Math 1B, lecture 9: Partial fractions

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1 Introduction

The fundamental functions in calculus are polynomials. Many, or even most, of the integrals that we can successfully evaluate at this stage are computed by performing transformations (substitution or parts) that turn them into polynomials, whose integrals are entirely understood. The purpose of this lecture is to understand how to integrate a broader class of functions closely related to polynomials, called rational functions. A rational function is simply a quotient of two polynomials. Integration of rational functions is completely understood, and we shall describe here how it is done.

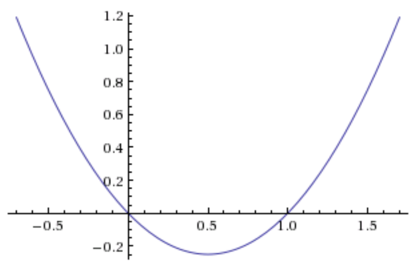
The most basic example of an integral of a rational function that will concern us in this course is $\int \frac{1}{x(x-1)}$. It can be computed by means of the observation that $\frac{1}{x(x-1)} = \frac{1}{x-1} - \frac{1}{x}$, from which it follows that the integral is $\ln|x-1| - \ln|x| + C$. This example, and others like it, will be central to the third part of the course, when we study differential equations and population dynamics. The same basic idea behind this integral is used to integrate all rational functions: they must first be split into polynomials and "partial fractions," which are very simple rational functions like $\frac{1}{x}$ or $\frac{1}{x-1}$ whose integrals are easy to compute as logarithms (and in some cases, inverse trigonometric functions). All rational functions can be decomposed into sums of such partial fractions.

The reading for today is Gottlieb §29.3. The homework for today is problem set 7 and a topic outline.

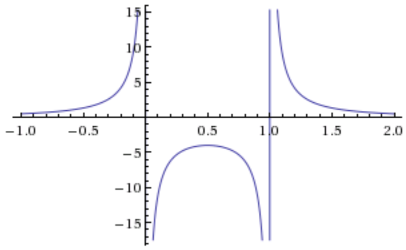
2 Rational functions

A *rational function* is simply a quotient of two polynomials. We begin by attempting to understand the basic behavior of these functions, which will shed light on why they should be able to decompose the way that they do.

Example 2.1. Graph the function $f(x) = \frac{1}{x(x-1)}$.
To begin, consider the graph of the function $y = x(x-1)$, which is as follows.

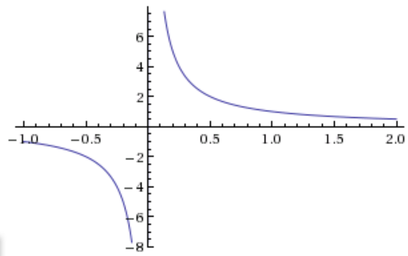


From this, it is easy to see what the graph of $\frac{1}{x(x-1)}$ will look like: where $x(x-1)$ approaches 0, its reciprocal will approach infinity. Similarly, where $x(x-1)$ goes to infinity, its reciprocal will go to 0. Reasoning in this fashion demonstrates that the graph will look like the following.

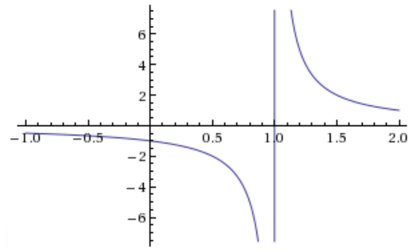


Now, notice that $\frac{1}{x}x - 1$ is in fact equal to $\frac{1}{x-1} - \frac{1}{x}$. It is worth noticing that this could have been predicted, without any algebra, by considering the shapes of the graphs $y = \frac{1}{x}$ and $y = \frac{1}{x-1}$.

$y = \frac{1}{x}$



$y = \frac{1}{x-1}$



Each function contributes one asymptote to $\frac{1}{x(x-1)}$; together they account for all the behavior of the function. This will be a general principle.

Graphing more complicated rational functions, such as $\frac{x+1}{x(x-1)}$ is more difficult to do just by looking at them, but we can understand them by thinking about some algebra.

