Math 2: Linear Algebra
Problems, Solutions and Tips

FOR THE ELECTRONICS AND TELECOMMUNICATION STUDENTS

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Contents

Preface

This is the complementary text to my Linear Algebra Lecture Notes for the telecommunication students at Technical University in Poznań.

It is designed to help you succeed in your linear algebra course, and shows you how to study mathematics, to learn new material, and to prepare effective review sheets for tests. This text guide you through each section, with summaries of important ideas and tables that connect related ideas. Detailed solutions to many of exercises allow you to check your work or help you get started on a difficult problem. Also, complete explanations are provided for some writing exercises Practical Problems point out important exercises, give hints about what to study, and sometimes highlight potential exam questions. Frequent warnings identify common student errors. Don't ever take an exam without reviewing these warnings! Good luck!

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1

Complex Numbers (Exercises)

Exercise 1.1 *Simplify the imaginary numbers below*

a) $\sqrt{-36}$

b) $\pm\sqrt{-49}$

c) $-\sqrt{-16}$

d) $11\sqrt{-81}$

e) i^9

f) i^{12}

g) $24i^{20}$

h) $16 - \sqrt{-169}$

i) $16 - \sqrt{-16}$

j) i^n *where* n *is positive even number.*

Exercise 1.2 *Solve the following problems. Answers are to be in simplest a+bi form.*

1. *Multiply:* $(3 + 5i)(3 - 5i)$

   a) $9 - 25i$
   b) 25
   c) 34

2. *Multiply:* $(8 + 9i)(7 - 3i)$

   a) $15 - 12i$
   b) $29 - 39i$
   c) $83 + 39i$

3. *Multiply:* $(4-3i)(3-4i)$

   a) 25
   b) $-25i$
   c) $12-12i$

4. *Simplify:* $(2+5i)^2$

   a) $21+20i$
   b) $-21+20i$
   c) $29+20i$

5. *Simplify:* $8+i(8-i)$

   a) $7+8i$
   b) $8+8i$
   c) $9+8i$

6. *Simplify:* $\dfrac{7-4i}{1-2i}$

   a) $5-2i$
   b) $3+2i$
   c) $15+10i$

7. *Simplify:* $\dfrac{6+i}{6-i}$

   a) $35/37+(12/37)i$
   b) $35+12i$
   c) $35/36+(12/36)i$

8. *Simplify:* $\dfrac{3-5i}{i}$

   a) $5+3i$
   b) $-5-3i$
   c) $5-3i$

9. *Simplify:* $\dfrac{1}{6-3i}$

a) $2/15 + i/15$

b) $2/15 - i/15$

c) $1/45 + i/15$

10. What is the multiplicative inverse of $\frac{1}{2} + i\frac{1}{2}$

a) $2/(1+i)$

b) $(1/2) - (1/2)i$

c) $(1+i)/2$

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Exercise 1.3 *Verify that*

a) $\left(\sqrt{3}+i\right) + i(1+\sqrt{3}i) = 2i$;

b) $(1,-3)(-2,3) = (7,9)$;

c) $(3,2)\,(3,-2)\,(1,2) = (13,26)$

Exercise 1.4 *Show that*

a) $\mathrm{Re}(iz) = -\,\mathrm{Im}(z)$;

b) $\mathrm{Im}(iz) = \mathrm{Re}(z)$.

Exercise 1.5 *Show that* $(1+z)^3 = z^3 + 3z^2 + 3z + 1$.

Exercise 1.6 *Verify that each of the two numbers* $z = 1 \pm i\sqrt{2}$ *satisfies the equation* $z^2 - 2z + 3 = 0$*;*

Exercise 1.7 *Prove that multiplication of complex numbers is commutative.*

Exercise 1.8 *Verify*

a) *the associative law for addition of complex numbers,*

b) *the distributive law .*

Exercise 1.9 *Use the associative law for addition and the distributive law to show that*

$$z\left(z_1 + z_2 + z_3 + z_4\right) = zz_1 + zz_2 + zz_3 + zz_4.$$

Exercise 1.10 **a)** *Write* $(x,y)+(u,v)=(x,y)$ *and point out how it follows that the complex number* $0=(0,0)$ *is unique as an additive identity.*

b) *Likewise, write* $(x,y)(u,v)=(x,y)$ *and show that the number* $1=(1,0)$ *is a unique multiplicative identity.*

Exercise 1.11 *Solve the equation* $z^2-2z+2=0$*, for* $z=(x,y)$ *by writing*

$$(x,y)(x,y)-2(x,y)+(2,0)=(0,0)$$

and then solving a pair of simultaneous equations in x *and* y*.*

HINT: Use the fact that no real number x *satisfies the given equation to show that* $y\neq 0$*.*

Answer: *Solution is:* $1+i,\ 1-i$*.*

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Exercise 1.12 *Reduce each of these quantities to a real number:*

a) $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}$

b) $\dfrac{5i}{(i-1)(2-i)(3-i)}$

c) $(1-i)^8$

Answer: *a)* $-\frac{2}{5}$, *b)* $\frac{1}{2}$, *c)* 16.

Exercise 1.13 *Show that*

$$\frac{1}{1/z}=z \qquad (z\neq 0)\,.$$

Exercise 1.14 *Use the associative and commutative laws for multiplication to show that*

$$(z_1z_2)(z_3z_4)=(z_1z_3)(z_2z_4).$$

Exercise 1.15 *Prove that if* $z_1z_2z_3=0$*, then at least one of the three factors is zero.*

HINT: Write $(z_1z_2)z_3=0$ *and use a similar result involving two factors.*

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Exercise 1.16 *Locate the numbers* z_1+z_2 *and* z_1-z_2 *vectorially when*

a) $z_1 = 3i, \quad z_2 = \frac{4}{3} - i,$

b) $z_1 = \left(-\sqrt{5}, 1\right), \quad z_2 = \left(\sqrt{2}, 1\right),$

c) $z_1 = (-2, 1), \quad z_2 = \left(\sqrt{3}, 1\right),$

d) $z_1 = x_1 + iy_{1,} \quad z_2 = x_1 - iy_1.$

Exercise 1.17 *Verify inequalities* **??***, involving* $\mathrm{Re}(z)$, $\mathrm{Im}(z)$*, and* $|z|$*.*

Exercise 1.18 *Use established properties of moduli to show that when* $|z_3| \neq |z_4|$*,*

$$\frac{\mathrm{Re}(z_1 + z_2)}{|z_3 + z_4|} \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}.$$

Exercise 1.19 *Verify that* $\sqrt{2}|z| \geq |\,\mathrm{Re}\, z| + |\,\mathrm{Im}\, z|$.
`HINT:` *Reduce this inequality to*$(|x| - |y|)^2 \geq 0$.

Exercise 1.20 *In each case, sketch the set of points determined by the given condition:*

a) $|z - 2 + i| = 1;$

b) $|z + i| \leq 2;$

c) $|z - 4i| \geq 3.$

Exercise 1.21 *Using the fact that* $|z_1 - z_2|$ *is the distance between two points* z_1 *and* z_2*, give a geometric argument that*

a) $|z - 4i| + |z + 4i| = 10$ *represents an ellipse whose foci are* $(0, \pm 4)$*;*

b) $|z - 1| = |z + i|$ *represents the line through the origin whose slope is* -1*.*

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Exercise 1.22 *Use properties of conjugates and moduli to show that*

a) $\overline{\bar{z} + 4i} = z - 4i;$

b) $\overline{iz} = -iz;$

c) $\overline{(2 + i)^2} = 3 - 4i;$

d) $\left|(2\bar{z} + 5)(\sqrt{2} - i)\right| = \sqrt{3}\,|2z + 5|\,.$

Exercise 1.23 *Sketch the set of points determined by the condition*

a) $\mathrm{Re}(\bar{z} - i) = 2$;

b) $|2\bar{z} + i| = 4$.

Exercise 1.24 *Verify properties*

$$\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

and

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

of conjugates.

Exercise 1.25 *Show that*

a) $\overline{z_1 z_2 z_3} = \bar{z}_1 \bar{z}_2 \bar{z}_3$ *;*

b) $\overline{z^4} = \bar{z}^4$.

Exercise 1.26 *Verify property*

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \qquad (z_2 \neq 0)\,.$$

of moduli.

