $\Delta y$ ?*

**Information 32.** *If  $P = (a, b)$  and  $Q = (c, d)$ , then the length of  $\overline{PQ}$  is given by  $\sqrt{(c-a)^2 + (d-b)^2}$ . This is known as the **distance formula between two points**.*

**Question 33.** *On a new set of coordinate axes, label the following points:*

$$P_1 = (1, 0), P_2 = (0, 1), P_3 = (-2, 2), P_4 = (1, -2), \text{ and } P_5 = (-3, -1)$$

- Draw the following line segments:  $\overline{P_1P_2}$ ,  $\overline{P_4P_2}$ ,  $\overline{P_2P_1}$ , and  $\overline{P_5P_3}$*
- What is the length of  $\overline{P_1P_2}$ ?*
- What is the length of  $\overline{P_4P_2}$ ?*

d) What is the length of  $\overline{P_2P_1}$ ?

e) What is the length of  $\overline{P_5P_3}$ ?

**Question 34.** On a new set of coordinate axes, label the following points:

$$P_1 = (1, 0), P_2 = (0, 1), P_3 = (-2, 2), P_4 = (1, -2), \text{ and } P_5 = (-3, -1)$$

If you put the origin of a new coordinate system at the point  $P_4$ , give the new coordinates, which we will call  $(\hat{x}, \hat{y})$ , of the following points. Note that the other points are still in the same location relative to  $P_4$  but their coordinates have changed because the location of the axes has changed.

a)  $P_1 =$

b)  $P_2 =$

c)  $P_3 =$

d)  $P_4 =$

e)  $P_5 =$

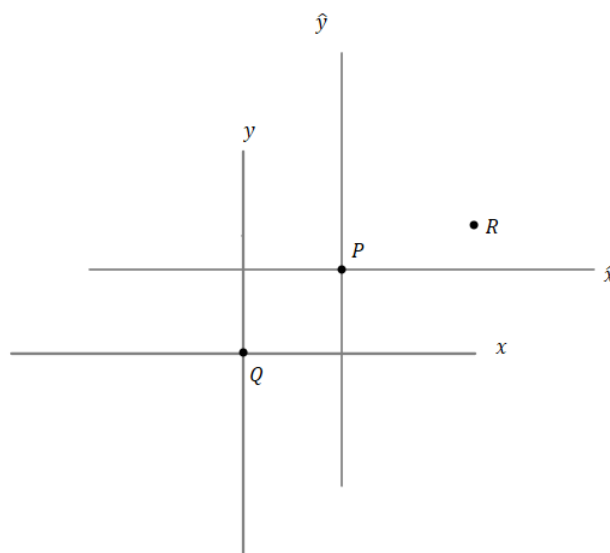
**Question 35.** Given the graph below, answer the following questions:

a) What are the  $xy$ -coordinates of point  $Q$ ?

b) What are the  $\hat{x}\hat{y}$ -coordinates of the point  $P$ ?

c) If the point  $P$  has  $xy$ -coordinates of  $(h, k)$ , give the  $\hat{x}\hat{y}$ -coordinates of the point  $Q$ ?

d) If the point  $R$  has  $xy$ -coordinates of  $(a, b)$ , give the  $\hat{x}\hat{y}$ -coordinates of the point  $R$ ?



**Question 36.** a) Using your work from Question 35, how could you write the  $xy$ -coordinates of a point in terms of the  $\hat{x}\hat{y}$ -coordinates? In other words, write  $x = \underline{\hspace{2cm}}$  and  $y = \underline{\hspace{2cm}}$  in terms of  $\hat{x}$ ,  $\hat{y}$ ,  $h$ , and  $k$ .

b) Using your work from Question 35, how could you write the  $\hat{x}\hat{y}$ -coordinates of a point in terms of the  $xy$ -coordinates? In other words, write  $\hat{x} = \underline{\hspace{2cm}}$  and  $\hat{y} = \underline{\hspace{2cm}}$  in terms of  $x$ ,  $y$ ,  $h$ , and  $k$ .

**Question 37.** What point is one third of the way from the origin to  $(5, 0)$ ? Draw a picture to make sure your answer makes sense.

**Question 38.** What point is one third of the way from the origin to  $(0, -2)$ ? Draw a picture to make sure your answer makes sense.

**Question 39.** What point is one third of the way from the origin to  $(5, -2)$ ? Draw a picture to make sure your answer makes sense.

**Question 40.** What point is one third of the way from  $(5, -2)$  to the origin? This answer should be different than the previous problem. Draw a picture to make sure your answer makes sense.

**Question 41.** What point is one third of the way from  $(x_1, y_1)$  to  $(x_2, y_2)$ ? Be sure to state your answer in terms of  $x_1$ ,  $y_1$ ,  $x_2$ , and  $y_2$ .

**Question 42.** What point is the fraction  $t$  of the way from  $(x_1, y_1)$  to  $(x_2, y_2)$ ? Be sure to state your answer in terms of  $x_1$ ,  $y_1$ ,  $x_2$ , and  $y_2$ .

**Question 43.** Use your answer to the previous question to find the following:

- a) *The point that is two fifths of the way from  $(-1, 3)$  to  $(6, -5)$ . Draw a picture to make sure your answer makes sense.*
- b) *The point that is one quarter of the the way from  $(4, -2)$  to  $(9, 0)$ . Draw a picture to make sure your answer makes sense.*
- c) *The point on the line through  $(4, -7)$  and  $(3, 0)$  that is twice as far from  $(4, -7)$  as it is from  $(3, 0)$ . Draw a picture to make sure your answer makes sense.*
- d) *Find a second point that satisfies the previous statement. Draw a picture to make sure your answer makes sense.*

**Question 44.** a) *Given any triangle show how you can choose coordinate axes such that all vertices of the triangle are on either the horizontal or vertical axis. Give the coordinates of the vertices of the triangle.*

- b) *Prove that the segment joining the midpoints of two sides of a triangle is parallel to the third side and is half of the length of the parallel side. (Hint: Remember your work on how to take any triangle and put axes on the triangle to have the vertices on the axes. If you have done this correctly, some of the coordinates for the vertices will be zero which will make your computation much simpler).*



### 1.3 Change and Vectors

We have already seen several places where measuring a change is useful. In this section, we will talk about vectors and how they are a great computational tool for describing change and computing various properties.

**Question 45.** a) On a new set of coordinate axes, label the following points:

$$P_1 = (1, 0), P_2 = (0, 1), P_3 = (-2, 2), P_4 = (1, -2), \text{ and } P_5 = (-3, -1)$$

b) Use a sentence to describe the change that occurs going from  $P_1$  to  $P_2$ ? Be sure to describe the directions in which the change occurs and how much of a change occurs in each direction.

c) What about from  $P_1$  to  $P_3$ ?

d) What about from  $P_4$  to  $P_5$ ?

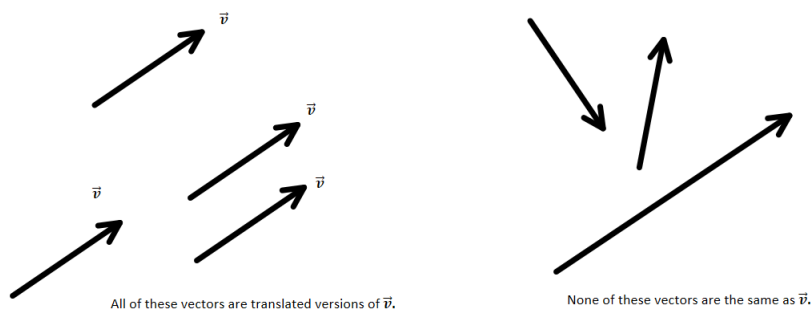
**Information 46.** The **directed line segment** from  $P$  to  $Q$  is denoted by  $\overrightarrow{PQ}$  and measures the change in coordinates that occurs going from  $P$  to  $Q$ . Directed line segments  $\overrightarrow{AB}$  and  $\overrightarrow{CD}$  are **equivalent** if they represent the same change (in both length and direction).

**Question 47.** If  $P_1 = (1, 2)$ ,  $P_2 = (-1, 4)$ , and  $Q_1 = (2, -3)$ , for what point  $Q_2$  will the directed line segments  $\overrightarrow{P_1P_2}$  and  $\overrightarrow{Q_1Q_2}$  be equivalent? Draw a picture to make sure your answer makes sense.

**Question 48.** If  $P_1 = (-2, 3)$  and the midpoint of  $\overrightarrow{P_1P_2}$  is  $(1, 0)$ , what are the coordinates of  $P_2$ ?

**Question 49.** If  $P_2 = (4, 3)$  and the midpoint of  $\overrightarrow{P_1P_2}$  is  $(-1, 2)$ , what are the coordinates of  $P_1$ ?

**Information 50.** A **vector** is an equivalence class of directed line segments (a set of directed line segments that all measure the same change). A vector is a way of measuring change that doesn't have a fixed starting or ending point. Because of this property, vectors can be translated (moved up, down, left, or right) and will remain the same vector. Stretching or rotating a vector will give a different vector though.



Directed line segments are said to be a representative of a vector. Vectors are denoted by bold face lower case letters (usually  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$ ). Since we can't write in bold font with handwriting, vectors can also be written as  $\vec{u}$ ,  $\vec{v}$ , and  $\vec{w}$ .

Vectors are specified by their components which are the changes in the horizontal and vertical coordinates. For instance, the vector  $\vec{v} = \langle a, b \rangle$  corresponds to a change of  $a$  in the horizontal coordinate and  $b$  in the vertical coordinate.

**Question 51.** If  $\vec{v}$  is represented by the change from  $(0, 1)$  to  $(3, -2)$ , what are the components of  $\vec{v}$ ? What is the length of  $\vec{v}$ ?

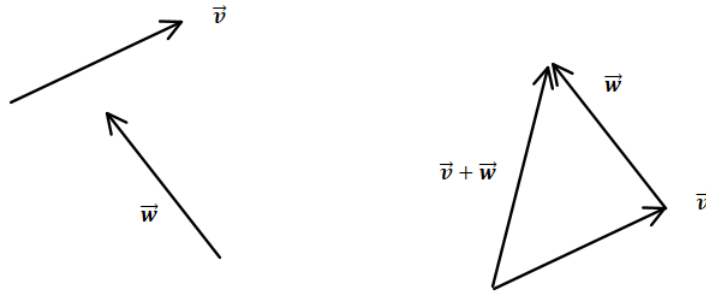
**Question 52.** If  $\vec{u}$  is represented by the change from  $(2, 0)$  to  $(-1, 1)$ , what angle does  $\vec{u}$  make with the positive horizontal axis? Give an exact answer; by this I mean don't use a calculator at all. Exact answers can include square roots or inverse trig functions. For example,  $\sqrt{2}$ ,  $\sin(1/2)$ , and  $\tan^{-1}(2) + \pi$  are examples of exact answers. Remember that angles are measured counterclockwise.

**Question 53.** If  $\vec{v} = \langle -2, 2 \rangle$  with initial point  $(9, -3)$ , what is the endpoint? What is the length of  $\vec{v}$ ?

**Question 54.** If  $\vec{v} = \langle a, b \rangle$ , what is the length of  $\vec{v}$ ? The length of  $\vec{v}$  is denoted  $|\vec{v}|$ .

**Information 55.** Vectors are represented visually by an arrow with the proper change in the horizontal and vertical coordinates given by their components. A vector in **standard position** is a vector that has initial point at the origin. Two vectors are equal if they have the same components.

The sum of two vectors is computed as the sum of their components. Graphically, vectors are added using the tip to tail rule. If you are adding vector  $\vec{v}$  to  $\vec{w}$ , then you would translate  $\vec{w}$  so that the start of  $\vec{w}$  (the tail) is at the same point as the end of  $\vec{v}$  (the tip). The sum of  $\vec{v}$  and  $\vec{w}$  corresponds to the change from the tail of  $\vec{v}$  to the tip of  $\vec{w}$ .



The scalar multiplication of  $\vec{v}$  by a number  $c$  is given by  $c\vec{v} = c\langle a, b \rangle = \langle ca, cb \rangle$ . The zero vector,  $\vec{0}$ , represents no change.

**Question 56.** What are the components of  $\vec{0}$ ?

**Question 57.** Let  $\vec{u} = \langle 3, 2 \rangle$ ,  $\vec{v} = \langle -2, 1 \rangle$  and  $\vec{w} = \langle 1, -2 \rangle$ .

a)  $\vec{u} + \vec{v} =$

b) Draw a graph showing the tip to tail addition of  $\vec{u} + \vec{v}$ . Make sure you start  $\vec{v}$  where  $\vec{u}$  ends.

c)  $\vec{v} + \vec{u} =$

d) Draw a graph showing the tip to tail addition of  $\vec{v} + \vec{u}$ . Make sure you start  $\vec{u}$  where  $\vec{v}$  ends.

e) Compute the following:

(a)  $3\vec{w} =$

(b)  $2\vec{u} =$

(c)  $3\vec{w} + 2\vec{u} =$

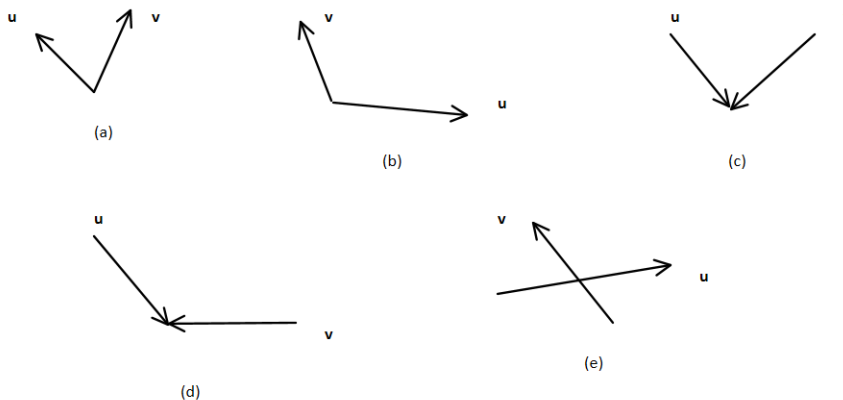
f) Graph the three vectors from part e) using the proper tip to tail to verify you have the correct answer.

**Question 58.** a) What vector would you need to add to  $\langle 3, 4 \rangle$  to get  $\langle -2, 7 \rangle$ ?

b) What vector would you need to add to  $\langle -2, 7 \rangle$  to get  $\langle 3, 4 \rangle$ ?