First consider the long-term behavior of a rational function $f(x) = P(x)/Q(x)$ (where $P(x)$ and $Q(x)$ are polynomials). For the simple example $f(x) = \frac{1}{x(x-1)}$, $f(x)$ decays to 0 as x becomes very positive or very negative. In other words, the horizontal asymptote of $y = f(x)$ is $y = 0$. This will be true in general whenever $f(x)$ is what is called a proper rational function.

Definition 2.2. A rational function $f(x) = P(x)/Q(x)$ is called *proper* if the degree of Q exceeds the degree of P.

To see why any proper rational function has a horizontal asymptote, consider the example $f(x) = \frac{x^2+1}{x^3-1}$. Dividing both the numerator and denominator by x^2 shows that $f(x) = \frac{1+1/x^2}{x-1/x^2}$. As x goes to $\pm\infty$, both terms $1/x^2$ become 0, so that the function resembles $1/x$, which of course goes to 0. The same argument will work for any proper rational function: divide the numerator and denominator by x^d, where d is the degree of the numerator, and then it becomes clear that $f(x)$ goes to 0 as x goes to $\pm\infty$.

To apply the technique of partial fractions, it is first necessary to transform rational functions into proper rational functions. This is accomplished by long division, synthetic division, or any other technique you happen to know for dividing polynomials. In most of the problems we give you, most of these techniques will not be needed, since the result can be done more or less by hand. The result of this process will be that $f(x) = P(x)/Q(x)$ is rewritten as $S(x) + \frac{R(x)}{Q(x)}$, where the degree of R is les than the degree of Q, and S is another polynomial. The polynomial S is usually called the *quotient*, and R is called the *remainder*. This process is very much analogous to long division of integers as you learned it in elementary school.

Example 2.3. Express $\frac{x^3+1}{x^2-x}$ as a polynomial plus a proper rational function.
To do this "by hand," first notice that the leading term will be $x^3/x^2 = x$ (divide the leading terms of the numerator and denominator). To pull out a factor of x, rewrite the numerator to include x times the

denominator:

$$\begin{aligned}
\frac{x^3+1}{x^2-x} &= \frac{x^3-x^2+x^2+1}{x^2-x} \\
&= \frac{x(x^2-x)+x^2+1}{x^2-x} \\
&= x+\frac{x^2+1}{x^2-x}
\end{aligned}$$

Now, again dividing leading terms, the next term will be $x^2/x^2 = 1$.

$$\begin{aligned}
\frac{x^2+1}{x^2-x} &= \frac{x^2-x+x+1}{x^2-x} \\
&= 1+\frac{x+1}{x^2-x}
\end{aligned}$$

Putting these two steps together, we have the required from.

$$\frac{x^3+1}{x^2-x} = x+1+\frac{x+1}{x^2-x}$$

Notice that since a proper rational function goes to 0 as x goes to $\pm\infty$, the polynomial (quotient) found by performing division described the long-term behavior of the function, while the remainder describes is behavior at special points (the vertical asymptotes). So this prices effectively splits a rational function into two pieces describing its long-term and short-term behavior.

Since we know how to integrate polynomials, we know how to integrate the "long-term" part of the function. What is left is to decompose the short-term behavior, that is, the proper rational function, into something that can be integrated. The basic technique is to break it into sum of functions, each of which describes the behavior around some asymptote. This is the partial fraction decomposition.

3 Partial fraction decomposition

In order to rewrite rational functions in a form that allows them to be easily integrated, we shall rewrite them as a sum of simpler rational functions, each of which accounts for one of the factors of the denominator. This process is called *expansion by partial fractions.*

Suppose that $f(x) = P(x)/Q(x)$ is a proper rational function (i.e. the degree of P is less than the degree of Q). Then f has some number of vertical asymptotes, each given by a place where $Q(x) = 0$, and hence f spikes to infinity. If $Q(a) = 0$ then this spike is due to the fact that $(x-a)$ is a factor of $Q(a)$. This same asymptote is produced by the very simple function $\frac{1}{x-a}$, so we might attempt to account for the asymptote by a multiple of this function. In fact, this works in many cases.