Exercise 1.27 *Show that when z_2 and z_3 are nonzero,*

a) $\overline{\left(\dfrac{z_1}{z_2 z_3}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2 \bar{z}_3}$;

b) $\left|\dfrac{z_1}{z_2 z_3}\right| = \dfrac{|z_1|}{|z_2|\,|z_3|}$.

Exercise 1.28 *Show that*

$$\left|\mathrm{Re}(2 + \bar{z} + z^3)\right| \leq 4 \qquad \textit{when} \quad |z| \leq 1.$$

Exercise 1.29 *Give an alternative proof that if $z_1 z_2 = 0$, then at least one of the numbers z_1 and z_2 must be zero. Use the corresponding result for real numbers and the identity $|z_1 z_2| = |z_1|\,|z_2|$.*

Exercise 1.30 *Prove that*

a) *z is real if and only if $\bar{z} = z$;*

b) *z is either real or pure imaginary if and only if $\bar{z}^2 = z^2$.*

Exercise 1.31 *Use mathematical induction to show that when $n = 2, 3, \ldots,$*

a) $\overline{z_1 + z_2 + \cdots + z_n} = \bar{z}_1 + \bar{z}_2 + \cdots + \bar{z}_n$;

b) $\overline{z_1 z_2 \cdots z_n} = \bar{z}_1 \bar{z}_2 \cdots \bar{z}_n$.

Exercise 1.32 *Let $a_0, a_1, a_2, \ldots, a_n$ $(n \geq 1)$ denote real numbers, and let z be any complex number. With the aid of the results in previous, show that*

$$\overline{a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n} = a_0 + a_1 \bar{z} + a_2 \bar{z}^2 + \cdots + a_n \bar{z}^n.$$

Exercise 1.33 *Show that the equation $|z - z_0| = R$ of a circle, centered at z_0 with radius R, can bewritten*

$$|z|^2 - 2\,\mathrm{Re}(\overline{z z_0}) + |z_0|^2 = R^2.$$

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Exercise 1.34 *Find the principal argument* Arg z *when*

a) $z = \dfrac{i}{-2 - 2i}$;

b) $z = \left(\sqrt{3} - i\right)^6$.

Answer: a) $-3\pi/4$, b) π.

Exercise 1.35 *Show that a) $|e^{i\theta}| = 1$; b) $\overline{e^{i\theta}} = e^{-i\theta}$.*

Exercise 1.36 *Use mathematical induction to show that*

$$e^{i\theta_1} e^{i\theta_2} \cdots e^{i\theta_n} = e^{i(\theta_1 + \theta_2 + \cdots + \theta_n)} \qquad (n = 2, 3, \ldots).$$

Exercise 1.37 *Using the fact that the modulus $|e^{i\theta} - 1|$ is the distance between the points $e^{i\theta}$ and 1,give a geometric argument to find a value of θ in the interval $0 \leq \theta < 2\pi$ that satisfies the equation $|e^{i\theta} - 1|$.*

Answer: π

Exercise 1.38 *By writing the individual factors on the left in exponential form, performing the needed operations, and finally changing back to rectangular coordinates, show that*

a) $i\left(1-\sqrt{3}i\right)\left(\sqrt{3}+i\right)=2\left(1+i\sqrt{3}\right);$

b) $5i/(2+i)=1+2i;$

c) $(-1+i)^7=-8-8i=-8(1+i);$

d) $\left(1+\sqrt{3}i\right)^{-10}=\frac{1}{2048}i\sqrt{3}-\frac{1}{2048}=\frac{1}{2^{11}}(-1+\sqrt{3}i).$

Exercise 1.39 *Show that if* $\operatorname{Re} z_1>0$ *and* $\operatorname{Re} z_2>0$, *then*

$$Arg(z_1z_2)=Arg(z_1)+Arg(z_2),$$

where principal arguments are used.

Exercise 1.40 *Let* z *be a nonzero complex number and* n *a negative integer* $(n=-1,-2,...)$. *Also, write* $z=re^{i\theta}$ *and* $m=-n=1,2,....$ *Using the expressions*

$$z^m=r^me^{im\theta}\qquad \text{and}\qquad z^{-1}=\left(\frac{1}{r}\right)e^{i(-\theta)}$$

verify that $(z^m)^{-1}=\left(z^{-1}\right)^m$.

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Exercise 1.41 *Find the square roots of*

a) $2i;$

b) $1-\sqrt{3i}$

and express them in rectangular coordinates.

Answer: a) $\pm(1+i)$; *b)* $\pm\frac{\sqrt{3}-i}{\sqrt{2}}$.

Exercise 1.42 *In each case, find all the roots in rectangular coordinates, exhibit them as vertices of certain squares, and point out which is the principal root:*

a) $(-16)^{1/4};$

b) $\left(-8-8\sqrt{3}i\right)^{1/4};$

Answer: a) $\pm\sqrt{2}(1+i),\ \pm\sqrt{2}(1-i)$ *b)* $\pm\left(\sqrt{3}-i\right),\ \pm\left(1+\sqrt{3}i\right).$

Exercise 1.43 *The three cube roots of a nonzero complex number* z_0 *can be-written* c_0, $c_0\omega_3$, $c_0\omega_3^2$ *where* c_0 *is the principal cube root of* z_0 *and*

$$\omega_3 = \exp\left(i\frac{2\pi}{3}\right) = \frac{-1+\sqrt{3}i}{2}.$$

Show that if $z_0{=}-4\sqrt{2}+4\sqrt{2}i$, *then* $c_0 = \sqrt{2}\,(1+i)$ *and the other two cube roots are, in rectangular form, the numbers*

$$c_0\omega_3 = \frac{-\left(\sqrt{3}+1\right)+\left(\sqrt{3}-1\right)i}{\sqrt{2}}, \qquad c_0\omega_3^2 = \frac{\left(\sqrt{3}-1\right)+\left(\sqrt{3}+1\right)i}{\sqrt{2}}.$$

Exercise 1.44 *Find the four zeros of the polynomial* z^4+4, *then use those zeros to factor* z^4+4 *into quadratic factors with real coefficients.*

<<< * >>>

Exercise 1.45 *Use complex numbers to find the sum of the* $n-1$ *terms of the series*

$$S = 2\sin\theta + 3\sin 2\theta + 4\sin 3\theta + ... + n\sin(n-1)\theta.$$

Show that, if $\theta = 2\pi/n$, *then* $S = \frac{1}{2}n\cot\theta/2$.

Exercise 1.46 *Use De Moivre's formula to show that:*

$$\begin{aligned}\cos(n\theta) &= \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta + ...\\ \sin(n\theta) &= \binom{n}{1}\cos^{n-1}\theta\sin\theta - \binom{n}{3}\cos^{n-3}\theta\sin^3\theta + ...\end{aligned}$$

Exercise 1.47 *Find real and imaginary parts and the modulus of*

$$\frac{1+e^{i\theta}}{1+e^{i\phi}}.$$

Exercise 1.48 *If* $\omega = \cos\frac{2\pi}{N} + i\sin\frac{2\pi}{N}$ *(see* **??**, *p.***??***) prove that*

$$1 + \omega^k + \omega^{2k} + ... + \omega^{(n-1)k} = 0$$

for any integer k *which is not a multiple of* n.

2

Systems of Linear Equations (Exercises)

Get into the habit now of working the Practice Problems before you start the exercises. Probably, you should attempt all the Practice Problems before checking the solutions, because once you start reading the first solution, you might tend to read on through the other solutions and spoil your chance to benefit from those problems.