**Question 59.** If Homer's house is at  $(1, 0)$ , the nuclear plant is at  $(4, -1)$ , and Moe's Tavern is at  $(6, -4)$ , draw the vectors that represent Homer going from his house to the nuclear plant and from the nuclear plant to Moe's. What vector is the net result of Homer's travels (what is the total change that has occurred)? Draw a graph to illustrate your answer and give the components of each the vectors.

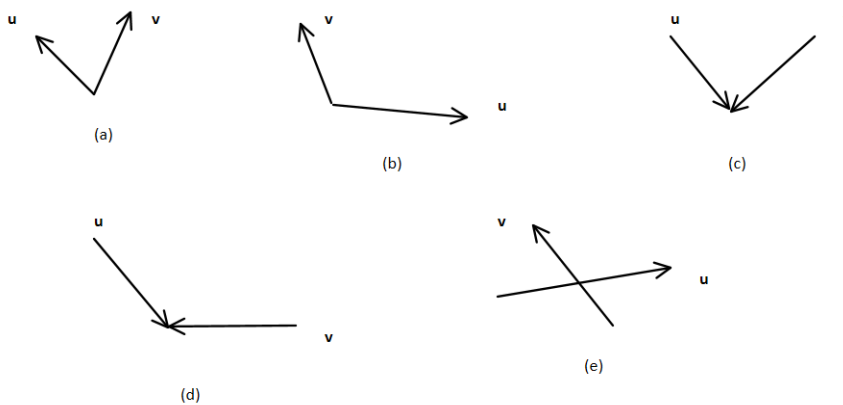
**Question 60.** Given the vectors below, draw  $\vec{v} + \vec{u}$ . Remember you can translate vectors to start or end where ever you want, but you can not rotate or stretch them.



**Question 61.** Given the vectors below, draw the vector  $\vec{x}$ , where  $\vec{x}$  is what you need to add to  $\vec{u}$  to get  $\vec{v}$ . Algebraically, this corresponds to

$$\underline{\hspace{2cm}} + \vec{x} = \underline{\hspace{2cm}}$$

Remember you can translate vectors to start or end where ever you want, but you can not rotate or stretch them.



**Question 62.** Make sure you give all possible answer for the following question. If  $\vec{v} = \langle -3, 1 \rangle$ ,

- what vector(s) has twice the length of  $\vec{v}$  and the same direction?
- what vector(s) has the same length of  $\vec{v}$  but is in the opposite direction?

- c) what vector(s) have the same length as  $\vec{v}$  but is in a perpendicular direction?
- d) what vector(s) do you need to add to  $\vec{v}$  to get your answer to part c)?
- e) what vector(s) is in the same direction as  $\vec{v}$  but has length 1?

**Information 63.** The vector  $\vec{u} - \vec{v}$  is the vector needed to add to  $\vec{v}$  in order to get  $\vec{u}$ . Algebraically, this looks like  $\vec{v} + (\vec{u} - \vec{v}) = \vec{u}$ .

**Question 64.** If  $\vec{u} = \langle 3, -2 \rangle$  and  $\vec{v} = \langle 2, 1 \rangle$ , draw the following vectors and give their components:

- a)  $\vec{u} + \vec{v}$
- b)  $2\vec{v} + 3\vec{u}$
- c)  $\vec{u} - \vec{v}$
- d)  $\vec{v} - \vec{u}$
- e)  $3\vec{u} - 2\vec{v}$

**Information 65.** Note here that all of the calculations for vectors have been very easy in terms of components since vector addition, vector subtraction, and scalar multiplication all work componentwise.

Vectors  $\vec{u}$  and  $\vec{v}$  are parallel if  $\vec{u} = c\vec{v}$  for some real number  $c$ .

**Question 66.** If  $\vec{u} = \langle 3, -2 \rangle$  and  $\vec{v} = \langle \clubsuit, 1 \rangle$ , for what value(s) of  $\clubsuit$  will  $\vec{v}$  and  $\vec{u}$  be parallel? Draw a picture to make sense of your answer.

**Question 67.** If  $\vec{u} = \langle 3, 0 \rangle$  and  $\vec{v} = \langle \clubsuit, 1 \rangle$ , for what value(s) of  $\clubsuit$  will  $\vec{v}$  and  $\vec{u}$  be parallel? Draw a picture to make sense of your answer.

**Question 68.** Are parallel vectors always in the same direction? Give examples to show why or why not.

**Information 69.** The dot product of  $\vec{u}$  and  $\vec{v}$  is given by

$$\vec{u} \cdot \vec{v} = \langle a_1, b_1 \rangle \cdot \langle a_2, b_2 \rangle = a_1a_2 + b_1b_2$$

Notice that the dot product of two vectors gives you a scalar (a number), not a vector. In the next problem, we will practice computing the dot product and try to figure out what the dot product measures.

**Question 70.** Let  $\vec{v}_1 = \langle 2, 1 \rangle$ ,  $\vec{v}_2 = \langle 3, 1 \rangle$ ,  $\vec{v}_3 = \langle -1, 2 \rangle$ , and  $\vec{v}_4 = \langle 4, 2 \rangle$ . Answer the following:

- a)  $\vec{v}_1 \cdot \vec{v}_1 =$

b)  $\vec{v}_2 \cdot \vec{v}_2 =$

c) In general, how is  $\vec{w} \cdot \vec{w}$  related to the length of  $\vec{w}$ ?

d)  $\vec{v}_1 \cdot \vec{v}_2 =$

e)  $\vec{v}_2 \cdot \vec{v}_1 =$

f) How is  $\vec{w} \cdot \vec{u}$  related to  $\vec{u} \cdot \vec{w}$ ?

g)  $\vec{v}_4 \cdot \vec{v}_2 =$

h) Since  $\vec{v}_4 = 2(\vec{v}_1)$ , how is  $\vec{u} \cdot \vec{w}$  related to  $(2\vec{u} \cdot \vec{w})$ ?

i)  $\vec{v}_1 \cdot \vec{v}_4 =$

j) If  $\vec{w}$  and  $\vec{u}$  are parallel, how is  $\vec{w} \cdot \vec{u}$  related to the lengths of  $\vec{w}$  and  $\vec{u}$ ?

k)  $\vec{v}_1 \cdot \vec{v}_3$

l)  $\vec{v}_4 \cdot \vec{v}_3$

m) Draw a graph of  $\vec{v}_1$ ,  $\vec{v}_3$ , and  $\vec{v}_4$  in standard position.

n) What is the angle between  $\vec{v}_1$  and  $\vec{v}_3$ ?

**Information 71.**  $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$ , where  $\theta$  is the angle between  $\vec{u}$  and  $\vec{v}$ .

Vectors  $\vec{u}$  and  $\vec{v}$  are orthogonal if  $\vec{u} \cdot \vec{v} = 0$ .

The horizontal unit vector is denoted by  $\vec{i} = \hat{i} = \mathbf{i} = \langle 1, 0 \rangle$  and the vertical unit vector is given by  $\vec{j} = \hat{j} = \mathbf{j} = \langle 0, 1 \rangle$

**Question 72.** a) Based on the formulas above, what geometric relationship between  $\vec{u}$  and  $\vec{v}$  will make  $\vec{u} \cdot \vec{v}$  be positive?

b) Based on the formulas above, what geometric relationship between  $\vec{u}$  and  $\vec{v}$  will make  $\vec{u} \cdot \vec{v}$  be negative?

c) Based on the formulas above, what geometric relationship between  $\vec{u}$  and  $\vec{v}$  will make  $\vec{u} \cdot \vec{v}$  be zero?

**Question 73.** What angle do  $\vec{u} = \langle 2, 1 \rangle$  and  $\vec{v} = \langle 3, -1 \rangle$  make? Be sure to draw a picture to make sense of your answer.

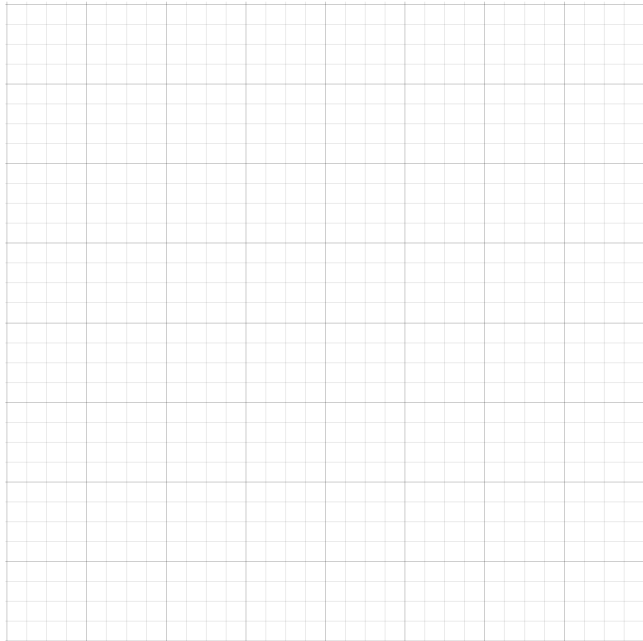
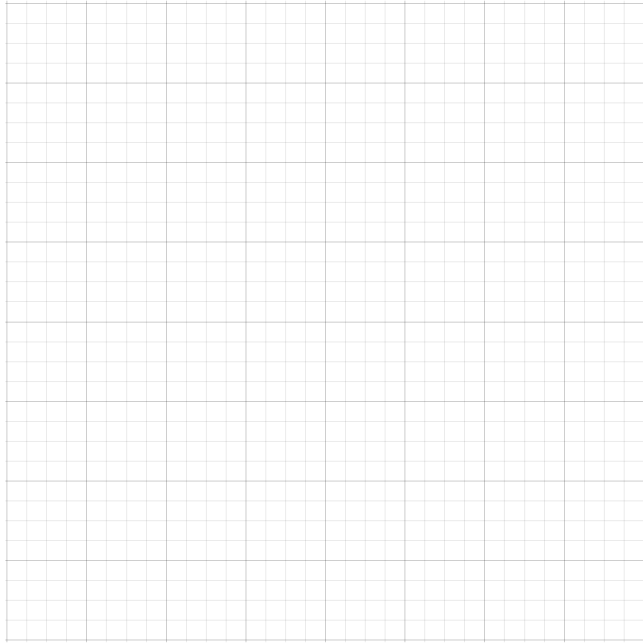
**Question 74.** What angle do  $\vec{u} = \langle 2, 6 \rangle$  and  $\vec{v} = \langle 3, -1 \rangle$  make? Be sure to draw a picture to make sense of your answer.

**Question 75.** If  $\vec{u} = \langle 3, -2 \rangle$  and  $\vec{v} = \langle s, 1 \rangle$ , for what value(s) of  $s$  will  $\vec{v}$  and  $\vec{u}$  be orthogonal?

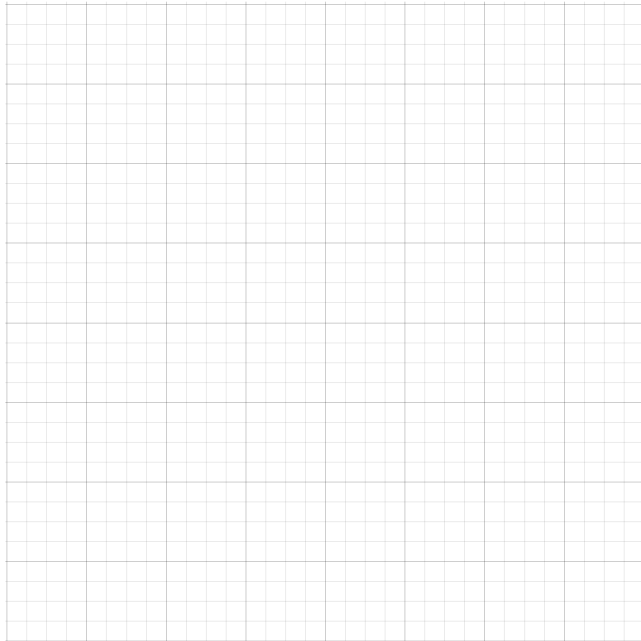
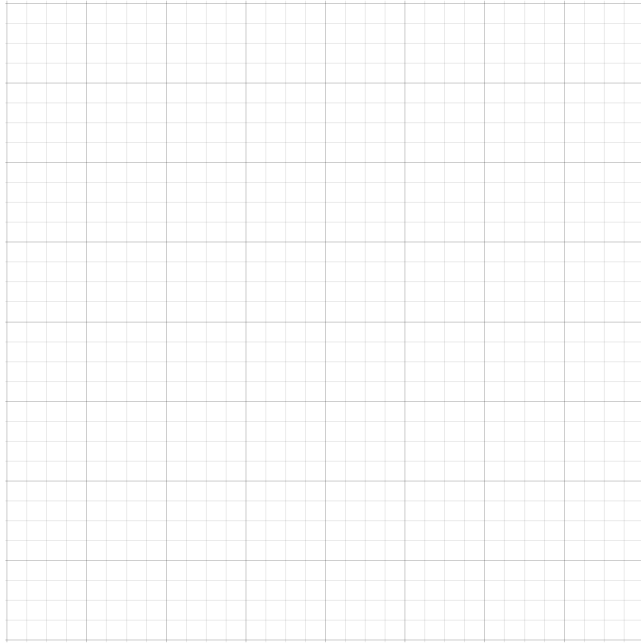
**Question 76.** If  $\vec{u} = \langle 3, -2 \rangle$  and  $\vec{v} = \langle \xi, 0 \rangle$ , for what value(s) of  $\xi$  will  $\vec{v}$  and  $\vec{u}$  be orthogonal?

**Question 77.** If  $\vec{u} = \langle 3, -2 \rangle$  and  $\vec{v} = \langle \xi, 1 \rangle$ , for what value(s) of  $\xi$  will  $\vec{v}$  and  $\vec{u}$  be at an angle of  $\frac{3\pi}{4}$  from each other? Draw a graph of your answers. Do your answers make sense? Why or why not?

**Question 78.** How much does  $\vec{v} = \langle 3, -1 \rangle$  move in the same direction as  $\mathbf{i}$ ? How much does  $\vec{v} = \langle 3, -1 \rangle$  move in the same direction as  $\mathbf{j}$ ? How can you write  $\vec{v}$  as a linear combination of  $\mathbf{i}$  and  $\mathbf{j}$ ? What would  $a$  and  $b$  be in the equation  $\vec{v} = a\mathbf{i} + b\mathbf{j}$ ?