Example 3.1. Consider the function $\frac{x+1}{x(x-1)}$, where a, b are any constants. Let us attempt to write this in the following way.

$$\frac{x+1}{x(x-1)} = \frac{A}{x}+\frac{B}{x-1}$$

By forming common denominators, this is equivalent to the following.

$$\begin{aligned}
\frac{x+1}{x(x-1)} &= \frac{A(x-1)+Bx}{x(x-1)} \\
&= \frac{(A+B)x-A}{x(x-1)}
\end{aligned}$$

Therfore, we must choose A and B so that $A+B=1$ and $-A=1$. Of course the only way to achieve this is $A=-1, B=2$. Therefore this shows that:

$$\frac{x+1}{x(x-1)} = -\frac{1}{x} + \frac{2}{x-1}.$$

Therefore, we have the following technique to rewrite proper rational functions: assume that they can be written as a sum of terms $\frac{A}{x-a}$, where $x=a$ is a root of the denominator, and A is a constant, add these fractions (in terms of the coefficients $A, B, \dots$) and use the result to find linear equations in the constants. If these equations can be solved, then the rational function can indeed by written as a sum of the terms $\frac{A}{x-a}$. This technique works in many cases.

Unfortunately, there are cases where this technique will fail. In fact, it will fail in two cases: if the denominator has complex roots and if the denominator has repeated roots. The following two examples illustrate these possibilities.

Example 3.2. Consider $f(x) = \frac{x-1}{x^3+x}$. The denominator factors as $x(x^2+1)$, which cannot be factored further without using complex numbers. Suppose we try to write this in the following form, similar to the technique above.

$$\begin{aligned}\frac{x-1}{x^3+x} &= \frac{A}{x} + \frac{B}{x^2+1} \\ &= \frac{Ax^2+A+Bx}{x(x^2+1)}\end{aligned}$$

In order for this equation to be true, we would need $A=0$ for the quadratic term to be 0, but also $A=-1$ for the constant term. So this is not going to work. However, the following approach will work instead.

$$\begin{aligned}\frac{x-1}{x^3+x} &= \frac{A}{x} + \frac{Bx+X}{x^2+1} \\ &= \frac{Ax^2+A+Bx^2+Cx}{x(x^2+1)} \\ &= \frac{(A+B)x^2+Cx+A}{x^3+x}\end{aligned}$$

Now this will work as long as:

$$\begin{aligned}A+B &= 0 \\ C &= 1 \\ A &= -1\end{aligned}$$

And this can be solved: $A=-1, B=1, C=1$. Therefore:

$$\frac{x-1}{x^3+x} = -\frac{1}{x} + \frac{x+1}{x^2+1}$$

In fact, this is the the best we can do, but it is good enough to compute the integral, as we will see in the next section.

Example 3.3. Consider $f(x) = \frac{x^2+1}{x^3+x^2}$. The denominator factors as $x^2(x+1)$, so the vertical asymptotes are at $x=0$ and $x=-1$. Let us try to do the technique above.

$$\begin{aligned}
\frac{x^2+1}{x^3+x^2} &= \frac{A}{x}+\frac{B}{x+1} \\
&= \frac{Ax+A+Bx}{x(x+1)}
\end{aligned}$$

This is simply not going to work, because the denominator is wrong. However, the following method will work: introduce a term for $\frac{1}{x}$, and also a term for $\frac{1}{x^2}$.

$$\begin{aligned}
\frac{x^2+1}{x^3+x^2} &= \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1} \\
&= \frac{Ax+B}{x^2}+\frac{C}{x+1} \\
&= \frac{(Ax+B)(x+1)+Cx^2}{x^2(x+1)} \\
&= \frac{Ax^2+(A+B)x+B+Cx^2}{x^3+x^2} \\
&= \frac{(A+C)x^2+(A+B)x+B}{x^3+x^2}
\end{aligned}$$

Now, this equation will hold as long as:

$$\begin{aligned}
A+C &= 1 \\
A+B &= 0 \\
B &= 1
\end{aligned}$$

Solving these equations gives $A=-1, B=1, C=2$. Therefore the following is obtained.

$$\frac{x^2+1}{x^3+x^2} = -\frac{1}{x}+\frac{1}{x^2}+\frac{2}{x+1}$$

The reason that these sort of decompositions are useful is that the resulting terms are easy to integrate, as will be seen in the following section.

The general process of expansion by partial fractions, for any rational function $f(x)=P(x)/Q(x)$ is the following.