2.1 Practice Problems

Problem 1 *Determine if the following system is consistent:*

$$\begin{array}{rcrcrcl} & & x_2 & - & 4x_3 & = & 8 \\ 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ 5x_1 & - & 8x_2 & + & 7x_3 & = & 1 \end{array} \tag{2.1}$$

Solution: The augmented matrix is

$$\begin{bmatrix} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{bmatrix} \tag{2.2}$$

To obtain an x_1 in the first equation, interchange rows 1 and 2:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{bmatrix} \tag{2.3}$$

To eliminate the $5x_1$ term in the third equation, add $-5/2$ times row 1 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -1/2 & 2 & -3/2 \end{bmatrix} \tag{2.4}$$

Next, use the x_2 term in the second equation to eliminate the $-1/2x_2$ term from the third equation. Add $1/2$ times row 2 to row 3:

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix} \tag{2.5}$$

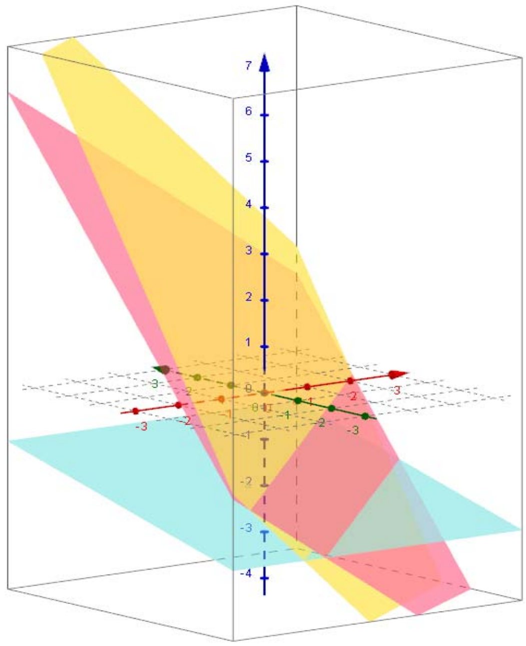


Fig. 2.1. The system 2.1 is inconsistent because there is no point that lies in all three planes (yellow,pink and blue)

The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation:

$$\begin{array}{rcrcrcr} 2x_1 & - & 3x_2 & + & 2x_3 & = & 1 \\ & & x_2 & - & 4x_3 & = & 1 \\ & & & & 0 & = & 5/2 \end{array} \tag{2.6}$$

The equation $0 = 5/2$ is a short form of $0x_1 + 0x_2 + 0x_3 = 5/2$. This system in triangular form obviously has a built-in contradiction. There are no values of $x_1; x_2; x_3$ that satisfy (2.6) because the equation $0 = 5/2$ is never true. Since (2.6) and (2.1) have the same solution set, the original system is inconsistent (i.e., has no solution).

Problem 2 *State in words the next elementary row operation that should be performed on the system in order to solve it. [More than one answer is possible.]*

a)

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & 4x_3 & + & 6x_4 & = & 8 \\ & & x_2 & + & 2x_3 & - & 3x_4 & = & 1 \\ & & & & 7x_3 & + & x_4 & = & 1 \\ & & & & x_3 & - & -3x_4 & = & 5 \end{array}$$

b)

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & 4x_3 & + & 6x_4 & = & 8 \\ & & x_2 & + & 2x_3 & & & = & 1 \\ & & & & 2x_3 & & & = & 1 \\ & & & & & & x_4 & = & 5 \end{array}$$

Solution:

a) For "hand computation," the best choice is to interchange equations 3 and 4. Another possibility is to multiply equation 3 by 1/7. Or, replace equation 4 by its sum with $-1/7$ times row 3. (In any case, do not use the x_2 in equation 2 to eliminate the x_2 in equation 1.)

b) The system is in triangular form. Further simplification begins with the x_4 in the fourth equation. Use the x_4 to eliminate all x_4 terms above it. The appropriate step now is to add -6 times equation 4 to equation 1. (After that, move to equation 3, multiply it by 1/2, and then use the equation to eliminate the x_3 terms above it.)

Problem 3 *The augmented matrix of a linear system has been transformed by row operations into the form below. Determine if the system is consistent.*

$$\begin{bmatrix} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{bmatrix}. \tag{2.7}$$

Solution: The system corresponding to the augmented matrix is

$$\begin{array}{rcrcrcr} x_1 & + & 5x_2 & + & 2x_3 & = & -6 \\ & & 4x_2 & - & 7x_3 & = & 2 \\ & & & & 5x_3 & = & 0 \end{array} \tag{2.8}$$

The third equation makes $x_3 = 0$, which is certainly an allowable value for x_3. After eliminating the x_3 terms in equations 1 and 2, you could go on to solve for unique values for x_2 and x_1. Hence a solution exists, and it is unique (see Figure 2.2).

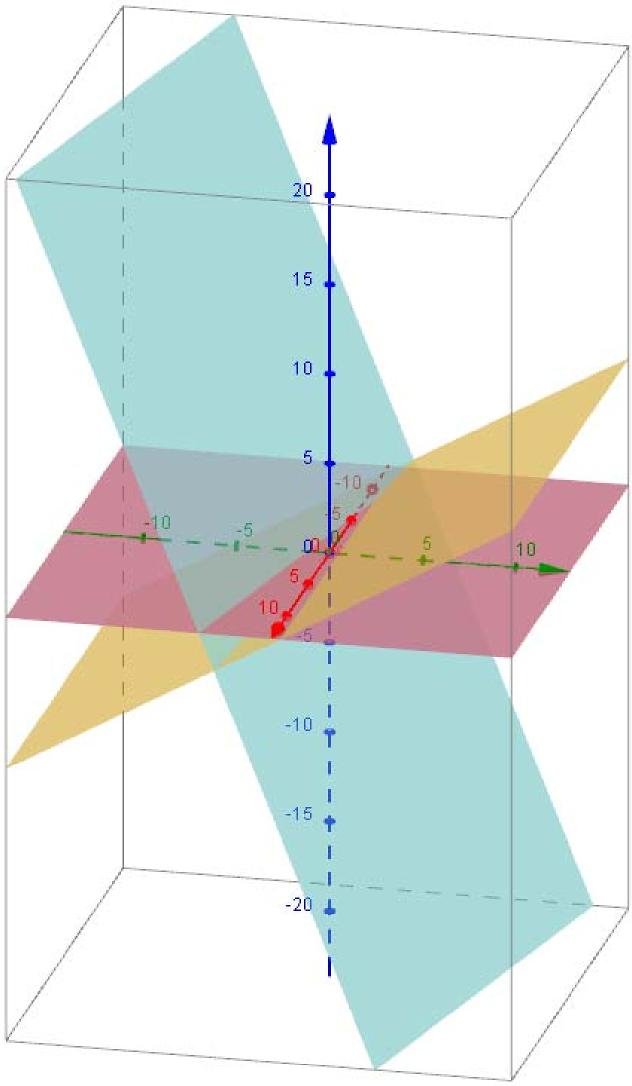


Fig. 2.2. Each of the equations 2.8 determines a plane in three-dimensional space. The solution lies in all three planes.

Problem 4 *Is* $(3, 4, -1)$ *a solution of the following system?*

$$\begin{array}{rcrcrcl} 5x_1 & - & x_2 & + & 2x_3 & = & 9 \\ -2x_1 & + & 6x_2 & + & 9x_3 & = & 9 \\ -7x_1 & + & 5x_2 & - & 3x_3 & = & 1 \end{array} \tag{2.9}$$

Solution: It is easy to check if a specific list of numbers is a solution. Set $x_1 = 3$, $x_2 = 4$, and $x_3 = -1$, and find that

$$\begin{array}{rcl} 5(3) - (4) + 2(-1) & = & 9 \\ -2(3) + 6(4) + 9(-1) & = & 9 \\ -7(3) + 5(4) - 3(-1) & = & 2 \end{array}$$

Although the first two equations are satisfied, the third is not, so $(3, 4, -1)$ is not a solution of the system. Notice the use of parentheses when making the substitutions. They are strongly recommended as a guard against arithmetic errors.

Problem 5 *For what values of* h *and* k *is the following system consistent?*

$$\begin{array}{rcrcl} 2x_1 & - & x_2 & = & h \\ -4x_1 & + & 2x_2 & = & k \end{array}$$

Solution: When the second equation is replaced by its sum with 2 times the first equation, the system becomes

$$\begin{array}{rcrcr} 2x_1 & - & x_2 & = & h \\ & & 0 & = & k + 2h \end{array}$$

If $k + 2h$ is nonzero, the system has no solution. The system is consistent for any values of h and k that make $k + 2h = 0$.