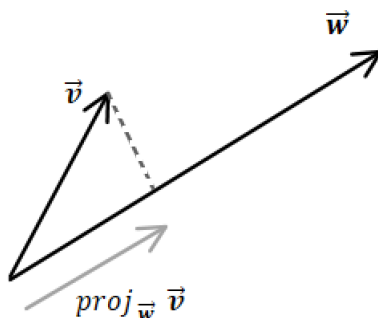


**Information 79.** The previous problem shows how the components of  $\vec{v}$  are really just how much  $\vec{v}$  moves in the direction of  $\vec{i}$  (horizontally) and  $\vec{j}$  (vertically). It will often be useful to know how much a vector moves in a particular direction that is not just up/down, or left/right.

The projection of  $\vec{v}$  onto  $\vec{w}$  is given by

$$\text{proj}_{\vec{w}} \vec{v} = \left( \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w}$$

The vector  $\text{proj}_{\vec{w}} \vec{v}$  is the piece of  $\vec{v}$  that is parallel to  $\vec{w}$ . In fact, it is possible that  $\text{proj}_{\vec{w}} \vec{v}$  is in the opposite (but still parallel) direction of  $\vec{w}$ . Also notice that  $\vec{v} \cdot \vec{w}$  will be a scalar measuring how much  $\vec{v}$  and  $\vec{w}$  move together, but  $\text{proj}_{\vec{w}} \vec{v}$  is the largest vector piece of  $\vec{v}$  that is parallel to  $\vec{w}$ .



**Question 80.** What is the projection of  $\vec{v} = \langle 3, -1 \rangle$  onto  $\vec{i}$ ? What is the projection of  $\vec{v} = \langle 3, -1 \rangle$  onto  $\vec{j}$ ? How does your answer compare to your answer to Question 78?

**Question 81.** Compute the projection of  $\langle 2, 3 \rangle$  onto  $\langle 4, -1 \rangle$ . Draw all these vectors in standard position to see if your work makes sense.

**Question 82.** Compute the projection of  $\langle 4, -1 \rangle$  onto  $\langle 2, 3 \rangle$ . Draw all these vectors in standard position to see if your work makes sense.

**Question 83.** Compute the projection of  $\langle 6, 2 \rangle$  onto  $\langle -3, -1 \rangle$ . Draw all these vectors in standard position to see if your work makes sense.

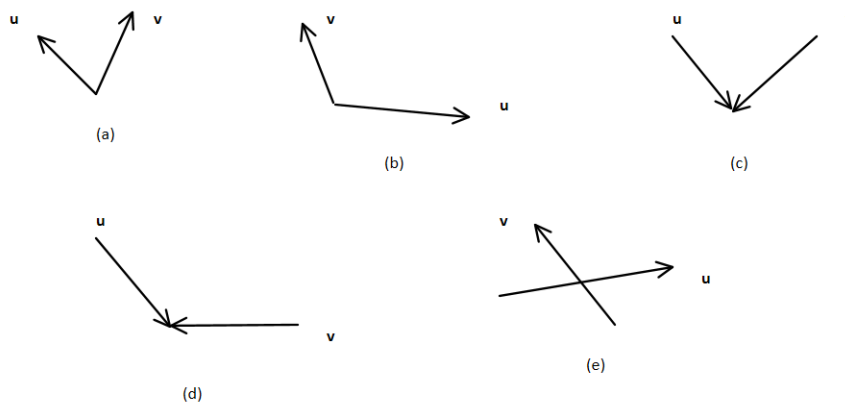
**Question 84.** Compute the projection of  $\langle 2, -1 \rangle$  onto  $\langle 4, 8 \rangle$ . Draw all these vectors in standard position to see if your work makes sense.

**Question 85.** How much of  $\langle 2, -1 \rangle$  is orthogonal to  $\langle 4, 8 \rangle$ ? Explain.

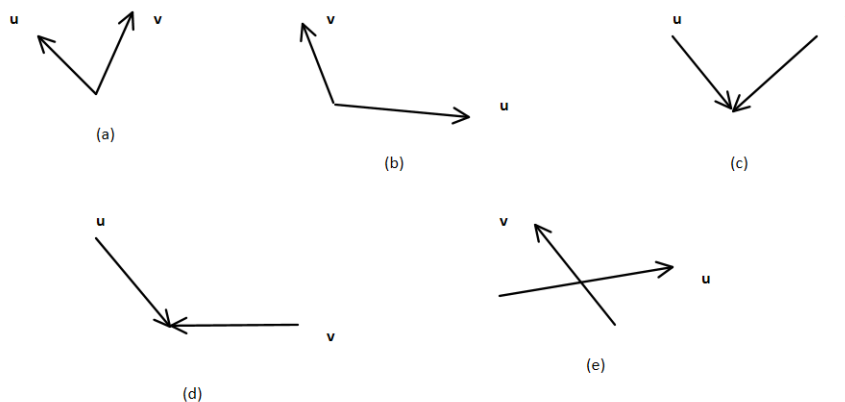
**Question 86.** How much of  $\langle 2, 3 \rangle$  is orthogonal to  $\langle 4, -1 \rangle$ ? Explain.

**Question 87.** In general, how do you find the piece of  $\vec{w}$  that is orthogonal to  $\vec{v}$ ? You can give your answer as a formula in terms of  $\vec{w}$  and  $\vec{v}$

**Question 88.** Given the vectors below, draw the vector  $\text{proj}_{\vec{u}}\vec{v}$ . Remember you can translate vectors to start or end where ever you want, but you can not rotate or stretch them.

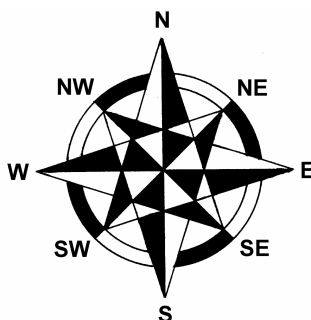


**Question 89.** Given the vectors below, draw the vector  $\text{proj}_{\vec{v}}\vec{u}$ . Remember you can translate vectors to start or end where ever you want, but you can not rotate or stretch them.



**Information 90.** Vectors are very useful in physics because they can represent many different quantities like velocity, force, or electric fields. When the forces on an object are balanced (add to  $\vec{0}$ ), the object is said to be in equilibrium.

Directions can be given in terms of vectors or in terms of angles or directions. **Bearing** refers to the angle a direction makes clockwise from north. The direction given by East-Southeast (abbreviated ESE) is half way between East and Southeast. On maps, the directions are shown by a fancy symbol called a compass rose.



**Question 91.** a) What is the bearing of the east direction?

b) What is the bearing of the southeast direction?

c) What is the bearing of ESE?

d) What direction has bearing of  $292.5^\circ$ ?

e) What are the components of a vector in the SE direction?

The remaining application problems are based on knowing information about some set of vectors in terms of their magnitude and direction, where as our calculations for all of the previous problems have been based on knowing the components of the vectors. In the next set of problems, we will talk about how to go from magnitude and direction to components. Each of the following problems uses a different meaning of a vector (force, displacement, and velocity) but the rules of vectors still apply to each which shows how versatile vectors are in physical situations.

**Question 92.** a) Jimbo Jones pushes on the statue of Jebediah Springfield in the NE direction with 10 pounds of force. What are the components of Jimbo's force vector?

b) Kearney Zyzwicz pushes on the statue of Jebediah Springfield in the S direction with 20 pounds of force. What are the components of Kearney's force vector?

c) Dolph Starbeam pushes on the statue of Jebediah Springfield in the WSW direction with 15 pounds of force. What are the components of Dolph's force vector?

d) What is resulting force (of all three bullies) on the statue?

e) How much of the resulting force is in the SE direction? Hint: look at the projection of your answer to the part d) onto a vector in the SE direction.

f) *What additional force should be put on the statue of Jebediah Springfield to make sure the statue is at equilibrium?*

**Question 93.** *If you are driving 50 mph at a direction 30 degrees north of east, stop to eat after 3 hours, and begin driving again at 60 mph due north for 2 hours, what would the resulting displacement vector be? Be sure to draw a picture to see if your answer makes sense.*

**Question 94.** *A boat is traveling at 30 mph at a bearing of 35 degrees. The river current is flowing 5 mph at a bearing of 115 degrees. What will the overall velocity of the boat be?*

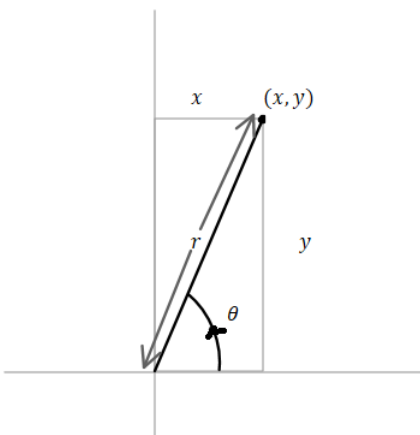
## Chapter 2

### **Polar Coordinates, Lines, Polynomials, and Rational Polynomials**

In this chapter we will be talking about graphing and the relationship between algebra and geometry for several types of objects that you may have seen before. Be sure to check on d2l to complete your second Individual Investigation assignment.

#### **2.1 Polar Coordinates: Another Way to Measure Location**

**Information 95.** *Previously, we specified the location of a point in the plane using the signed distance to the vertical and horizontal axes (giving the horizontal and vertical coordinates respectively). This is not the only way to use two measurements to specify location. Polar coordinates use two different measurements to specify the location of a point. Specifically the polar coordinates of a point  $P$  are given by  $(r, \theta)$  where  $r$  is the distance from  $P$  to the origin and  $\theta$  is the angle (measured counterclockwise) from the positive horizontal axis to the segment between  $P$  and the origin.*



**Question 96.** Plot the following points and give their rectangular coordinates:

- a)  $P_1 : (r, \theta) = (1, 0)$
- b)  $P_2 : (r, \theta) = (3, \pi)$
- c)  $P_3 : (r, \theta) = (2, \frac{\pi}{2})$
- d)  $P_4 : (r, \theta) = (4, \frac{4\pi}{3})$
- e)  $P_5 : (r, \theta) = (-2, \frac{3\pi}{4})$

**Question 97.** Draw the following points on the same set of axes and give polar coordinates for each point. Remember to leave your answer in exact form.

- a)  $P_1 : (x, y) = (1, 1)$
- b)  $P_2 : (x, y) = (0, -2)$
- c)  $P_3 : (x, y) = (\sqrt{3}, -1)$
- d)  $P_4 : (x, y) = (-5, -1)$
- e)  $P_5 : (x, y) = (2, 4)$

**Question 98.** In general, we use the following equations as a way to convert between polar and rectangular coordinates:

$$x = \underline{\hspace{2cm}}$$

$$y = \underline{\hspace{2cm}}$$

$$r^2 = \underline{\hspace{4cm}}$$

$$\tan(\theta) = \underline{\hspace{4cm}}$$

**Question 99.** *Why do you need to state the above expressions in terms of  $r^2$  and  $\tan(\theta)$  instead of  $r$  and  $\theta$ ?*

In the same way that we can transform points from rectangular  $(x, y)$  coordinates to polar  $(r, \theta)$  coordinates and viceversa, we can convert equations in  $x$  and  $y$  to equations in  $r$  and  $\theta$  such that if a point  $(x, y)$  satisfies the rectangular equation then the polar form of the point will satisfy the polar equation. In fact, we use the same substitutions in converting equations as we do in converting points. Just as you saw above, converting points from polar to rectangular was easy, converting rectangular equations to polar equations tends to be easier.

**Question 100.** *Convert the equation  $y + x = 0$  to polar coordinates. In other words, come up with an equation using  $r$  and  $\theta$  that describes the same set of points as  $y + x = 0$ . Solve for either  $r$  or  $\theta$  in your answer.*

**Question 101.** *Convert the equation  $x^2 = y$  to polar coordinates. In other words, come up with an equation using  $r$  and  $\theta$  that describes the same set of points as  $x^2 = y$ . Solve for either  $r$  or  $\theta$  in your answer.*

**Question 102.** *Convert  $r = 2$  to rectangular coordinates. In other words, come up with an equation using  $x$  and  $y$  that describes the same set of points as  $r = 2$ . Simplify your answer and explain what the graph will look like.*

**Question 103.** *Convert  $\theta = \frac{\pi}{4}$  to rectangular coordinates. Simplify your answer and explain what the graph will look like.*

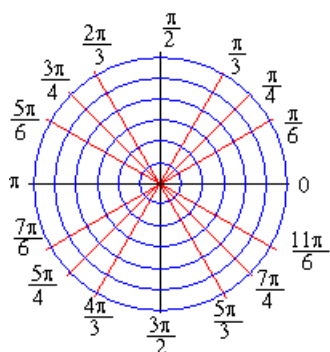
**Question 104.** *Convert  $r = 2 \cos(\theta)$  to rectangular coordinates. Simplify your answer and explain what the graph will look like.*

**Question 105.** *Convert  $r = 2 + \sin(\theta)$  to rectangular coordinates.*

**Information 106.** *The graph of an equation is a plot of all points whose coordinates make the equation a true statement (i.e. the left hand side equals the right hand side).*

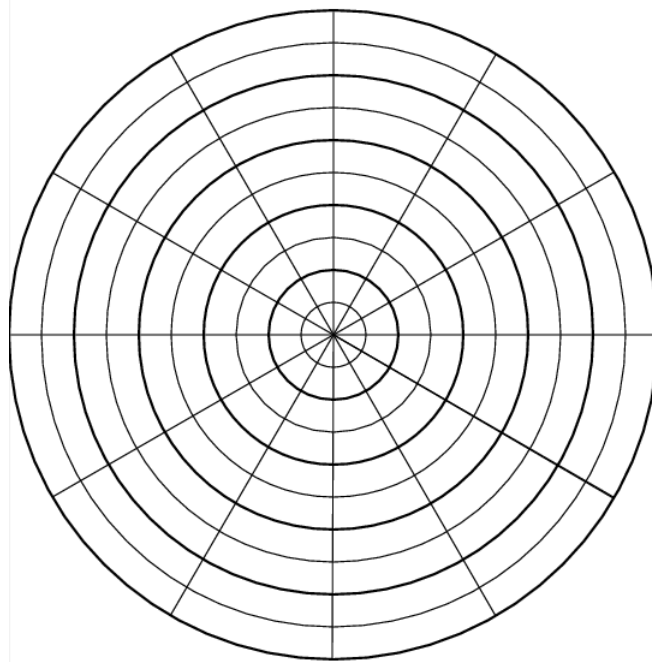
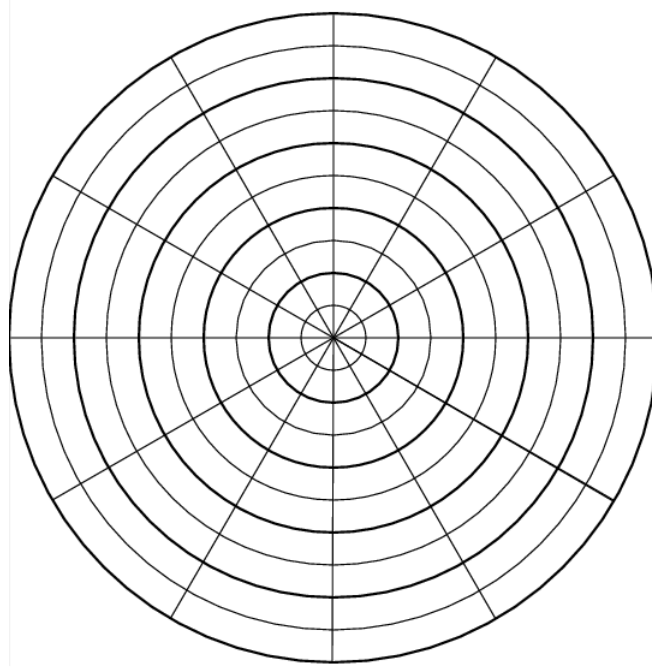
*Below is an example of a set of axes and a "grid" of polar coordinates.*





**Question 107.** Graph the set of points in the plane that satisfy each of the following polar equations. Two blank polar grids are given on the next page.

