Math 141: Section 4.1 Extreme Values of Functions - Notes

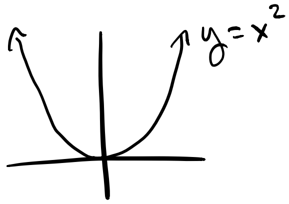
Definition: Let f be a function with domain D. Then f has an **absolute (global) maximum** value on D at a point c if

$$f(x) \leq f(c) \text{ for all } x \text{ in } D$$

and an **absolute (global) minimum** value on D at c if

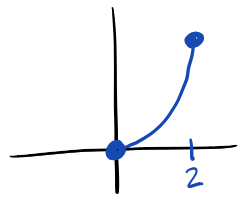
$$f(x) \geq f(c) \text{ for all } x \text{ in } D.$$

Example 1 Consider the function $y = x^2$ on the domains $(-\infty, \infty)$, $[0, 2]$, $(0, 2]$, and $(0, 2)$.



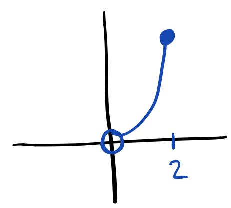
No global max
Absolute min
at $x=0$

[0,2]



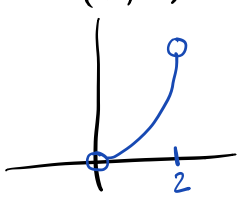
Absolute max
at $x=2$
Absolute min
at $x=0$

(0,2]



Absolute max
at $x=2$
No absolute
min

(0,2)

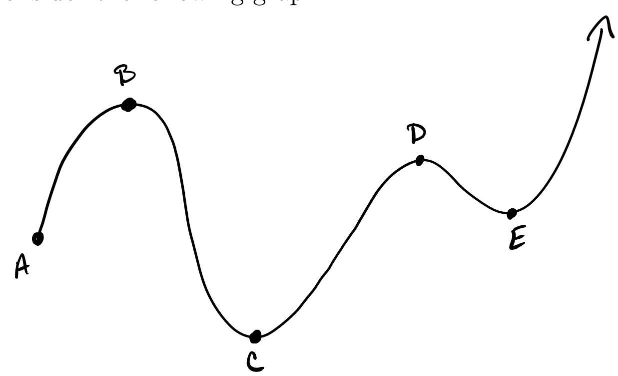


No absolute
max or min

Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

Local Extreme Values; Definition A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.
A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

Example 2 Consider the following graph:



Local max occurs at B, D
Local min occurs at A, C, E
Absolute min at C

The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

Definition: An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1) Evaluate f at all critical points and endpoints.

2) Take the largest and smallest of these values.

Example 3 Find the absolute maximum and minimum values of

$$f(x) = 10x(2 - \ln x)$$

on the interval $[1, e^2]$.

Since the interval $[1, e^2]$ is closed and finite, we are guaranteed an absolute max and an absolute min.

1) Find $f'(x)$

$$f'(x) = 10(2 - \ln x) + 10x(-1/x)$$
$$= 20 - 10\ln x - 10$$
$$= 10 - 10\ln x$$

2) Find critical points

$f'(x) = 0$

$$10 - 10\ln x = 0$$
$$-10\ln x = -10$$
$$\ln x = 1$$
$$x = e$$

3) Evaluate the original function $f(x) = 10x(2 - \ln x)$ at the critical point(s) and the endpoints of the interval

$f(e) = 10(e)(2 - \ln e) = 10e$
≈ 27.18

$f(e^2) = 10(e^2)(2 - \ln(e^2))$
$= 10(e^2)(2-2)$
$= 0$

$f(1) = 10(1)(2 - \underbrace{\ln(1)}_{=0})$
$= 20$

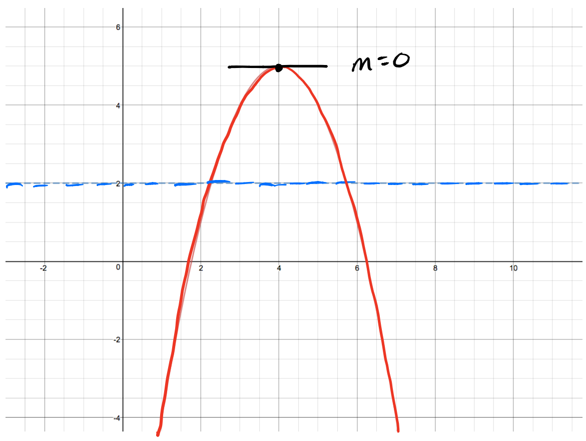
Absolute max value is ≈ 27.18 $(10e)$ and occurs at $x = e$

Absolute min value is 0 and occurs at $x = e^2$

Math 141: Section 4.2 The Mean Value Theorem - Notes

Rolle's Theorem

Consider the following graph:



Rolle's Theorem Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

Example 1 Show that the equation $x^3 + 3x + 1 = 0$ has exactly one real solution.