Exercise 2.1 (True or False) *Mark each statement True or False, and justify your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.) Similar true/false questions will appear in many next lectures.*

a) *Every elementary row operation is reversible.*

b) *A* 5×6 *matrix has six rows.*

c) *The solution set of a linear system involving variables $x_1, ..., x_n$ is a list of numbers $s_1, ..., s_n$ that makes each equation in the system a true statement when the values $s_1, ..., s_n$ are substituted for $x_1, ..., x_n$, respectively.*

d) *Two fundamental questions about a linear system involve existence and uniqueness.*

Exercise 2.2 *The augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.*

a)

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 4 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix},$$

b)

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

Exercise 2.3 *Solve the following system*

$$\begin{array}{rcrcrcr} x_1 & & & - & 3x_3 & = & 8 \\ 2x_1 & + & 2x_2 & + & 9x_3 & = & 7 \\ & & x_2 & + & 5x_3 & = & -2 \end{array}$$

Exercise 2.4 *Construct three different augmented matrices for linear systems whose solution set is $x_1 = 3$, $x_2 = -2$, $x_3 = -1$.*

Exercise 2.5 *Determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.*

$$\begin{bmatrix} 1 & 2 & h \\ 3 & 4 & -2 \end{bmatrix}.$$

3

Row Reduction and Echelon Forms (Exercises)

3.1 Practice problems

Example 6 *Find the general solution of the linear system whose augmented matrix is*

$$\begin{bmatrix} 1 & -3 & -5 & 0 \\ 0 & 1 & 1 & 3 \end{bmatrix} \tag{3.1}$$

Solution: The reduced echelon form of the augmented matrix and the corresponding system are

$$\begin{bmatrix} 1 & 0 & -2 & 9 \\ 0 & 1 & 1 & 3 \end{bmatrix} \quad \text{and} \quad \left\{ \begin{array}{rcrcl} x_1 & & - & 2x_3 & = & 9 \\ & x_3 & + & x_3 & = & 3 \end{array} \right. \tag{3.2}$$

The basic variables are x_1 and x_2, and the general solution is

$$\left\{ \begin{array}{lcl} x_1 & = & 9 + 2x_3 \\ x_2 & = & 3 - x_3 \\ x_3 & & \text{is free} \end{array} \right.$$

See Figures 3.1 and 3.2 □

Example 7 *Find the general solution of the system*

$$\left\{ \begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & = & 0 \\ -2x_1 & + & 4x_2 & + & 5x_3 & - & 5x_4 & = & 3 \\ 3x_1 & - & 6x_2 & - & 6x_3 & + & 8x_4 & = & 2 \end{array} \right.$$

Solution: Row reduce the system's augmented matrix:

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{bmatrix}$$

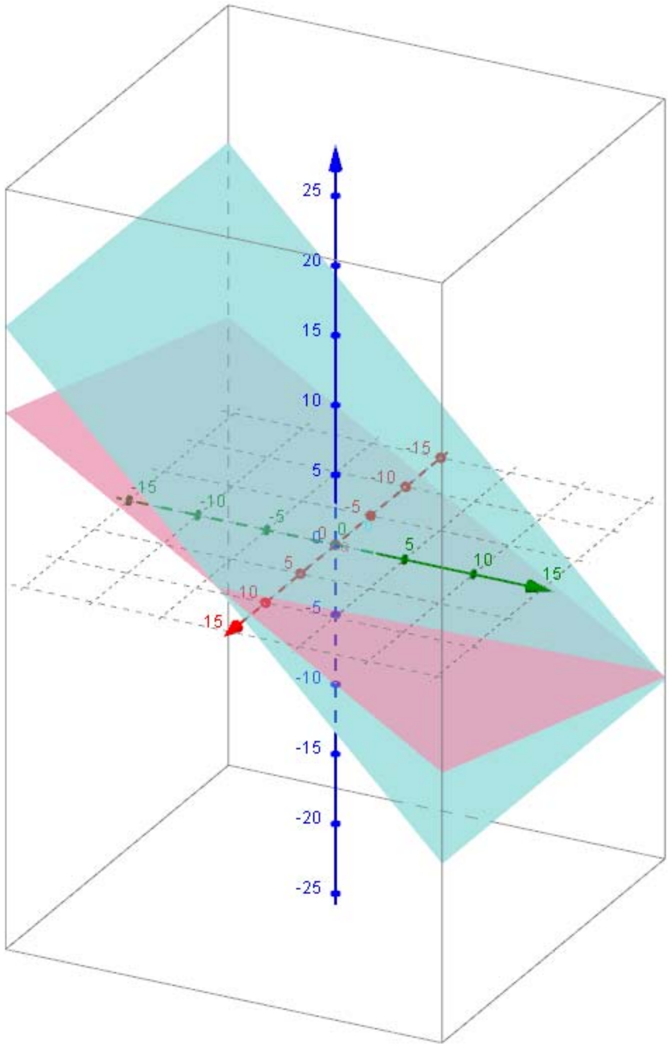


Fig. 3.1. The general solution of the oryginal system of equations 3.1 is the line of intersection of the two planes.

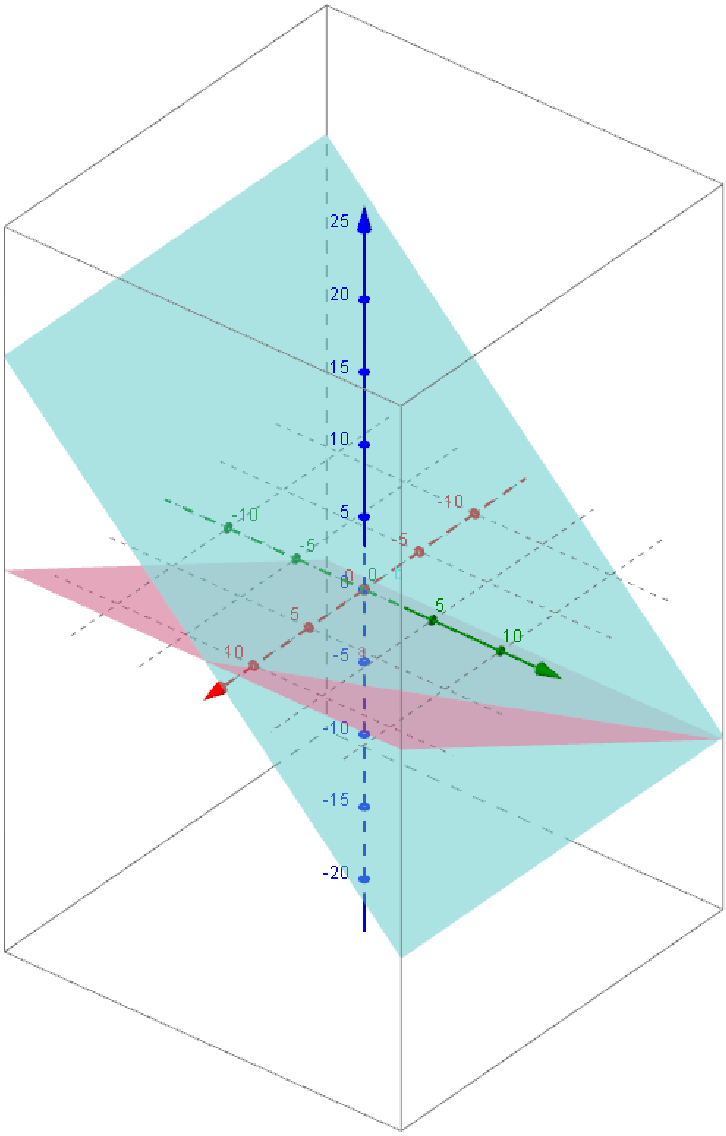


Fig. 3.2. Line of intersection of the two planes which correspond to the system 3.2 in rref. The solution sets for the system 3.1 and 3.2 are identical.

$$\sim \begin{bmatrix} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

This echelon matrix shows that the system is *inconsistent*, because its rightmost column is a pivot column; the third row corresponds to the equation $0 = 5$. There is no need to perform any more row operations. Note that the presence of the free variables in this problem is irrelevant because the system is inconsistent.

3.2 Solving Several Systems Simultaneously

In many cases, we need to solve two or more systems having the same coefficient matrix. Suppose we wanted to solve both of the systems:

$$\begin{cases} 3x_1 + x_2 - 2x_3 = 1 \\ 4x_1 - x_3 = 7 \\ 2x_1 - 3x_2 + 5x_3 = 18 \end{cases} \quad \text{and} \quad \begin{cases} 3x_1 + x_2 - 2x_3 = 8 \\ 4x_1 - x_3 = -1 \\ 2x_1 - 3x_2 + 5x_3 = -32 \end{cases}$$

It is wasteful to do two almost identical row reductions on the augmented matrices

$$\left[\begin{array}{ccc|c} 3 & 1 & -2 & 1 \\ 4 & 0 & -1 & 7 \\ 2 & -3 & 5 & 18 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{ccc|c} 3 & 1 & -2 & 8 \\ 4 & 0 & -1 & -1 \\ 2 & -3 & 5 & -32 \end{array}\right]$$

Instead,we can create the following "simultaneous" matrix containing the information from both systems:

$$\left[\begin{array}{ccc|cc} 3 & 1 & -2 & 1 & 8 \\ 4 & 0 & -1 & 7 & -1 \\ 2 & -3 & 5 & 18 & -32 \end{array}\right]$$

Row reducing this matrix completely yields

$$\left[\begin{array}{ccc|cc} 1 & 0 & 0 & 2 & -1 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 1 & -3 \end{array}\right]$$

By considering both of the right-hand columns separately,we discover that the unique solution of the first system is $x_1 = 2$, $x_2 = -3$, and $x_3 = 1$ and that the unique solution of the second system is $x_1 = -1$, $x_2 = 5$, and $x_3 = -3$.