- a)  $r = 3$ .
- b)  $\theta = \frac{\pi}{3}$
- c)  $y = 2 \cos(x)$  This problem needs to be plotted on the  $xy$ -plane and not in the polar plane.
- d)  $r = 2 \cos(\theta)$  Hint: Use points from the plot of the previous part to plot this graph on a polar grid.
- e)  $r = 2 \sin(\theta) - 1$
- f)  $r = 3 \cos(2\theta)$



**Information 108.** *The graph of an equation is a plot of all points whose coordinates make the equation a true statement (i.e. the left hand side equals the right hand side).*

**Question 109.** *Which of the following points are on the graph of  $2xy + 3x = 4y^2 - x^2 + 2$ :  $(0, 0)$ ,  $(1, 1)$ ,  $(-1, 1)$ ,  $(\frac{3}{2}, 2)$*

**Question 110.** *What are the  $x$ -intercepts for the graph of  $2xy + 3x = 4y^2 - x^2 + 2$ ?*

**Question 111.** *What are the  $y$ -intercepts for the graph of  $2xy + 3x = 4y^2 - x^2 + 2$ ?*

**Question 112.** *Find two other points on the graph of  $2xy + 3x = 4y^2 - x^2 + 2$ .*

**Question 113.** *Plot the graphs of each equation on a separate set of axes:*

a)  $x = 2$

b)  $3x - 5y = 15$

c)  $x^2 + y^2 = 10$

d)  $y = 1/x$

**Question 114.** *At what point(s) do  $3x - 5y = 15$  and  $y = 1/x$  intersect? Be sure to give exact answers. Draw both graphs on the same axes to check if your answer makes sense (you may want to look a decimal approximation to make sure your answer is in the same place as your graph suggests).*

**Question 115.** a) *What is the distance from  $(x, y)$  to  $(2, 1)$ ?*

b) *What is the distance from  $(x, y)$  to  $(-4, 5)$ ?*

c) *Give an equation for the set of points that are equidistant from  $(2, 1)$  and  $(-4, 5)$ .*

d) *Simplify your answer for the previous part as much as possible.*

e) *Graph your simplified equation and describe why the graph makes sense as an answer to part c).*

**Question 116.** *Give an equation for the set of points that satisfy the following statements. Simplify the equation as much as possible (expand and combine like terms, etc.). Hint: Make sure you can write out what distances you are comparing. It may help to write out the problem just as we did in the previous question.*

- a) the set of points  $(x,y)$  whose distance from  $(0,0)$  is three times its distance from  $(0,5)$ .
- b) the set of points  $(x,y)$  whose distance to the  $x$ -axis is twice the distance to  $(2,5)$ .
- c) the set of points  $(x,y)$  whose sum of distances to  $(3,0)$  and  $(-2,1)$  is 12.
- d) the set of points  $(x,y)$  that are twice as far from  $(1,1)$  as they are from  $(4,-2)$ .

## 2.2 Lines in the Plane

In the next set of problems, we will come up with several different ways to give the equation for the set of points on a line. It is important for you to understand when each of the forms works and when it does not work.

**Information 117.** Given any two points, a **line** is the set of points through which the segment between the points can be extended. Recalling your work on Questions 37-43, the point that is the fraction  $t$  of the way from  $(x_1, y_1)$  to  $(x_2, y_2)$  is  $(x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$ . This is exactly what we have defined a line to be but is not the familiar way you have dealt with lines before. The following form is called the **Parametric Form of a Line** through  $(x_1, y_1)$  and  $(x_2, y_2)$ .

$$(x, y) = (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$$

The term *parametric* comes from the use of the parameter  $t$ , which tells us where we are on the line relative to  $(x_1, y_1)$  and  $(x_2, y_2)$ , but is not part of the  $(x, y)$  points on the graph.

**Question 118.** Using  $(x_1, y_1) = (1, -2)$  and  $(x_2, y_2) = (5, 4)$ , compute the point on the line when:

- a)  $t = 0$
- b)  $t = 1$
- c)  $t = 1/3$
- d)  $t = 2$
- e)  $t = -1$
- f)  $t = 10/3$

**Question 119.** Confirm your answers for the previous problem by graphing all of those points on the same axes. Does this parametric form make sense as the line through  $(x_1, y_1) = (1, -2)$  and  $(x_2, y_2) = (5, 4)$ ?

**Question 120.** Give the parametric form of the line through each pair of points and simplify your answer.

a)  $(3, -2)$  and  $(1, 1)$

b)  $(4, 1)$  and  $(1, 1)$

c)  $(1, -2)$  and  $(3, 2)$

d)  $(4, 1)$  and  $(4, -5)$

**Question 121.** Are there any lines for which the parametric form of that line does not exist?

**Information 122.** The parametric form of a line can really be split into two coordinate equations, namely  $x = x_1 + t(x_2 - x_1)$  and  $y = y_1 + t(y_2 - y_1)$ .

**Question 123.** Let's try to get rid of that  $t$ . Solve the  $x$ -coordinate equation for  $t$  and plug your answer into the  $y$ -coordinate equation to get a relationship between  $x$  and  $y$  using just  $x_1$ ,  $x_2$ ,  $y_1$ , and  $y_2$ . This form is called the **Two Point Form of a Line**. Hint: Solve for  $(y - y_1)$  and see if you can recognize a familiar fraction.

**Question 124.** Does the Two Point Form of a line always exist? If not, then tell when the Two Point Form of a Line does not exist. In other words, for which kind of lines does the Two Point Form not exist?

**Question 125.** Give the Two Point Form of the line through  $(-2, 5)$  and  $(1, 1)$ .

**Information 126.** We commonly abbreviate the expression  $\frac{y_2 - y_1}{x_2 - x_1}$  as  $m$  and call it **slope**.

**Question 127.** State the Two Point Form using the slope. (This is called the **Point-Slope Form of a Line**). For what kind of lines does the Point-Slope Form of a Line not exist?

**Question 128.** Give the Point-Slope Form of the line through  $(0, 2)$  and  $(-7, 5)$ .

**Question 129.** State the Point-Slope Form of a Line using the  $y$ -intercept  $(0, b)$  and solve for  $y$ . (This is the **Slope-Intercept Form of a Line**). For which lines does the Slope-Intercept Form of a Line not exist?

**Question 130.** Give the Slope-Intercept Form of the line through  $(1, 2)$  and  $(4, 0)$ .

**Question 131.** The **Intercept Form of a Line** is  $\frac{x}{a} + \frac{y}{b} = 1$  where  $(a, 0)$  and  $(0, b)$  are the  $x$ - and  $y$ -intercepts of the line. When does the Intercept Form of a Line not exist?

**Question 132.** Give the Intercept Form of the line through  $(2, -5)$  and  $(1, 1)$ .

**Question 133.** Give a form for horizontal lines.

**Question 134.** Give a form for vertical lines.

**Information 135.** All of the forms we have talked about can be rewritten in some way as  $Ax + By + C = 0$ . This is called the **General Form of a Line**. It turns out that anything written in the form  $Ax + By + C = 0$  will have a graph as a line and the equation of any line can be expressed in the form  $Ax + By + C = 0$ .

**Question 136.** Give the equation of the line through  $(x_1, y_1) = (1, -2)$  and  $(x_2, y_2) = (5, 4)$  in General Form.

**Question 137.** Give the equation of a line with slope 5 through  $(3, -7)$  in General Form.

**Question 138.** Take each of your answers to Question 123 through Question 134 and write them in General Form. Note that the coefficients of  $x$  and  $y$  mean different things for each form. For instance, the coefficient of  $x$  is not always the slope.

We now have many forms of a line. In general, you should use the most convenient form to solve your problem, unless the problem says to use a specific form. For instance, if you know the slope of a line and a point on the line, you should probably start with Point-Slope Form and go from there.

**Question 139.** What angle do the following lines make with the horizontal axis?

a)  $x + y + 1 = 0$

b)  $4x - 2y + 6 = 0$

c)  $-3x + 5y - 4 = 0$

d)  $-3x + 5y + 2 = 0$

e)  $x - 4 = 0$

$$f) y + 1 = 0$$

**Information 140.** *The inclination of a line is the angle (measured counter-clockwise) from the positive horizontal axis to the line. A horizontal line has inclination of 0. Inclinations are given as an angle between 0 and  $\pi$  radians. This may be different than what you measured on the previous problem.*

**Question 141.** *Give the inclination of the following lines.*

$$a) x + y + 1 = 0$$

$$b) 4x - 2y + 6 = 0$$

$$c) -3x + 5y - 4 = 0$$

$$d) -3x + 5y + 2 = 0$$

$$e) x - 4 = 0$$

$$f) y + 1 = 0$$

**Question 142.** *Using your previous work as a guide, what is the relationship between slope ( $m$ ) and inclination ( $\theta$ )?*

**Question 143.** *For what lines does slope not exist? For what lines does inclination not exist?*

**Information 144.** *Two lines are **parallel** if they do not intersect. Two lines are **perpendicular** if they meet at a right angle. The **perpendicular bisector** of two points  $P$  and  $Q$  is the perpendicular line through the midpoint of  $\overline{PQ}$ .*

**Question 145.** *How are the inclinations of two parallel lines related? How are the inclinations of two perpendicular lines related?*

**Question 146.** *How are the slopes of two parallel lines related? How are the slopes of two perpendicular lines related?*

**Question 147.** *Give the slope of a line that is perpendicular to the line through  $(3, 2)$  and  $(-5, 1)$*

*Remember how set builder notation works and look back at our work on Question 11. In general set builder notation looks like:*

*{ the kinds of objects that are in the set | restrictions on the description }*

*The set of points in the plane that have the same horizontal and vertical coordinate can be written several ways. Two possible ways are  $\{(x, y) \in \mathbb{R}^2 | x = y\}$  and  $\{(a, a) | a \in \mathbb{R}\}$ . Note that the first part of set builder notation tells what kind of objects are in the set and the second part tells us any restrictions.*

**Question 148.** *Set Builder Practice: For each of the sets below, tell which kind of object (scalar, vector, line, point, letter, etc.) each set contains:*

- a)  $\{x \in \mathbb{R} | x > 0\}$
- b)  $\{(x, 2x) | x \in (0, 1]\}$
- c)  $\{\langle a, 2a \rangle | a \in \mathbb{R}\}$
- d)  $\{3x + By + C = 0 | B, C \in \mathbb{R}\}$
- e)  $\{3 + B + C | B, C \in \mathbb{R}\}$
- f)  $\{y = Ax^2 + C | A, C \in \mathbb{R}\}$

*When writing a set of lines, two questions should guide how you write the set using set builder notation:*

- a) *What do the lines have in common?*
- b) *Do all the lines in the set have the property?*

**Question 149.** *Write a sentence that describes the following set of lines:  $\{y = 2x + b | b \in \mathbb{R}\}$ . Hint: What geometric property do all of the lines have in common? Are there any lines with that property not included in the set?*

*The set  $\{y = 2x + b | b \in \mathbb{R}\}$  contains lines with \_\_\_\_\_.*

**Question 150.** *Write a sentence that describes the following set of lines:  $\{x - 2y + C = 0 | C \in \mathbb{R}\}$ .*

**Question 151.** *Give the set of lines that are perpendicular to the line through  $(3, 2)$  and  $(-5, 1)$ .*

**Question 152.** *Give the equation of the perpendicular bisector of  $(3, 2)$  and  $(-5, 1)$  in General Form.*

**Question 153.** *Give the set of lines that are parallel to the line through  $(3, 2)$  and  $(-5, 1)$ .*

**Question 154.** *Give the equation (in General Form) of the line that contains  $(-2, -2)$  and is parallel to the line through  $(3, 2)$  and  $(-5, 1)$ .*

**Question 155.** *Write a sentence that describes the following set of lines:  $\{y - 2 = m(x - 3) | m \in \mathbb{R}\}$*

**Question 156.** *Write a sentence that describes the following set of lines:  $\{y - 2 = m(x - 3) | m \in \mathbb{R}\} \cup \{x = 3\}$ .*

**Question 157.** *Give the set of lines through the origin in General Form.*



**Question 158.** Write a sentence that describes the following set of lines:  
 $\{x + C = 0 \mid C \in \mathbb{R}\}$ .

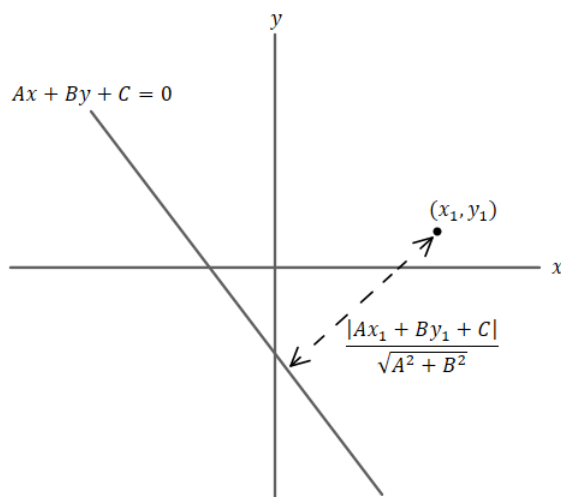
**Question 159.** Write a sentence that describes the following set of lines:  
 $\{Ax - By - (2A - 3B) = 0 \mid A, B \in \mathbb{R}\}$ .