Expansion by partial fractions

1. Perform division to rewrite $f(x)$ as a polynomial $S(x)$ plus a proper rational function $R(x)/Q(x)$, where the degree of $R(x)$ is less than the degree of $Q(x)$.

2. Factor the denominator $Q(x)$.

3. "Set up the partial fraction expansion": write $R(x)/Q(x)$ as a sum of simple rational functions, of the following forms.

   - If $(x-a)$ is a factor of $Q(x)$, include a constant times $\frac{1}{x-a}$.
   - If $(x-a)$ occurs more than once when $Q(x)$ is factored, also include a constant times $\frac{1}{(x-a)^2}$, a constant times $\frac{1}{(x-a)^3}$, and so on, stopping at the highest power of $(x-a)$ that occurs in $Q(x)$.

- If $x^2 + bx + c$ is a factor of $Q(x)$ and is an irreducible quadratic, include a term of the form $\frac{Ax+B}{x^2+bx+c}$, where A, B are some variable symbols that have not been used yet.
- If $x^2 + bx + c$ occurs more than once when $Q(x)$ is factored, also include terms of the form $\frac{Ax+B}{(x^2+bx+c)^k}$ for all k from 1 to the number of times this quadratic occurs in the factorization.

4. Add up the resulting simple functions (in terms of the unchosen constants $A, B, C, \ldots$ to obtain equations that must be satisfied by these constants.

5. Solve the equations to express $f(x)$ as a sum of a polynomial and the simple rational functions described above.

The description of the simple rational functions sought is a bit of a mouthful, but it is not as complicated as it seems, after looking at some examples.

You may be wondering why step 1 is necessary. Essentially, it is because all of the simple rational functions we describe are proper, hence all sums of them are also proper. If you wish, think of polynomials as one kind of atom of rational functions, an these simple rational functions as the other atoms.

4 Integration of rational functions

Once a rational function has been expanded by partial fractions, it can be integrated. Rather than speaking in generalities, I shall give some examples of how this looks.

Example 4.1. Evaluate $\displaystyle\int \frac{x^3+1}{x^2-x}dx$.

From example 2.3, this function is equal to $x + 1 + \frac{x+1}{x^2-x}$. From example 3.1, this is in turn equal to $x + 1 - \frac{1}{x} + \frac{2}{x-1}$. Now each of these terms can be evaluated to obtain:

$$\begin{aligned} \int \frac{x^3+1}{x^2-x} &= \int \left(x + 1 - \frac{1}{x} + \frac{2}{x-1} \right) dx \\ &= \frac{1}{2}x^2 + x - \ln|x| + 2\ln|x-1| + C. \end{aligned}$$

Example 4.2. Evaluate $\displaystyle\int \frac{x-1}{x^3+x}dx$.

From example 3.2, the integrand can be expanded as $-\frac{1}{x} + \frac{x+1}{x^2+1}$. Therefore:

$$\begin{aligned} \int \frac{x-1}{x^3+x}dx &= \int \left(-\frac{1}{x} + \frac{x+1}{x^2+1} \right) dx \\ &= -\ln|x| + \int \frac{x}{x^2+1}dx + \int \frac{1}{x^2+1}dx \\ &= -\ln|x| + \frac{1}{2}\ln|x^2+1| + \tan^{-1} x + C. \end{aligned}$$

Note that the middle integral is performed by the substation $u = x^2 + 1$.

Example 4.3. Evaluate $\displaystyle\int \frac{x^2+1}{x^3+x^2}dx$.

From example 3.3, the integrand can be rewritten $-\frac{1}{x} + \frac{1}{x^2} + \frac{2}{x+1}$. Therfore:

$$\begin{aligned} \int \frac{x^2+1}{x^3+x^2}dx &= \int \left(-\frac{1}{x} + \frac{1}{x^2} + \frac{2}{x+1} \right) \\ &= -\ln|x| - \frac{1}{x} + 2\ln|x+1| + C. \end{aligned}$$

From these examples, it is apparent that the sort of functions that will occur in the integral of a rational function are:

- Polynomials
- Rational functions
- Logarithms
- The inverse tangent function

In fact, later in the course we may discuss why logarithms and inverse trigonometric functions are really just variants of the same concept, once complex numbers enter the picture. In any case, they often occur in very similar contexts.