Any number of systems with the same coefficient matrix can be handled similarly, with one column on the right side of the augmented matrix for each system.

Example 8 *Find the general solutions of the system whose augmented matrix is given by*

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.3}$$

Solution:

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 0 & -1 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Corresponding system:

$$\begin{cases} x_1 & & & - & 3x_5 & = & 5 \\ & x_2 & & - & 4x_5 & = & 1 \\ & & x_4 & + & 9x_5 & = & 4 \\ & & & & 0 & = & = \end{cases}$$

Basic variables: x_1, x_2, x_4; free variables: x_3, x_5. General solution:

$$\begin{cases} x_1 & = & 5 + 3x_3 \\ x_2 & = & 1 + 4x_5 \\ x_3 & = & \text{is free} \\ x_4 & = & 4 - 9x_5 \\ x_5 & = & \text{is free} \end{cases}$$

Note: A common error in this exercise is to assume that x_3 is zero. Another common error is to say *nothing* about x_3 and write only x_1, x_2, x_4, and x_5, as above. **To avoid these mistakes, identify the basic variables first. Any remaining variables are free.** □

Exercise 3.1 *Solve the systems* $A\mathbf{x} = \mathbf{b}_1$ *and* $A\mathbf{x} = \mathbf{b}_2$ *simultaneously, as illustrated above, where*

$$A = \begin{bmatrix} 9 & 2 & 2 \\ 3 & 2 & 4 \\ 27 & 12 & 22 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} -6 \\ 0 \\ 12 \end{bmatrix}, \quad \mathbf{b}_2 == \begin{bmatrix} -12 \\ -5 \\ 8 \end{bmatrix}.$$

Exercise 3.2 *Solve the systems* $A\mathbf{x} = \mathbf{b}_1$ *and* $A\mathbf{x} = \mathbf{b}_2$ *simultaneously, as illustrated above, where*

$$A = \begin{bmatrix} 12 & 2 & 0 & 3 \\ -24 & -4 & 1 & -6 \\ -4 & -1 & -1 & 0 \\ -30 & -5 & 0 & -6 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 3 \\ 8 \\ -4 \\ 6 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} 2 \\ 4 \\ -24 \\ 0 \end{bmatrix}.$$

Exercise 3.3 *Find the values of* A, B, C *(and* D *in part (b)) in the following partial fractions problems:*

a)

$$\frac{5x^2 + 23x - 58}{(x-1)(x-3)(x+4)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x+4}$$

b)

$$\frac{-3x^3 + 29x^2 - 91x + 94}{(x-2)^2(x-3)^2} = \frac{A}{(x-2)^2} + \frac{B}{x-2} + \frac{C}{(x-3)^2} + \frac{D}{x-3}$$

Exercise 3.4 (True of False) *Mark each statement True or False. Justify each answer.*

a) *In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.*

b) *The row reduction algorithm applies only to augmented matrices for a linear system.*

c) *A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.*

d) *Finding a parametric description of the solution set of a linear system is the same as solving the system.*

e) *If one row in an echelon form of an augmented matrix is* $[0\ 0\ 0\ 5\ 0]$*, then the associated linear system is inconsistent.*

Exercise 3.5 (True or False) *Mark each statement True or False. Justify each answer.*

a) *The reduced echelon form of a matrix is unique.*

b) *If every column of an augmented matrix contains a pivot, then the corresponding system is consistent.*

c) *The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.*

d) *A general solution of a system is an explicit description of all solutions of the system.*

e) *Whenever a system has free variables, the solution set contains many solutions.*

f) *If a linear system is consistent, then the solution is unique if and only if every column in the coefficient matrix is a pivot column; otherwise there are infinitely many solutions.*

4

Vector equations (Exercises)

4.1 Practice problems

Example 9 *Compute* $\mathbf{u}+2\mathbf{v}$ *and* $\mathbf{u}-\mathbf{v}$ *when*

$$\mathbf{u}=\begin{bmatrix}-1\\2\end{bmatrix}\qquad \textit{and}\qquad \mathbf{v}=\begin{bmatrix}-3\\-1\end{bmatrix}.$$

Solution:

$$\begin{aligned}\mathbf{u}+2\mathbf{v} &= \begin{bmatrix}-1\\2\end{bmatrix}+2\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}2(-3)\\2(-1)\end{bmatrix}\\ &= \begin{bmatrix}-1\\2\end{bmatrix}+\begin{bmatrix}-6\\-2\end{bmatrix}=\begin{bmatrix}-1-6\\2-2\end{bmatrix}=\begin{bmatrix}-7\\0\end{bmatrix}.\end{aligned}$$

$$\mathbf{u}-\mathbf{v}=\begin{bmatrix}-1\\2\end{bmatrix}-\begin{bmatrix}-3\\-1\end{bmatrix}=\begin{bmatrix}-1-(-3)\\2-(-1)\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix}.$$

Example 10 *Compute* $\mathbf{u}+2\mathbf{v}$ *and* $\mathbf{u}-\mathbf{v}$ *when*

$$\mathbf{u}=\begin{bmatrix}1\\3\end{bmatrix}\qquad \textit{and}\qquad \mathbf{v}=\begin{bmatrix}-2\\1\end{bmatrix}.$$

Solution:

$$\mathbf{u}+2\mathbf{v}=\begin{bmatrix}-3\\5\end{bmatrix}.$$

$$\mathbf{u}-\mathbf{v}=\begin{bmatrix}3\\2\end{bmatrix}.$$

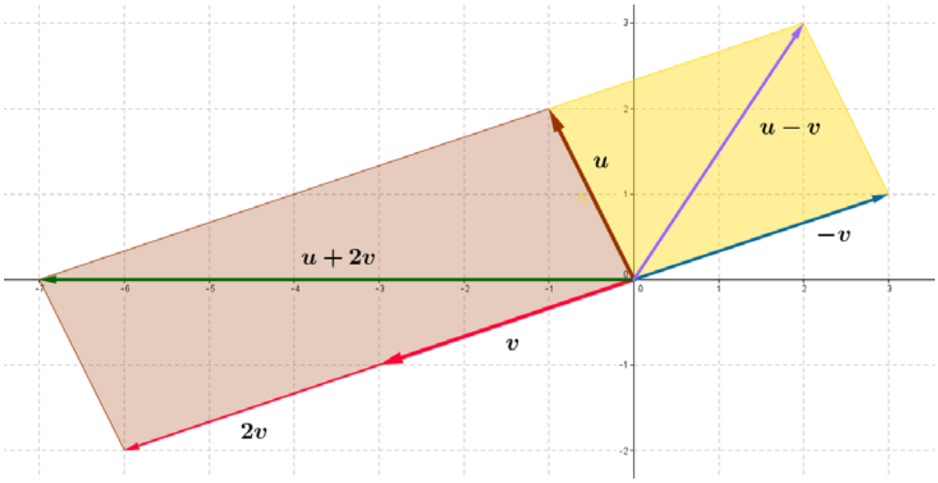
□

Example 11 *Display the following vectors using arrows on an xy-graph:*

$$\mathbf{u},\ \mathbf{v},\ -\mathbf{v},\ \mathbf{2v},\ \mathbf{u}+\mathbf{2v},\ \mathbf{u}-\mathbf{v}.$$

Notice that $\mathbf{u}-\mathbf{v}$ *is the vertex of a parallelogram whose other vertices are* $\mathbf{u}$*,* $\mathbf{0}$*, and* $-\mathbf{v}$*. Take vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *as in Example 9*

Solution:



Example 12 *Prove that* $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *for any* u *and* v *in* $\mathbb{R}^n$.