**Question 160.** You should be able to draw a graph to help solve each part of this problem and you should not use the formula from future problems.

- How far is the point  $(3, -4)$  from the vertical axis? Draw a graph to justify your answer.
- How far is the line  $x + y + 1 = 0$  from the origin? Draw a graph to justify your answer.

**Information 161.** The distance from a **line** given by  $Ax + By + C = 0$  to the **point**  $(x_1, y_1)$  is

$$\frac{|A(x_1) + B(y_1) + C|}{\sqrt{A^2 + B^2}}$$



**Question 162.** Use this formula to check your answer to the previous question.

**Question 163.** How far is the point  $(7, -8)$  from the line through  $(5, 2)$  with slope  $-1$ ?

**Question 164.** a) Going back to Question 26, what is the equation of Pennsylvania Avenue? Hint: Use the White House and the Capitol Building as points to determine this line.

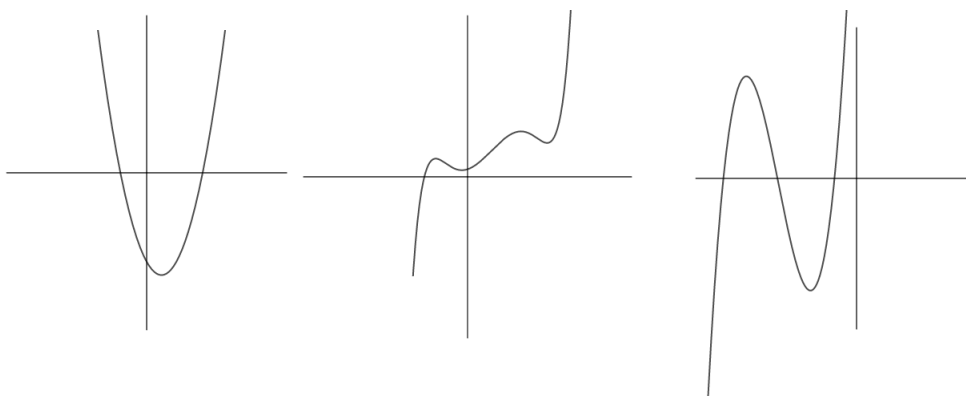
- b) How far is the International Spy Museum from Pennsylvania Avenue?  
This should be the shortest possible distance (as the crow flies).
- c) How far is the center of Dupont Circle from Pennsylvania Avenue?
- d) How far is Ben's Chili Bowl from Pennsylvania Avenue?

**Question 165.** Using set builder notation, give the set of lines that are a distance 1 from the origin.

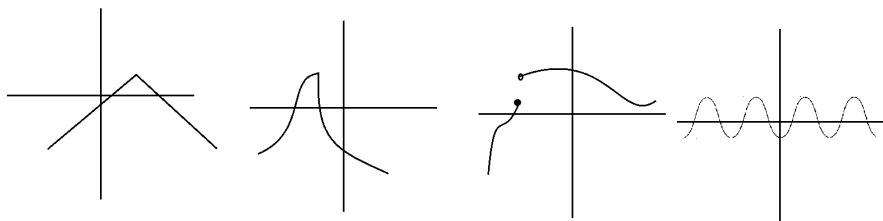
## 2.3 Polynomials and Their Graphs

The goal for this section is to be able to draw a good graph of polynomials with the proper features from a minimal set of information. We have already done this same kind of idea when we found that in order to draw any line, you need two points on the line. We will spend some time looking at what features polynomials have and how to find them from the algebraic form of a polynomial. Polynomials are of the form  $y = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  and the degree of a polynomial is the largest exponent of  $x$ . The highest degree term of a polynomial is not always the first term written. For instance,  $y = 3x^2 - 5 + 4x^5$  has degree 5, highest degree term of  $4x^5$ , and thus has a coefficient of the highest degree term of 4. Remember that the graphs of polynomials will be smooth, they will not have cusps, sharp turns, missing points, or jumps.

The following graphs are examples of polynomials.

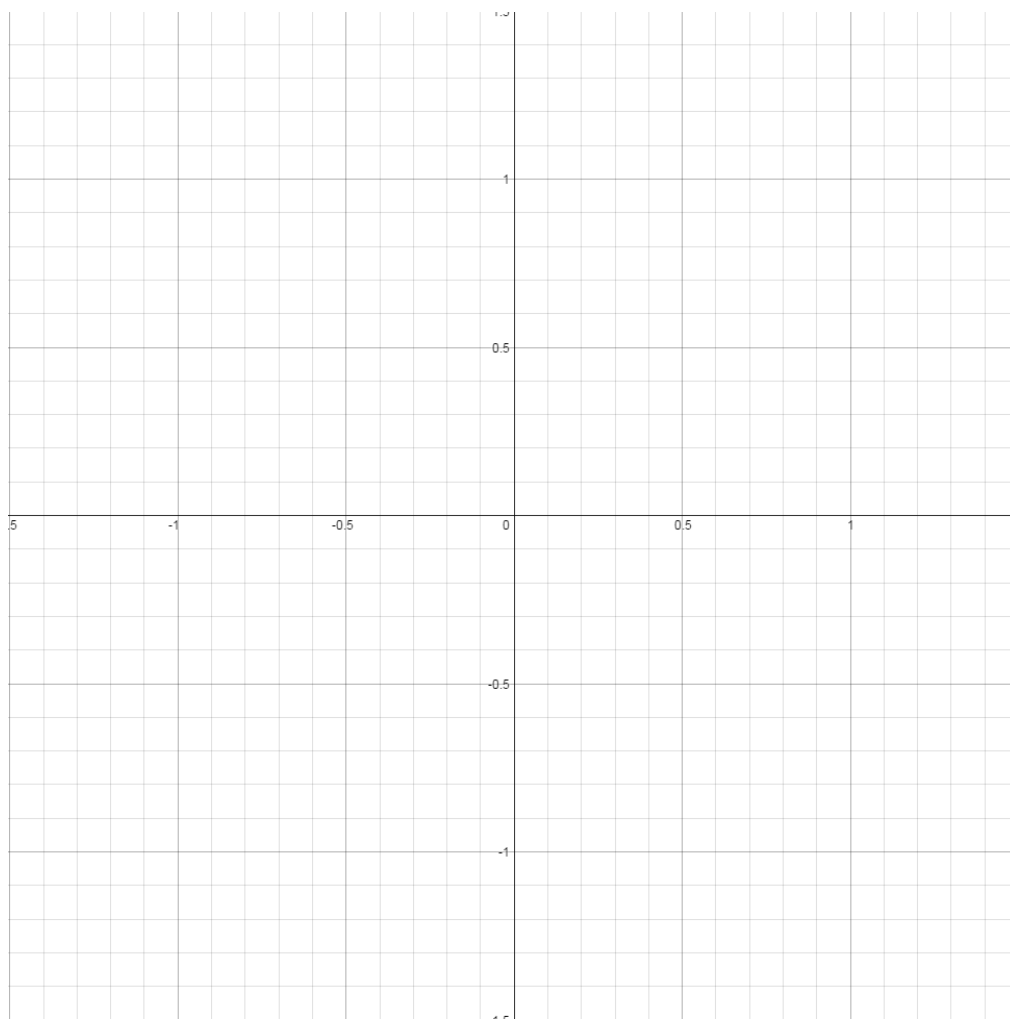


The following graphs are not examples of polynomials.



**Question 166.** Draw the graphs for the following monomials (one-term) on the same set of axes below. If you don't remember these graphs, use Desmos.

$$y = x, y = x^2, y = x^3, y = x^4, y = x^5$$



**Question 167.** For each polynomial, graph (using Desmos) the polynomial and its highest degree term.

a)  $y = 4x^2 - 3x^3$

b)  $y = 4x - 10$

c)  $y = 6x + 10x^2 - 4x^3 + 8x^4$

How do your two graphs for each polynomial compare at the right and left hand sides of the graph?

**Question 168.** How does the degree of the polynomial effect the end behavior of graph?

**Information 169.** The end behavior of the highest degree term determines the end behavior of a polynomial. So knowing each different monomial's end behavior will allow you to figure out what the end behavior of any polynomial is.

**Question 170.** What would you expect for the right end behavior (up or down) of the following polynomials?

a)  $y = 8x - 2$

b)  $y = 3x^2 - 4$

c)  $y = -3x^2 + 4$

d)  $y = 3x - 4x^3 + 2$

e)  $y = -7x^5 + 3x + 100$

**Question 171.** What would you expect for the left end behavior (up or down) of the following polynomials?

a)  $y = 8x - 2$

b)  $y = 3x^2 - 4$

c)  $y = -3x^2 + 4$

d)  $y = 3x - 4x^3 + 2$

e)  $y = -7x^5 + 3x + 100$

**Information 172.** The most efficient method of plotting polynomials by hand is to plot intercepts (including the behavior at x-intercepts), plot end behaviors, and then connect the pieces as simply as possible.

An  $x$ -intercept coming from a factor of the form  $(x - a)^k$  is of multiplicity  $k$  and has local behavior like  $x^k$  does at the origin. A  $x$ -intercept with multiplicity 1 looks linear, multiplicity 2 looks parabolic, multiplicity 3 looks cubic, and so on.

**Question 173.** a) Factor  $x^2 - x - 2$ .

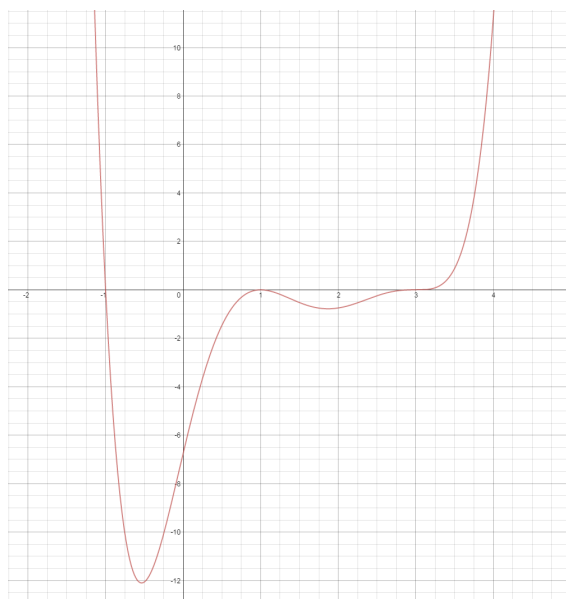
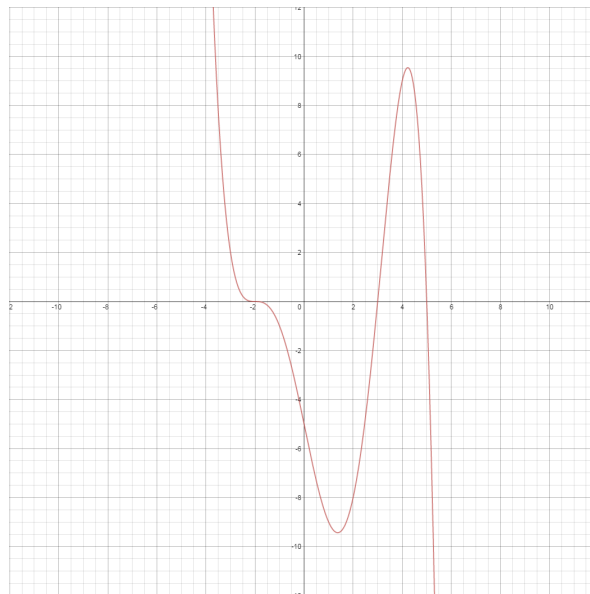
- b) What are the  $x$ -intercepts of  $y = x^2 - x - 2$ ? What are the multiplicities of each of these  $x$ -intercepts?
- c) What is the  $y$ -intercept of  $y = x^2 - x - 2$ ? (There is no multiplicity associated with  $y$ -intercepts.)
- d) What is the right end behavior of  $y = x^2 - x - 2$ ?
- e) What is the left end behavior of  $y = x^2 - x - 2$ ?
- f) Draw each of the pieces of information from the preceding parts on a set of axes and connect them in the simplest possible way.

**Question 174.** Repeat the previous question for the following polynomials and make sure your graphs are at least a quarter page in order to label and see the details of the graph.

- a)  $y = x^3 - x^2$
- b)  $y = x^2 - x^4$
- c)  $y = (x + 1)^2(3 - 2x)(x - 4)^3$
- d)  $y = -3(x - 1)^2(x + 1)^2$

In the preceding problems, we used algebra to figure out the geometry. As it turns out, it is just as easy to go backwards once you understand where each piece comes from.

**Question 175.** Give  $y$  as a polynomial in  $x$  that fits each of the graphs below.



## 2.4 Rational Polynomials and Their Graphs

**Information 176.** Rational polynomial functions are of the form. Rational comes from the same place as ratio, so rational polynomials are ratios of polynomials and are of the form  $y = \frac{f(x)}{g(x)}$ , where  $f$  and  $g$  are polynomials

in  $x$ . Rational polynomials have many of the same behaviors as polynomials, such as  $x$ - and  $y$ - intercepts, right and left end behaviors, etc. One of the features that rational polynomials can have that do not occur in polynomials are asymptotes (vertical, horizontal, or slant).