Intermediate Value Theorem:

Since $f(-1) = (-1)^3 + 3(-1) + 1 = -3$ and

$f(0) = 1$

the IVT says there is at least one real solution to $x^3 + 3x + 1 = 0$.

Rolle's Theorem says if there were another point where $x^3 + 3x + 1 = 0$, then there would exist a point $x = c$ where $f'(c) = 0$.

Note $f'(x) = 3x^2 + 3$ which is always positive. ($3x^2 + 3 = 0 \to 3x^2 = -3$, no real solution)

So, there is no such value for c and there is only one real solution to $x^3 + 3x + 1 = 0$.

The Mean Value Theorem Suppose $y = f(x)$ is continuous over a closed interval $[a, b]$ and differentiable on the interval's interior, (a, b). Then there is at least one point c in (a, b) at which

Slope of the secant line between x=a x=b

$$\underline{\frac{f(b) - f(a)}{b - a}} = \underline{f'(c)}.$$

Slope of the tangent line at x = c

Example 2 If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is 352/8=44 ft/sec. The Mean Value Theorem says that at some point during the acceleration the speedometer must read exactly 30 mph (44 ft/sec).

Corollary 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Corollary 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

Example 3 Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

If $g(x) = -\cos x$ then $g'(x) = \sin x$

$f(x) = g(x) + c$

$f(x) = -\cos x + c$ To solve for c, use (0,2)

$f(0) = -\cos(0) + c = 2$

$-1 + c = 2$

$c = 3$

$$\boxed{f(x) = -\cos x + 3}$$

Ex. Finding values of c that satisfy the MVT.

1) Find all values of c that satisfy the conclusion of the MVT for

$$f(x) = x^3 - x - 1 \quad \text{on } [-1, 3].$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad a \quad b$

$$\frac{f(b)-f(a)}{b-a} = \frac{f(3)-f(-1)}{3--1} = \frac{23+1}{4} = 6$$

$f'(c) = 6 \qquad f'(x) = 3x^2 - 1$

$3x^2 - 1 = 6 \qquad 3x^2 = 7$

$x = \sqrt{7/3}$ ($-\sqrt{7/3}$ is outside the interval)

2) Find all values of c that satisfy the conclusion of the MVT for

$$f(x) = \sin(x) \quad \text{on } [0, 7].$$

$$\frac{f(b)-f(a)}{b-a} = \frac{\sin(7)}{7}, \quad f'(x) = \cos x$$

$$\cos(x) = \frac{\sin(7)}{7}$$

$$x = \arccos\left(\frac{\sin(7)}{7}\right)$$

(Use technology to approximate the two solutions in the interval)

Math 141: Section 4.3 Monotonic Functions and the First Derivative Test - Notes

Increasing and Decreasing Functions As another corollary to the Mean Value Theorem, we can show that function with positive derivatives are increasing functions and functions with negative derivatives are decreasing functions.

Definition: A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

Corollary 3: Suppose that f is continuous on $[a.b]$ and differentiable on (a, b).
If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Example 1 Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

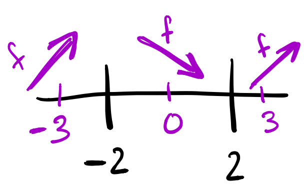
$f'(x) = 3x^2 - 12$ CPs occur when f' is 0 or undefined

$f'(x)$ is always defined so only consider $f'(x) = 0$

$3x^2 - 12 = 0$

$3x^2 = 12$

$x = 2$ or $x = -2$ critical points



$f'(-3) = 3(-3)^2 - 12 > 0$

$f'(0) = 3(0)^2 - 12 < 0$

$f'(3) > 0$

f is

Increasing on $(-\infty, -2) \cup (2, \infty)$

Decreasing on $(-2, 2)$

(Note the open intervals)

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across the interval from left to right,
1. If f' changes from negative to positive at c, then f has a local minimum at $x = c$;
2. If f' changes from positive to negative at c, then f has a local maximum at $x = c$;
3. If f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at $x = c$.

Example 2 Find the critical points of

$$f(x) = x^{1/3}(x-4).$$

Identify the open intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

$$f(x) = x^{4/3} - 4x^{1/3}$$

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

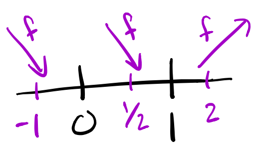
$$= \frac{4x^{1/3}}{3} - \frac{4}{3x^{2/3}} = \frac{4x-4}{3x^{2/3}}$$

$f'(x)$ is undefined at $x=0$

$f'(x) = 0$ when $4x-4=0$ or $x=1$

CPs $x=0$, $x=1$

$$f'(-1) = \frac{4(-1)-4}{3(-1)^{2/3}} < 0$$



$$f'(1/2) = \frac{4(1/2)-4}{3(1/2)^{2/3}} < 0$$

$$f'(2) = \frac{4(2)-4}{3(2)^{2/3}} > 0$$

Increase: $(1, \infty)$

Decrease: $(-\infty, 0)$, $(0, 1)$

Local min occurs at $x=1$

Furthermore, $x=1$ corresponds to the absolute min