Solution: Take arbitrary vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \quad \text{in } \mathbb{R}^n,$$

and compute

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix} = \begin{bmatrix} v_1 + u_1 \\ v_2 + u_2 \\ \vdots \\ v_n + u_n \end{bmatrix} = \mathbf{v} + \mathbf{u}.$$

□

Example 13 *Write a system of equations that is equivalent to the given vector equation.*

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Solution:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ -2x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_2 \\ x_1 - 2x_2 \\ x_1 + 3x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

System of equations that is equivalent to the given vector equation is of the following form:

$$\begin{cases} x_1 + x_2 & & 1 \\ x_1 - 2x_2 & = & -2 \\ x_1 + 3x_2 & & 1 \end{cases}$$

Usually the intermediate steps are not displayed. □

Example 14 *Determine if* **b** *is a linear combination of* $\mathbf{a}_1$*,* $\mathbf{a}_2$*, and* $\mathbf{a}_3$*.*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}$$

Solution: The question

Is **b** a linear combination of $\mathbf{a}_1, \mathbf{a}_2,$ and $\mathbf{a}_3$?

is equivalent to the question

Does the vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$ have a solution?

The equation

$$x_1 \underbrace{\begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}}_{\mathbf{a}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ -3 \\ 2 \end{bmatrix}}_{\mathbf{a}_2} + x_3 \underbrace{\begin{bmatrix} 5 \\ -1 \\ 5 \end{bmatrix}}_{\mathbf{a}_3} = \underbrace{\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}}_{\mathbf{b}} \tag{4.1}$$

has the same solution set as the linear system whose augmented matrix is

$$M = \begin{bmatrix} 1 & 0 & 5 & 2 \\ -3 & -3 & -1 & -1 \\ 0 & 2 & 5 & 5 \end{bmatrix}.$$

Row reduce M until the pivot positions are visible:

$$M \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & -3 & 14 & 5 \\ 0 & 2 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -19 & -10 \\ 0 & 2 & 5 & 5 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & -19 & -10 \\ 0 & 0 & 43 & 25 \end{bmatrix}$$

The linear system corresponding to M has a solution, so the vector equation (4.1) has a solution, and therefore $\mathbf{b}$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$,and $\mathbf{a}_3$. □

Example 15 *Let*

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$$

For what value(s) of h is $\mathbf{b}$ in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$?

Solution:

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & -2 & 4 \\ 4 & -3 & 1 \\ -2 & 7 & h \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 5 & -15 \\ 0 & 3 & h+8 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 3 & h+8 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & h+17 \end{bmatrix}$$

The vector b is in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$ when $h+17$ is zero, that is, when $h = -17$. □

Example 16 *Let $\mathbf{v}_1, ..., \mathbf{v}_k$ be points in $\mathbb{R}^3$ and suppose that for $j = 1..., k$ an object with mass m_j is located at point $\mathbf{v}_j$. Physicists call such objects* point masses. *The total mass of the system of point masses is*

$$m = m_1 + m_2 + ... + m_k$$

The center of gravity *(or* center of mass*) of the system is*

$$\bar{\mathbf{v}} = \frac{1}{m}(m_1\mathbf{v}_1 + ... + m_k\mathbf{v}_k)$$

Compute the center of gravity of the system consisting of the following point masses (see the Figure 4.1):

Point	***Mass***
$\mathbf{v}_1 = (2, -2, 4)$	$4g$
$\mathbf{v}_2 = (-4, 2, 3)$	$2g$
$\mathbf{v}_3 = (4, 0, -2)$	$3g$
$\mathbf{v}_4 = (1, -6, 0)$	$5g$

Solution: The total mass is $4 + 2 + 3 + 5 = 14$. So

$$\bar{\mathbf{v}} = (4\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3 + 5\mathbf{v}_4)/14.$$

That is,

$$\bar{\mathbf{v}} = \frac{1}{14}\left(4\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} + 2\begin{bmatrix} -4 \\ 2 \\ 3 \end{bmatrix} + 3\begin{bmatrix} 4 \\ 0 \\ -2 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -6 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{17}{14} \\ -\frac{17}{7} \\ \frac{8}{7} \end{bmatrix}.$$

4.2 Exercises

Exercise 4.1 (True or False) a) *An example of a linear combination of vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *is the vector* $\frac{1}{3}\mathbf{v}_1$.

b) *The solution set of the linear system whose augmented matrix is*

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$$

is the same as the solution set of the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$.

c) *The set* $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ *is always visualized as a plane through the origin.*

d) *When* $\mathbf{u}$ *and* $\mathbf{v}$ *are nonzero vectors,* $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ *contains only the line through* $\mathbf{u}$ *and the origin, and the line through* $\mathbf{v}$ *and the origin.*

e) *Asking whether the linear system corresponding to an augmented matrix* $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{b} \end{bmatrix}$ *has a solution amounts to asking whether* $\mathbf{b}$ *is in* $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

f) *The weights* $c_1, ..., c_p$ *in a linear combination* $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + ...c_p\ \mathbf{a}_p$ *cannot all be zero.*

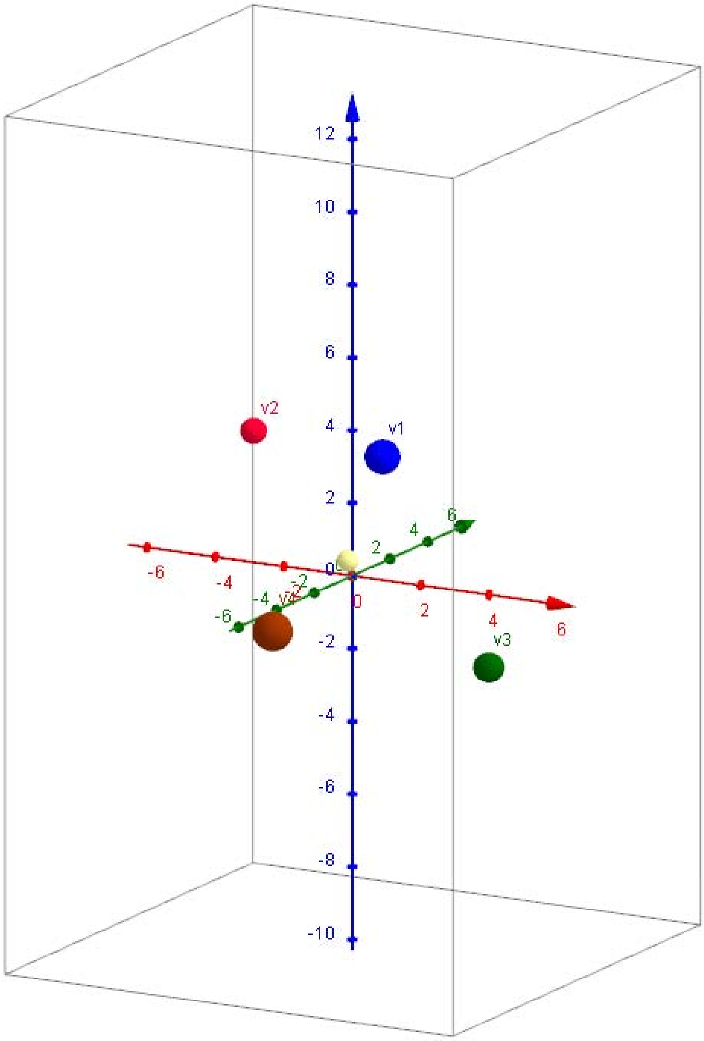


Fig. 4.1. Center of gravity shown in yellow.