**Question 177.** Give all intercepts for the following rational polynomials:

a)  $y = \frac{x^2-1}{x^2+1}$

b)  $y = \frac{1-x^2}{3x+1}$

c)  $y = \frac{x^3-125}{2x-1}$

d)  $y = \frac{5x}{(x-1)^2(3x+1)}$

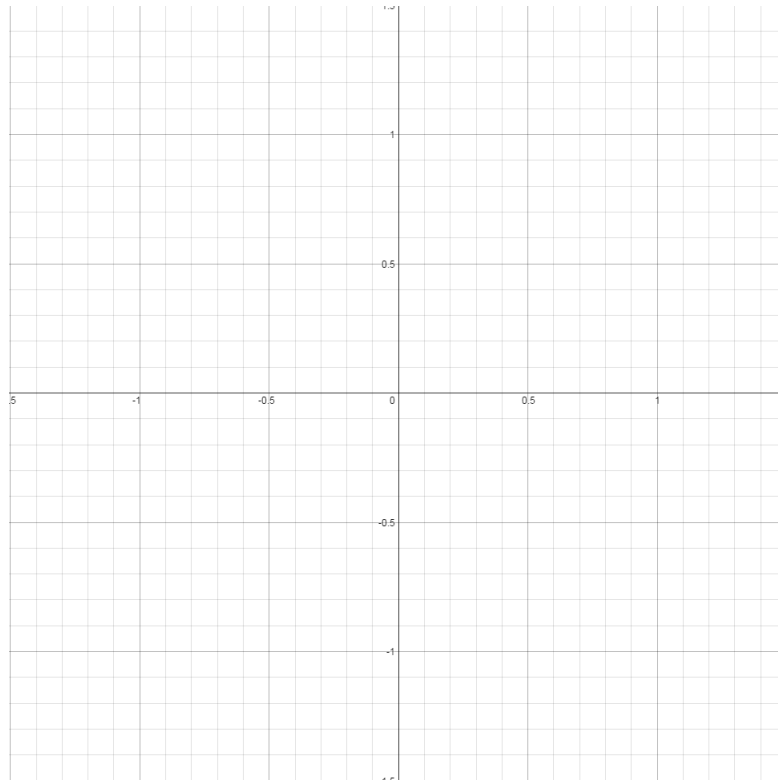
e)  $y = \frac{x^2+4x+4}{(x-2)^2}$

f)  $y = \frac{x^2+4x+4}{x-2}$

g)  $y = \frac{x-2}{x^2+4x+4}$

**Question 178.** Draw the graphs for the following basic rational polynomials on the same set of axes below:

$$y = 1/x, y = 1/x^2, y = 1/x^3, y = 1/x^4$$



**Question 179.** *What patterns do you see for the end behavior? What patterns do you see for the behavior around  $x = 0$ ?*

**Information 180.** *A rational polynomial of the form  $y = \frac{f(x)}{g(x)}$  will have a vertical asymptote at  $x = a$  if  $g(a) = 0$  and  $f(a) \neq 0$ . Vertical asymptotes have multiplicity the same way that  $x$ -intercepts do and you can determine the directions (same or opposite) in which the asymptote will be approached in the same way. In other words, a vertical asymptote  $x = a$  coming from a term of multiplicity  $k$  ( $(x - a)^k$ ) will behave like  $1/x^k$  does around  $x = 0$ .*

**Question 181.** *Give all vertical asymptotes (and multiplicity) for the following rational polynomials:*

a)  $y = \frac{x^2 - 1}{x^2 + 1}$

b)  $y = \frac{1 - x^2}{3x + 1}$

c)  $y = \frac{x^3 - 125}{2x - 1}$

d)  $y = \frac{5x}{(x - 1)^2(3x + 1)}$



$$e) y = \frac{x^2+4x+4}{(x-2)^2}$$

$$f) y = \frac{x^2+4x+4}{x-2}$$

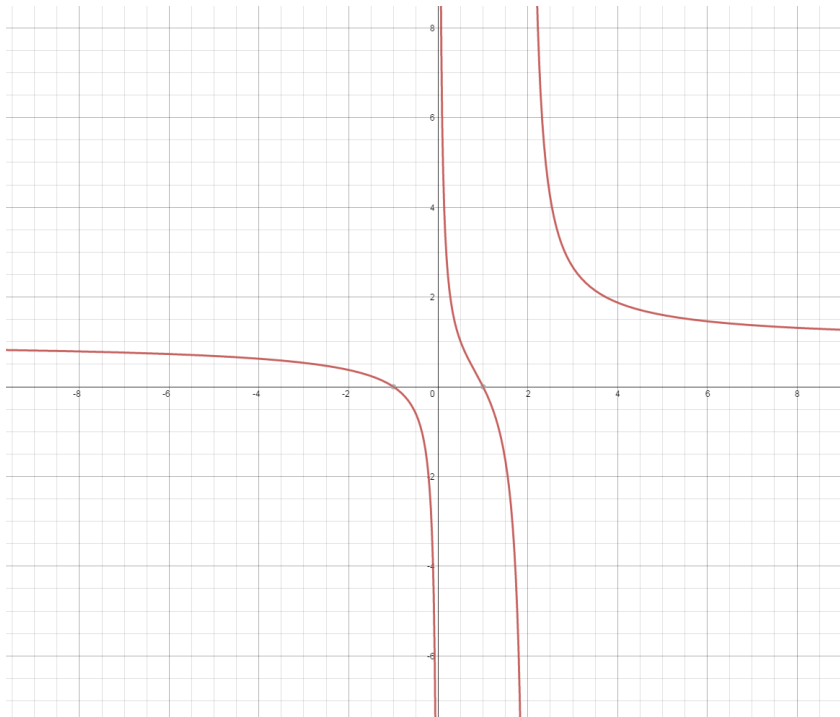
$$g) y = \frac{x-2}{x^2+4x+4}$$

**Information 182.** *Horizontal and slant asymptotes describe some of the possible end behaviors of rational polynomials. Because we are trying to describe end behavior, we need to look at the highest degree terms, specifically the **ratio of the highest degree terms** in the numerator and denominator. The ratio of the highest degree terms is different than the ratio of the degree of the numerator to the degree of the denominator.*

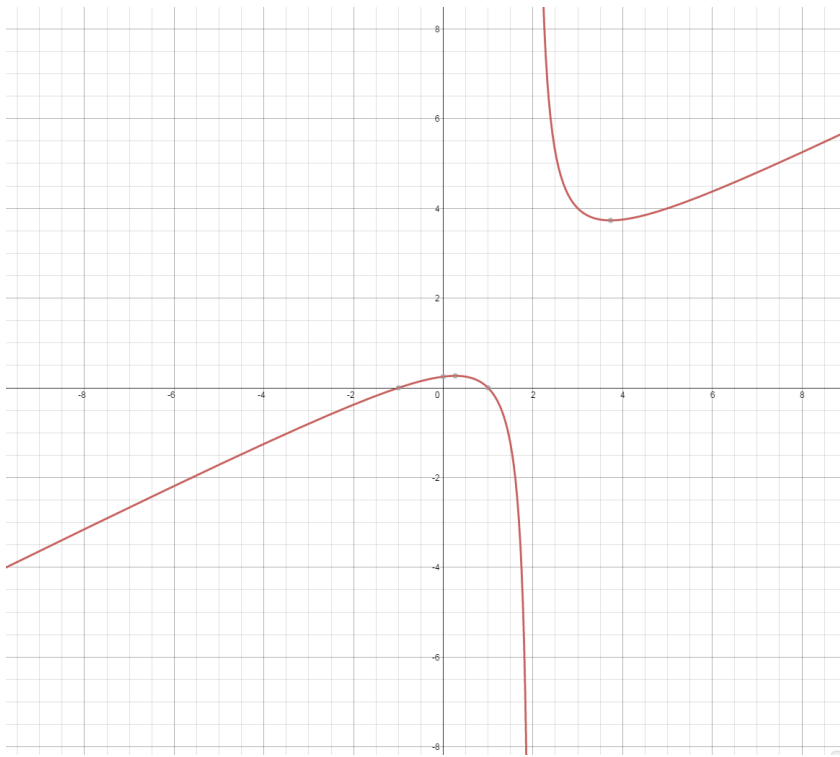
*Similar to a polynomial, a rational polynomial will have the same end behavior as the ratio of its highest degree terms. Horizontal asymptotes correspond to horizontal (or constant) behavior on the ends and slant asymptotes correspond to slanted (non-horizontal) linear behavior on the ends. If you find that the graph has a slant asymptote, then you need to perform polynomial division to find the slant asymptote.*

**Question 183.** *For each of the following graphs, state all vertical asymptotes.*

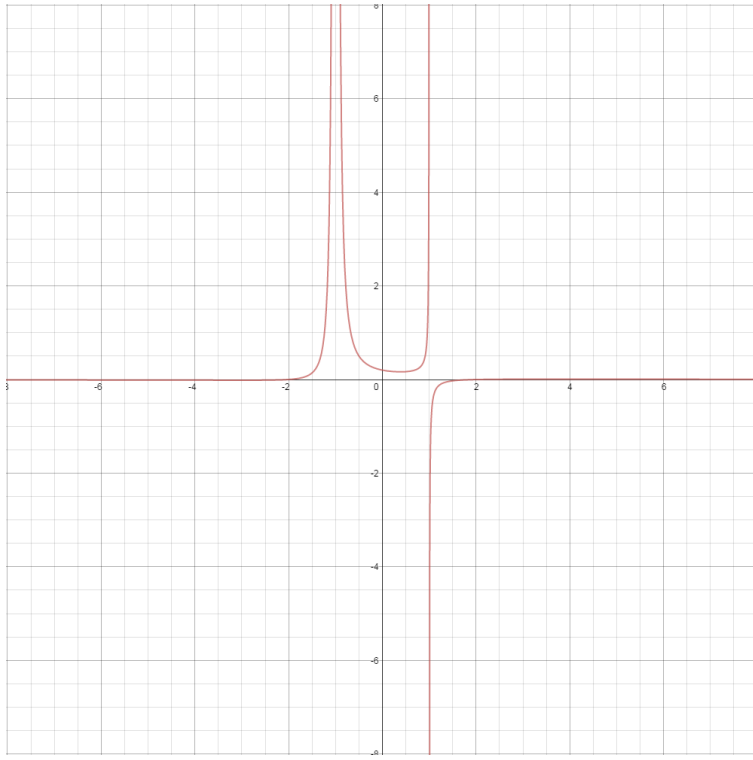
*If the graph has a horizontal or slant asymptote, give an equation for the horizontal or slant asymptote.*



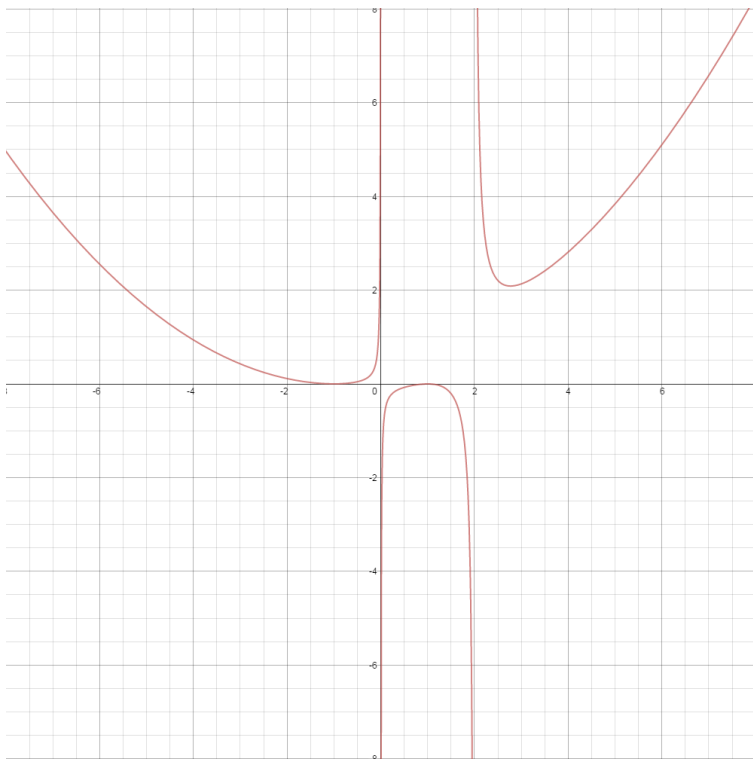
a)



b)



c)



d)

**Question 184.** a) What is the ratio of the highest degree terms for the following rational polynomials?

b) What is the right and left end behavior, in other words, what horizontal or slant asymptotes will the graph have?

a)  $y = \frac{x^2-1}{x^2+1}$

b)  $y = \frac{1-x^2}{3x+1}$

c)  $y = \frac{x^3-125}{2x-1}$

d)  $y = \frac{5x}{(x-1)^2(3x+1)}$

e)  $y = \frac{x^2+4x+4}{(x-2)^2}$

f)  $y = \frac{x^2+4x+4}{x-2}$

g)  $y = \frac{x-2}{x^2+4x+4}$

**Question 185.** Using the information you computed in previous questions, draw the graphs of the following rational polynomials. Be sure to label all intercepts, asymptotes and other features of your graph.

a)  $y = \frac{x^2-1}{x^2+1}$

b)  $y = \frac{1-x^2}{3x+1}$

c)  $y = \frac{x^3-125}{2x-1}$

d)  $y = \frac{5x}{(x-1)^2(3x+1)}$

e)  $y = \frac{x^2+4x+4}{(x-2)^2}$

f)  $y = \frac{x^2+4x+4}{x-2}$

g)  $y = \frac{x-2}{x^2+4x+4}$

## Chapter 3

### The General Second Degree Equation and Conic Sections

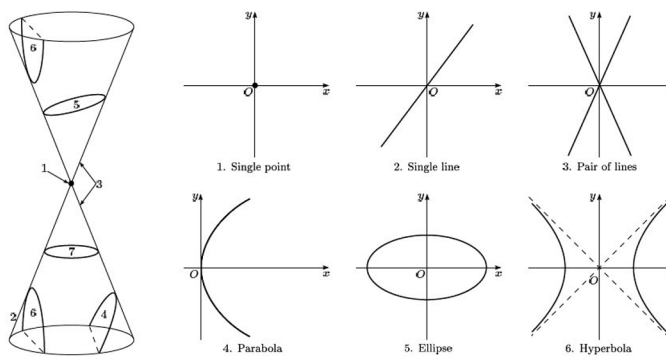
**Information 186.** Previously, we had the general form of a line as given by the general 1st degree equation:

$$Ax + By + C = 0$$

In the next several weeks we will be looking at the **general second degree equation**:

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The possible graphs of the general second degree equation are called **conic sections**.