Exercise 4.2 *Display the following vectors using arrows on an xy-graph:*

$$\mathbf{u},\ \mathbf{v},\ -\mathbf{v},\ \mathbf{2v},\ \mathbf{u}+\mathbf{2v},\ \mathbf{u}-\mathbf{v}.$$

Notice that $\mathbf{u}-\mathbf{v}$ *is the vertex of a parallelogram whose other vertices are* $\mathbf{u}$, $\mathbf{0}$, *and* $-\mathbf{v}$. *Take vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *as in Example 10*

Exercise 4.3 *Write a system of equations that is equivalent to the given vector equation.*

$$x_1\begin{bmatrix}1\\1\\1\end{bmatrix}+x_2\begin{bmatrix}1\\-2\\3\end{bmatrix}+x_3\begin{bmatrix}1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Exercise 4.4 *Determine if* $\mathbf{b}$ *is a linear combination of* $\mathbf{a}_1$, $\mathbf{a}_2$, *and* $\mathbf{a}_3$ *when*

$$\mathbf{a}_1=\begin{bmatrix}1\\0\\1\end{bmatrix},\quad \mathbf{a}_2=\begin{bmatrix}-2\\3\\-2\end{bmatrix},\quad \mathbf{a}_3=\begin{bmatrix}-6\\7\\5\end{bmatrix},\quad \mathbf{b}=\begin{bmatrix}11\\-5\\9\end{bmatrix}$$

Exercise 4.5 *Let*

$$\mathbf{v}_1=\begin{bmatrix}1\\0\\-2\end{bmatrix},\quad \mathbf{v}_2=\begin{bmatrix}-2\\1\\7\end{bmatrix},\quad \mathbf{y}=\begin{bmatrix}h\\-3\\-5\end{bmatrix}$$

For what value(s) of h *is* $\mathbf{y}$ *in the plane spanned by* $\mathbf{v}_1$ *and* $\mathbf{v}_2$*?*

Exercise 4.6 *Let* $\bar{\mathbf{v}}$ *be the center of mass of a system of point masses located at* $\mathbf{v}_1, ..., \mathbf{v}_k$ *as in Example 16. Is* $\bar{\mathbf{v}}$ *in* $\text{Span}\,\{\mathbf{v}_1, ..., \mathbf{v}_k\}$*? Explain.*

5

The Matrix Equation $\mathbf{Ax} = \mathbf{b}$ (Exercises)

5.1 Practice Problems

Example 17 *Write the system*

$$\begin{cases} 2x_1 - 3x_2 + 5x_3 &= 7 \\ 9x_1 + 4x_2 - 6x_3 &= 8 \end{cases}$$

in matrix form.

Solution: The coefficient matrix is

$$A = \begin{bmatrix} 2 & -3 & +5 \\ 9 & 4 & -6 \end{bmatrix} \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

The matrix form is

$$A\mathbf{x} = \mathbf{b}$$

or

$$\begin{bmatrix} 2 & -3 & +5 \\ 9 & 4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

□

Example 18 *Let*

$$A = \begin{bmatrix} 1 & -1 & 0 & 2 & -3 \\ 0 & 2 & 1 & 4 & -1 \\ 3 & 5 & -2 & 0 & 1 \end{bmatrix}, \qquad \mathbf{p} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} -5 \\ 9 \\ 17 \end{bmatrix}$$

It can be shown that $\mathbf{p}$ *is a solution of* $A\mathbf{x} = \mathbf{b}$. *Use this fact to exhibit* $\mathbf{b}$ *as a specific linear combination of the columns of* A.

Solution : The matrix equation

$$\begin{bmatrix} 1 & -1 & 0 & 2 & -3 \\ 0 & 2 & 1 & 4 & -1 \\ 3 & 5 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ 17 \end{bmatrix}$$

is equivalent to the vector equation

$$2\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} + 4\begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ 17 \end{bmatrix}$$

which expresses **b** as a linear combination of the columns of A. □

Example 19 *Let*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \qquad \mathbf{u} = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

Verify that

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}.$$

Solution:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix},$$

$$A(\mathbf{u} + \mathbf{v}) = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix} = \begin{bmatrix} 13 \\ 28 \\ 43 \end{bmatrix},$$

$$\begin{aligned} A\mathbf{u} + A\mathbf{v} &= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 18 \\ 36 \\ 54 \end{bmatrix} + \begin{bmatrix} -5 \\ -8 \\ -11 \end{bmatrix} = \begin{bmatrix} 13 \\ 28 \\ 43 \end{bmatrix}. \end{aligned}$$

□

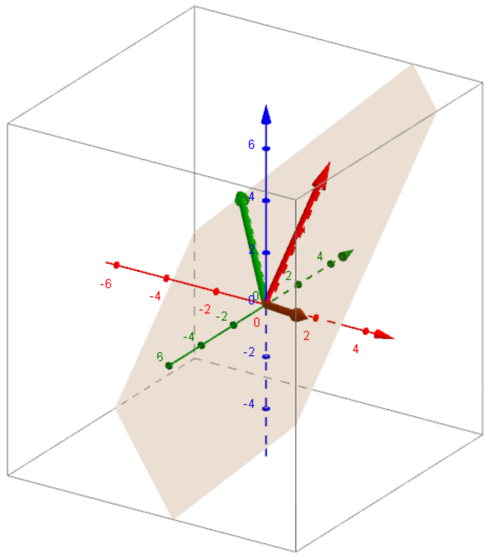


Fig. 5.1. Plane spanned by the columns of A.

Example 20 *Let*

$$\mathbf{u} = \begin{bmatrix} 0 \\ 4 \\ 4 \end{bmatrix} \qquad \textit{and} \qquad A = \begin{bmatrix} 3 & -5 \\ -2 & 6 \\ 1 & 1 \end{bmatrix}$$

Is $\mathbf{u}$ *(in red) in the plane in* $\mathbb{R}^3$ *spanned by the columns of* A*? (See the Figure 5.1) Why or why not?*

Solution : The vector $\mathbf{u}$ is in the plane spanned by the columns of A if and only if $\mathbf{u}$ is a linear combination of the columns of A. This happens if and only if the equation $A\mathbf{x} = \mathbf{u}$ has a solution. To study this equation, reduce the augmented matrix $[A \quad \mathbf{u}]$:

$$\begin{bmatrix} 3 & -5 & 0 \\ -2 & 6 & 4 \\ 1 & 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ -2 & 6 & 4 \\ 3 & -5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & -8 & -12 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 \\ 0 & 8 & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

The equation $A\mathbf{x} = \mathbf{u}$ has a solution, so $\mathbf{u}$ is in the plane spanned by the columns of A. □

Example 21 *Let*

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & 2 & -7 \end{bmatrix} \quad \textit{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

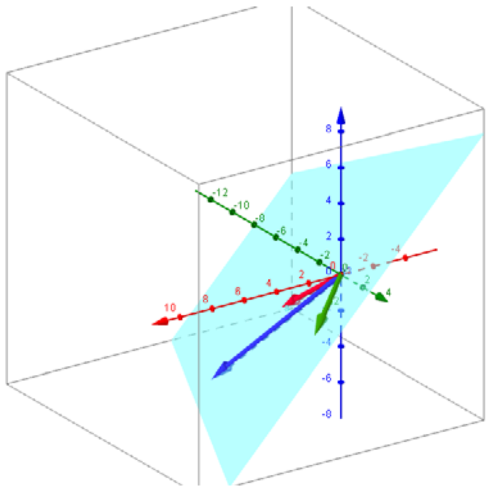


Fig. 5.2. In Example 21 the columns of $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ span a plane through $\mathbf{0}$,

Is the equation $A\mathbf{x} = \mathbf{b}$ consistent for all possible b_1, b_2, b_3?

Solution : Row reduce the augmented matrix for $A\mathbf{x} = \mathbf{b}$

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}.$$

The third entry in column 4 equals : $b_1 - \frac{1}{2}b_2 + b_3$. The equation $A\mathbf{x} = \mathbf{b}$ is not consistent for every $\mathbf{b}$ because some choices of $\mathbf{b}$ can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero. The columns of $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ span a plane through $\mathbf{0}$ (see Figure 5.2). □

Example 22 *For the following list of polynomials*

$$4x^3 + 2x^2 - 6, \qquad x^3 - 2x^2 + 4x + 1, \qquad 3x^3 - 6x^2 + x + 4$$

determine whether the first polynomial can be expressed as

$$s\left(x^3 - 2x^2 + 4x + 1\right) + t\left(3x^3 - 6x^2 + x + 4\right),$$

where $s, t \in \mathbb{R}$.