#### 3.1 Circles

**Information 187.** A circle is the set of all points in a plane at a fixed distance (called the radius) from a fixed point (called the center).

**Question 188.** Give the equation of a circle with center  $(-3, 7)$  with radius 4.

**Question 189.** What is the equation of a circle given a radius  $r$  and center  $(h, k)$ ? This is the **Standard Form of a Circle**.

**Question 190.** Expand the Standard Form of a Circle into a general second degree equation of the form  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ . What kinds of terms do you notice? Which terms are missing?

Remember our earlier discussion about how to complete the square from Chapter 1.

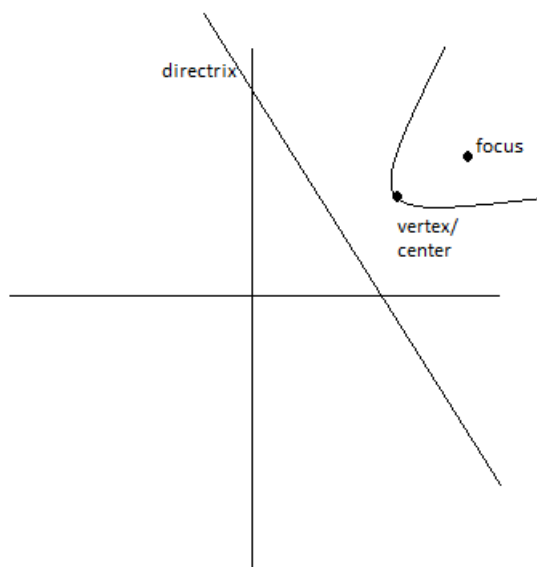
**Question 191.** What is the center and radius of the circle given by  $2x^2 + 2y^2 - 4x + 10y - 10 = 0$ ? Give a graph.

**Question 192.** What is the center and radius of the circle given by  $2x^2 + 2y^2 - 4x + 10y + 20 = 0$ ? Give a graph.

**Question 193.** What is the center and radius of the circle given by  $2x^2 + 2y^2 - 4x + 10y + \frac{29}{2} = 0$ ? Give a graph.

## 3.2 Parabolas

**Information 194.** A **parabola** is the set of all points in a plane equidistant from a fixed point (called the focus) and a fixed line (called the directrix)



- Question 195.** a) What is the distance from the point  $(x,y)$  to the point  $(c,0)$ ?
- b) What is the distance from the point  $(x,y)$  to the line  $x = -c$ ?
- c) Use your answer to the previous parts to give the equation of a parabola with focus  $(c,0)$  and directrix given by  $x = -c$ .
- d) Simplify the equation of the parabola as much as possible. This will require some algebra but we should have a nice form to use for parabolas of this type and we will not need to do this algebra again for a parabola. Hint: Solve for  $x^2$  at the end.

**Question 196.** How would your equation change if the focus of a parabola is at the point  $(0,c)$  and the directrix is given by  $y = -c$ ?

**Information 197.** A conic section in standard position means that the focus (or foci) is on either the  $x$ - or  $y$ -axis and the center is at the origin.

For a parabola, the **vertex** and the **center** are the same point and  $c$  is the distance from the center to the focus. You should not think of  $c$  as being the coordinate of the focus since not every parabola we will deal with is in standard position. When plotting a parabola, you need to label the focus, the directrix, and the vertex.

Note that the forms you found in Questions 195 and 196 will work for negative values of  $c$  as well.

**Question 198.** Graph  $y = x^2$ .

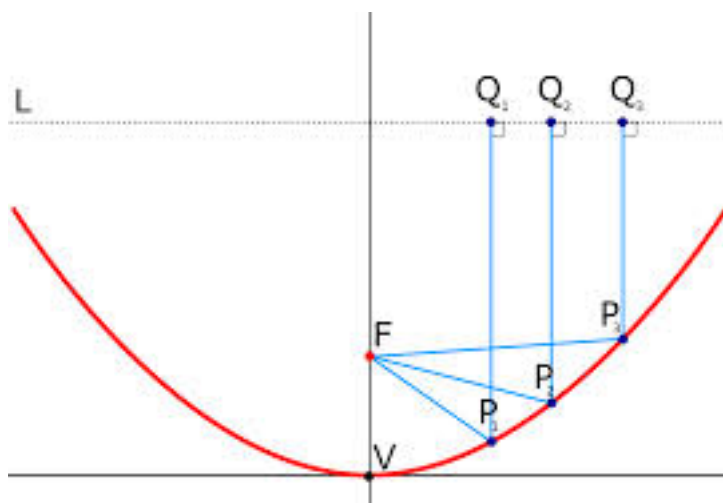
**Question 199.** Graph  $x^2 = -3y$

**Question 200.** What is the equation of the parabola in standard position with focus  $(0, -6)$ ?

**Question 201.** Give the equations of all parabolas in standard position that contain the point  $(2,4)$ .

**Question 202.** Give the equations of all parabolas in standard position that contain the points  $(3, -4)$  and  $(-3, -4)$

**Information 203.** Parabolas have many uses including as reflectors, as bridge supports, and in celestial mechanics. For instance, parallel beams of incoming light or sound will be reflected by a parabolic mirror through the focus.



When solving application/word problems make sure you draw a good picture and label all of the information you have before you try to put coordinates to the problem or use an equation.

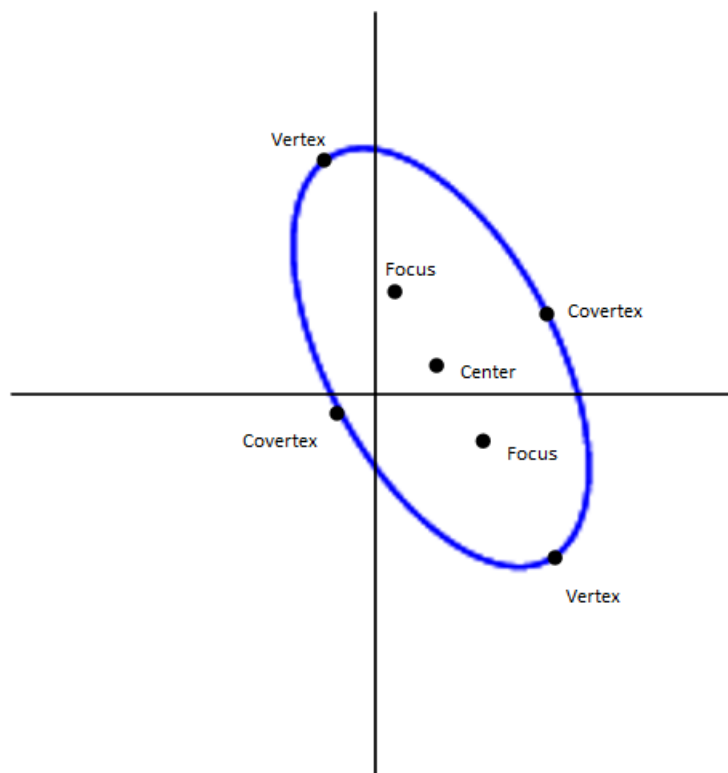
**Question 204.** While watching the Dallas Cowboys play the New England Patriots, you notice that the Patriots are using a parabolic reflector with a microphone at its focus to spy on Jerry Jones' drink orders. You notice that the microphone is in the same plane as the edge of the reflector and the reflector has a diameter of 3.5 feet. How far is the microphone from the vertex of the reflector?

**Question 205.** Later in the game you see another parabolic reflector being used to spy on Tony Romo's wife but this time you can tell that the microphone is in the same plane as the edge of the reflector and that the microphone is 8 inches from the vertex of the reflector. How wide is this second listening device?

### 3.3 Ellipses

**Information 206.** An *ellipse* is the set of all points such that the sum of the distances from the point  $(x,y)$  to a pair of distinct points (called foci) is a fixed constant. We will call this constant  $2a$ . Remember: The "nice" conic sections we will deal with first are in standard position. Standard position means that the focus (or foci) is on either the  $x$ - or  $y$ -axis and the center is at the origin.





**Question 207.** a) What is the distance from a point  $(x,y)$  to the point  $(c,0)$ ?

b) What is the distance from a point  $(x,y)$  to the point  $(-c,0)$ ?

c) Use your answer to the previous parts to give the equation of an ellipse with foci  $(c,0)$  and  $(-c,0)$ .

d) Simplify your equation as much as possible. This will require some algebra but we should have a nice form to use for ellipses of this type and we will not need to do this algebra again for an ellipse.

e) What are the  $x$ - and  $y$ -intercepts for this ellipse?

**Question 208.** a) How does your equation change if the foci are  $(0,c)$  and  $(0,-c)$ ?

b) What are the  $x$ - and  $y$ -intercepts for this ellipse?

**Information 209.** Whenever you draw an ellipse, you need to label the center, foci, vertices, and covertices.

The points on the ellipse farthest from the center (on the major axis) are the vertices and the points on the minor axis are the covertices. The vertices are a distance  $a$  from the center and the covertices are a distance  $b$  away from the center where  $a^2 = b^2 + c^2$ .

The eccentricity of an ellipse is  $\frac{c}{a}$ .

**Question 210.** Sketch and label the graph of  $25x^2 + 16y^2 = 400$ .

**Question 211.** Find an equation of an ellipse with vertices  $(\pm 6, 0)$  and eccentricity  $3/5$ .

**Question 212.** Sketch and label the graph of  $9x^2 + y^2 = 36$ .

**Question 213.** Find an equation of an ellipse in standard position with vertex  $(5, 0)$  and contains the point  $(\sqrt{15}, 2)$

**Information 214.** Ellipses are used as reflectors and are found in applications like planetary motion. For instance, the orbit of the Moon around Earth is an ellipse with the Earth at one of the foci and not at the center. Ellipses are useful as reflectors because any source of light or sound at one foci will be reflected directly at the other focus.

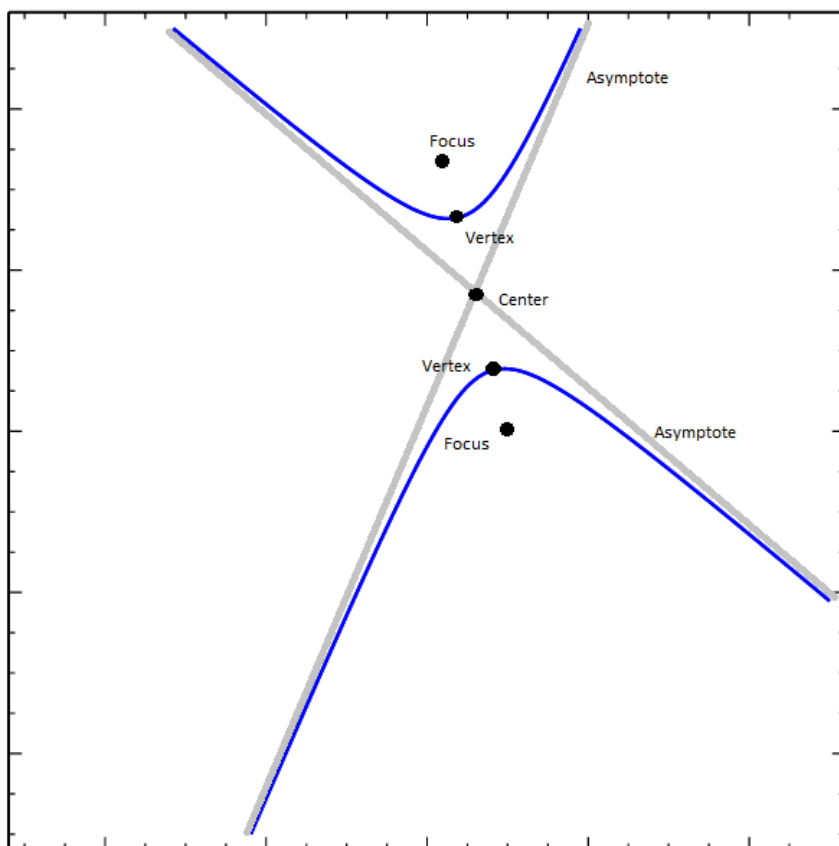
**Question 215.** A room is elliptic with vertical walls 6 feet high and an ellipsoid ceiling. If it is 40 feet long and 20 feet wide, where should two people stand (other than next to each other) so that they can whisper to each other without being heard by others in the room?

**Question 216.** If the orbit of the Moon around the Earth varies from 221,463 miles to 252,710 miles, find the eccentricity of the Moon's orbit and the lengths of the major and minor axes.

**Question 217.** The ellipse in Washington DC is 3525 feet by 1265 feet. Give an equation for the edge of the ellipse if the axes are placed in the center with the  $y$ -axis as the major axis.

### 3.4 Hyperbolas

**Information 218.** A **hyperbola** is the set of all points  $(x, y)$  in a plane such that the positive difference between the distances from  $(x, y)$  to a pair of distinct fixed points (called foci) is a fixed constant. Again, we will call this fixed constant  $2a$ . Remember: The "nice" conic sections we will deal with first are in standard position. Standard position means that the focus (or foci) is on either the  $x$ - or  $y$ -axis and the center is at the origin.



- Question 219.** a) What is the distance from a point  $(x,y)$  to the point  $(c,0)$ ?
- b) What is the distance from a point  $(x,y)$  to the point  $(-c,0)$ ?
- c) Use your answer to the previous parts to give the equation of a hyperbola with foci  $(c,0)$  and  $(-c,0)$ .
- d) Simplify your equation as much as possible. This will require some algebra but we should have a nice form to use for hyperbolas of this type and we will not need to do this algebra again for an hyperbola.
- e) What are the  $x$ - and  $y$ -intercepts for this hyperbola?
- Question 220.** a) How does your equation change if the foci are  $(0,c)$  and  $(0,-c)$ ?
- b) What are the  $x$ - and  $y$ -intercepts for this hyperbola?

**Information 221.** Whenever you draw a hyperbola, you need to label the center, foci, vertices, and asymptotes.

The points on the hyperbola closest to the center (on the major axis) are the vertices. The vertices are a distance  $a$  from the center and the asymptotes are of the form  $y = \pm(\frac{a}{b})x$  (if foci are on the  $y$ -axis) or  $y = \pm(\frac{b}{a})x$  (if foci are on the  $x$ -axis) where  $a^2 + b^2 = c^2$ .

The eccentricity of a hyperbola is  $\frac{c}{a}$ .

**Question 222.** Sketch and label the graph of  $25x^2 - 16y^2 = 400$ .

**Question 223.** Give the equation of a hyperbola with vertices  $(0, \pm 3)$  and  $e = 5/3$

**Question 224.** Sketch and label the graph of  $9x^2 - y^2 = -36$ .

**Question 225.** A proton is launched along the line  $y = \frac{1}{2}x$  toward the nucleus of an atom at the origin. If the proton is deflected upwards toward the line  $y = -\frac{1}{2}x$  along a hyperbolic path, give the equation for the path of the proton.

**Question 226.** Repeat the previous problem if the proton is deflected downward toward the line  $y = -\frac{1}{2}x$  along a hyperbolic path.

**Question 227.** A man standing at a point  $Q = (x, y)$  hears the crack of a rifle at point  $P_1 = (1000, 0)$  and the sound of the bullet hitting the target at  $P_2 = (1000, 0)$  at the same time. If the bullet travels at 2000 ft/s and sound travels at 1100 ft/s, find an equation relating  $x$  and  $y$ .

### 3.5 Translated Conic Sections

**Question 228.** a) Write out the forms of a parabola in standard position (there should be two). For each form, give the focus, the directrix, and the vertex.

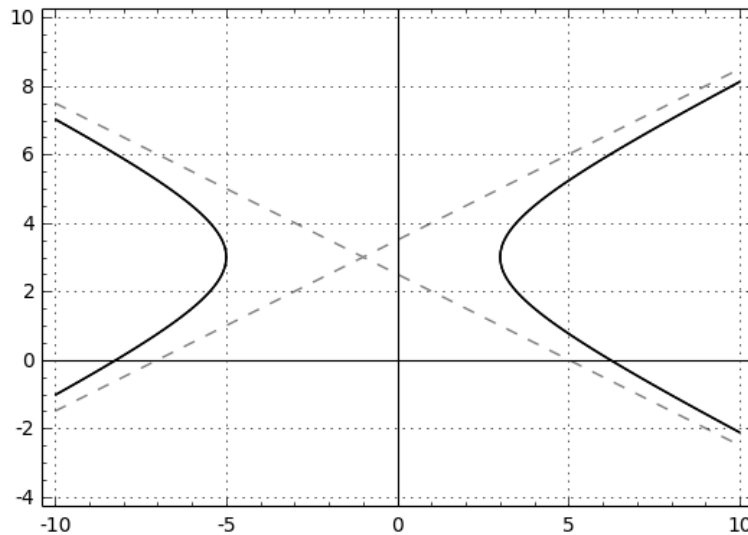
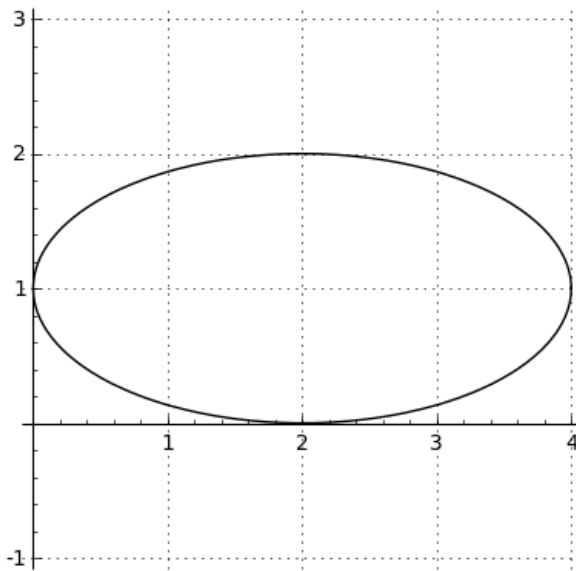
b) Write out the forms of an ellipse in standard position (there should be two). For each form, give the foci, the vertices, and the covertices.

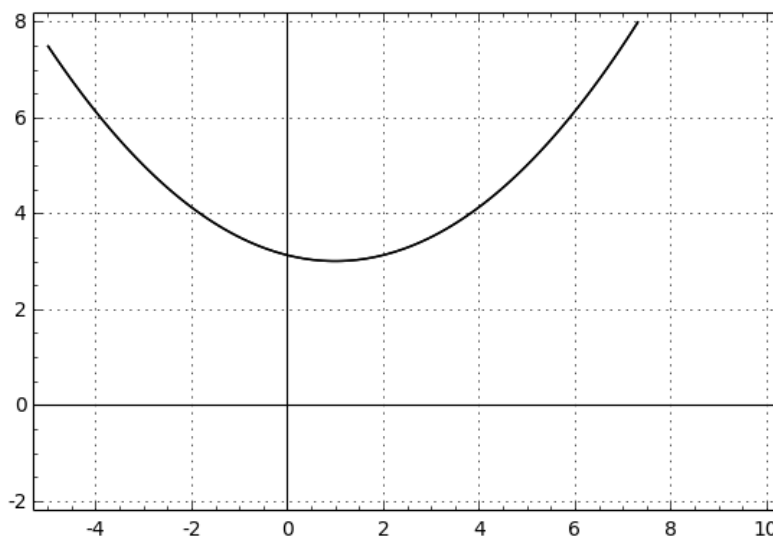
c) Write out the forms of a hyperbola in standard position (there should be two). For each form, give the foci, the vertices, and the asymptotes.

**Question 229.** a) For each of the shapes below, give the center of the conic section.

b) Draw a new set of axes ( $\hat{x}$ - and  $\hat{y}$ -axes) through the **center** of the conic section.

- c) Write the equation of the conic section in terms of the  $\hat{x}$ - and  $\hat{y}$ -axes.  
*Hint: In terms of  $\hat{x}$  and  $\hat{y}$  you have a conic section in standard position and can use your forms from Question 228.*
- d) Using your results of Questions 35 and 36, write the equation of the conic section in terms of the  $x$ - and  $y$ -axes.





**Question 230.** Expand each of the forms of a parabola, ellipse, and hyperbola in standard position (from Question 228) and compare the results to the general second degree equation ( $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ ). What kinds of terms do you notice? Which terms are missing? Just like we talked about with circles, the goal of this exercise is to be given a general second degree equation of the form,  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ , and identify if the graph will be one of the translated conic sections.

**Question 231.** For each of the following general second degree equations, identify what type of conic section is represented and complete the square or factor as necessary to obtain the translated form of the conic section. If it is not possible to identify the type of conic section, state why you can't identify the shape of the graph.

$$x^2 - 8x - 8y + 8 = 0$$

$$x^2 - xy + y^2 - 2 = 0$$

$$4x^2 + y^2 + 24x - 2y + 21 = 0$$

$$9x^2 - 4y^2 + 90x + 32y + 125 = 0$$

$$x^2 + 4xy - 2y^2 - 6 = 0$$

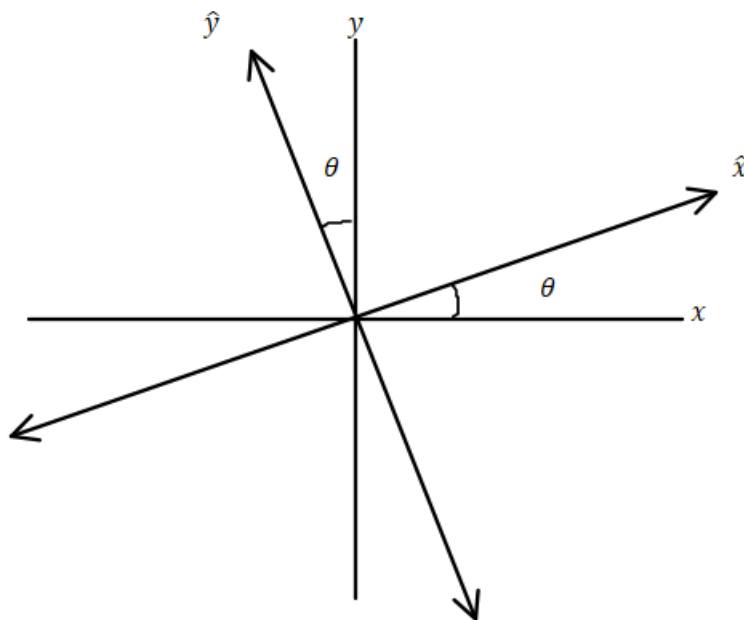
### 3.6 Rotated Conic Sections

**Information 232.** In our previous problems, we saw how translating our coordinate system to the center of our figure ( $\hat{x} = x - h$  and  $\hat{y} = y - k$ ) simplified our problem to standard position.

We still have to find equations and graph conics that are not oriented up/down or left/right. For this we will need to look at rotated coordinate systems. Specifically, if we would like to rotate our coordinate system counterclockwise by  $\theta$  around the origin, we use the substitution

$$x = \hat{x} \cos(\theta) - \hat{y} \sin(\theta)$$

$$y = \hat{x} \sin(\theta) + \hat{y} \cos(\theta)$$



**Question 233.** What does rotation by an angle of  $\frac{\pi}{2}$  give for the rotation equations above? Draw a picture to make sure your answer makes sense.

**Question 234.** a) Find a new representation of the line  $y = x + 1$  after rotating by an angle of  $\frac{\pi}{4}$ .

b) Draw  $y = x + 1$  on the  $x$ - and  $y$ - axes and draw your answer to part a) on a separate plot using  $\hat{x}$ - and  $\hat{y}$ - axes.

c) Now draw a graph that contains both the  $x$ - and  $y$ - axes and  $\hat{x}$ - and  $\hat{y}$ - axes (how are they related?). What does the graph of  $y = x + 1$  look like on this new plot?

**Question 235.** Find a new representation of  $x^2 - xy + y^2 - 2 = 0$  after rotating by  $45^\circ$ . Is your conic section in standard position? If so, graph the conic with both the original and rotated set of axes.

**Question 236.** Find a new representation for  $31x^2 + 10\sqrt{3}xy + 21y^2 - 144 = 0$  after rotating by  $\frac{\pi}{3}$ . Is your conic section in standard position? If so, graph the conic with both the original and rotated set of axes.

**Information 237.** Given a general second degree equation of the form  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ , rotation by an angle  $\theta$  that satisfies  $\tan(\theta) = \frac{(C-A) \pm \sqrt{(C-A)^2 + B^2}}{B}$  will give a representation without a  $xy$ -term.

**Question 238.** What angle of rotation is needed to eliminate the  $xy$ -term of  $x^2 - xy + y^2 - 2 = 0$ ?

**Question 239.** What angle of rotation is needed to eliminate the  $xy$ -term of  $31x^2 + 10\sqrt{3}xy + 21y^2 - 144 = 0$ ?

**Question 240.** What angle of rotation is needed to eliminate the  $xy$ -term of  $x^2 + 4xy - 2y^2 - 6 = 0$ ?

**Question 241.** Give the equation of the conic section given by  $31x^2 + 10\sqrt{3}xy + 21y^2 - 144 = 0$  after rotating to eliminate the  $xy$ -term. Graph the conic with both the original and rotated set of axes.

**Question 242.** Give the equation of the conic section given by  $x^2 + 4xy - 2y^2 - 6 = 0$  after rotating to eliminate the  $xy$ -term. Graph the conic with both the original and rotated set of axes.

**Information 243.** The shape of the conic section with equation  $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$  is given by the following table:

Conic Section	$B^2 - 4AC$
Hyperbola	Positive
Parabola	0
Ellipse	Negative

**Question 244.** What kind of conic section is:

- $x^2 - xy + y^2 - 2 = 0$
- $31x^2 + 10\sqrt{3}xy + 21y^2 - 144 = 0$
- $x^2 + 4xy - 2y^2 - 6 = 0$ ?



- d)  $3x^2 - 8xy + y^2 - 9 = 0$   
e)  $4x^2 + 3xy - y^2 - 9 = 0$   
f)  $-2x^2 + 12xy - 18y^2 - 9 = 0$   
g)  $9x^2 + 8xy - 6y^2 = 70$

### 3.7 Other Parametric Forms

**Information 245.** Recall the equation  $(x, y) = (x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1))$  is the **Parametric Form of a Line** through  $(x_1, y_1)$  and  $(x_2, y_2)$ . We would like to try to describe other shapes using parametric equations. Parametric equations can be given as  $x(t)$  and  $y(t)$  or as a vector valued function  $\vec{v}(t) = \langle x(t), y(t) \rangle$ .

**Question 246.** Graph each of the following parametric equations. It will probably be helpful to calculate the  $(x, y)$  points to correspond to several  $t$  values like you did in Question 118.

- a)  $(x, y) = (3t - 1, 4t + 2)$   
b)  $(x, y) = (t^2 - 1, t^2 + 1)$   
c)  $(x, y) = (t^2 - 2t, t)$   
d)  $(x, y) = (\cos(t), \tan(t))$

**Question 247.** Give parametric equations for the circle centered at the origin of radius 1. Be sure to give the bounds of your parameter  $t$ . Plot your parametric equations to confirm your answer.

**Question 248.** Give parametric equations for the circle centered at  $(h, k)$  of radius 1. Be sure to give the bounds of your parameter  $t$ . Plot your parametric equations to confirm your answer.

**Question 249.** Give parametric equations for  $\frac{x^2}{4} + \frac{y^2}{9} = 1$ . Be sure to give the bounds of your parameter  $t$ . Plot your parametric equations to confirm your answer.

**Question 250.** Give parametric equations for  $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ . Be sure to give the bounds of your parameter  $t$ . Plot your parametric equations to confirm your answer.

**Question 251.** Give parametric equations for  $y = x^3 - x^2$ . Be sure to give the bounds of your parameter  $t$ . Plot your parametric equations to confirm your answer.

**Question 252.** Give parametric equations for  $y = x^2 - x^4$ . Be sure to give the bounds of your parameter  $t$ . Plot your parametric equations to confirm your answer.

**Question 253.** Give parametric equations for  $y^2 - 8x - 8y + 8 = 0$ . Be sure to give the bounds of your parameter  $t$ . Plot your parametric equations to confirm your answer.

**Question 254.** Give parametric equations for the positive branch of  $9x^2 - 4y^2 + 90x + 32y + 125 = 0$ . Be sure to give the bounds of your parameter  $t$ . Plot your parametric equations to confirm your answer.