Solution: We need to verify that there exist $s, t \in \mathbb{R}$ such that

$$4x^3 + 2x^2 - 6 = s\left(x^3 - 2x^2 + 4x + 1\right) + t\left(3x^3 - 6x^2 + x + 4\right).$$

This yields the following system of equations:

$$\begin{cases} s + 3t = 4 \\ -2s + t = 2 \\ 4s + t = 0 \\ s + 4t = -6 \end{cases}$$

This system is inconsistent (check it !) and therefore has no solutions. We conclude that $4x^3 + 2x^2 - 6$ cannot be expressed as a linear combination of $x^3 - 2x^2 + 4x + 1$ and $3x^3 - 6x^2 + x + 4$. □

5.2 Exercises

Exercise 5.1 *Write the following system first as a vector equation and then as a matrix equation.*

$$\begin{cases} x_1 + 2x_2 - x_3 = 1 \\ \phantom{x_1 + {}} x_2 + 3x_3 = -2 \end{cases}.$$

Exercise 5.2 *Write the following system first as a vector equation and then as a matrix equation.*

$$\begin{cases} x_1 - x_2 = 0 \\ x_1 + 2x_2 = -1 \\ x_1 + 5x_2 = 2 \end{cases}.$$

Exercise 5.3 *Note that*

$$\begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ -2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 5 \\ 2 \end{bmatrix}$$

Use this fact (and no row operations) to find scalars c_1, c_2, c_3 such that

$$\begin{bmatrix} 12 \\ 5 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$$

Exercise 5.4 *Construct a 3×3 matrix, not in echelon form, whose columns span $\mathbb{R}^3$. Show that the matrix you construct has the desired property.*

Exercise 5.5 *Construct a* 3×3 *matrix, not in echelon form, whose columns do* not *span* $\mathbb{R}^3$. *Show that the matrix you construct has the desired property.*

Exercise 5.6 *Determine if the columns of the matrix*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}.$$

span $\mathbb{R}^4$.

`Answer: yes.`

Exercise 5.7 (True or False)

a) *The equation* $A\mathbf{x} = \mathbf{b}$ *is referred to as a* vector equation.

b) *A vector* $\mathbf{b}$ *is a linear combination of the columns of a matrix* A *if and only if the equation* $A\mathbf{x} = \mathbf{b}$ *has at least one solution.*

c) *The equation* $A\mathbf{x} = \mathbf{b}$ *is consistent if the augmented matrix* $[A \quad \mathbf{b}]$ *has a pivot position in every row.*

d) *If the columns of an* $m \times n$ *matrix* A *span* $\mathbb{R}^m$, *then the equation* $A\mathbf{x} = \mathbf{b}$ *is consistent for each* $\mathbf{b}$ *in* $\mathbb{R}^m$.

e) *If* A *is an* $m \times n$ *matrix and if the equation* $A\mathbf{x} = \mathbf{b}$ *is inconsistent for some* $\mathbf{b}$ *in* $\mathbb{R}^m$, *then* A *cannot have a pivot position in every row.*

f) *Every matrix equation* $A\mathbf{x} = \mathbf{b}$ *corresponds to a vector equation with the same solution set.*

g) *If the equation* $A\mathbf{x} = \mathbf{b}$ *is consistent, then* $\mathbf{b}$ *is in the set spanned by the columns of* A.

h) *Any linear combination of vectors can always be written in the form* $A\mathbf{x}$ *for a suitable matrix* A *and vector* $\mathbf{x}$.

i) *If the coefficient matrix* A *has a pivot position in every row, then the equation* $A\mathbf{x} = \mathbf{b}$ *is inconsistent.*

j) *The solution set of a linear system whose augmented matrix is* $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3 \quad \mathbf{b}]$ *is the same as the solution set of* $A\mathbf{x} = \mathbf{b}$, *if* $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$.

k) *If* A *is an* $m \times n$ *matrix whose columns do not span* $\mathbb{R}^m$, *then the equation* $A\mathbf{x} = \mathbf{b}$ *is consistent for every* $\mathbf{b}$ *in* $\mathbb{R}^m$

Exercise 5.8 *Solve the following system of nonlinear equations for x, y, and z.*

$$\begin{aligned} x^2 + y^2 + z^2 &= 6 \\ x^2 - y^2 + 2z^2 &= 2 \\ 2x^2 + y^2 - z^2 &= 3 \end{aligned}$$

HINT: Begin by making the substitutions $X = x^2$, $Y = y^2$, $Z = z^2$.
Answer: $x = \pm 1$, $y = \pm\sqrt{3}$, $z = \pm\sqrt{2}$.

6

Solutions Sets of Linear Systems (Exercises)

6.1 Practice Problems

Example 23 *Each of the following equations determines a plane in* $\mathbb{R}^3$*. Do the two planes intersect? If so, describe their intersection.*

$$\begin{array}{rcrcrcr} x_1 & + & 4x_2 & - & 5x_3 & = & 0 \\ 2x_1 & - & x_2 & + & 8x_3 & = & 9 \end{array}$$

Solution: Row reduce the augmented matrix:

$$\begin{bmatrix} 1 & 4 & 5 & 0 \\ 2 & -1 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 5 & 0 \\ 0 & -9 & 18 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -1 \end{bmatrix}$$

$$\begin{array}{rcrcrcr} x_1 & & & + & 3x_3 & = & 4 \\ & & x_2 & - & 2x_3 & = & -1 \end{array}$$

Thus $x_1 = 4 - 3x_3$; $x_2 = -1 + 2x_3$, with x_3 free. The general solution in parametric vector form is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 - 3x_3 \\ -1 + 2x_3 \\ x_3 \end{bmatrix} = \underset{\uparrow \atop \mathbf{p}}{\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}} + x_3 \underset{\uparrow \atop \mathbf{v}}{\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}}$$

The intersection of the two planes is the line through **p** in the direction of **v** (see Figure 6.1). □

Example 24 *Write the general solution of* $10x_1 - 3x_2 - 2x_3 = 7$ *in parametric vector form.*

Solution: The augmented matrix

$$\begin{bmatrix} 10 & -3 & -2 & 7 \end{bmatrix}$$

is row equivalent to

$$\begin{bmatrix} 1 & -.3 & -.2 & .7 \end{bmatrix}$$

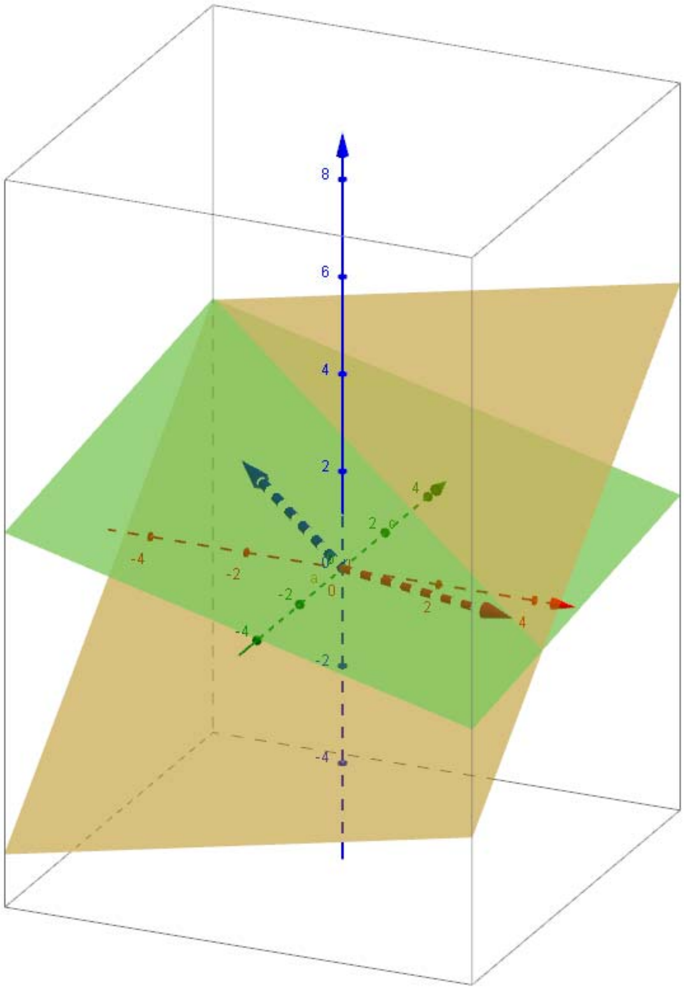


Fig. 6.1. The intersection of the two planes is the line through **p** (in red) in the direction of **v** (in blue).

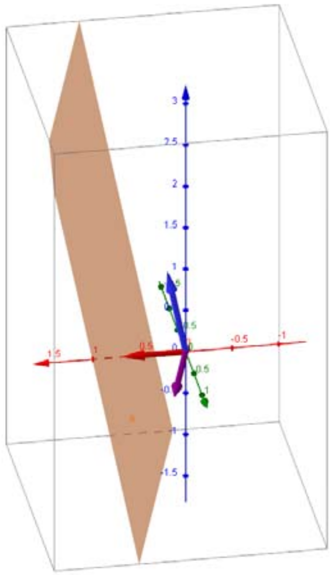


Fig. 6.2. The translated plane $\mathbf{p} + \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$, which passes through $\mathbf{p}$ (in red) and is parallel to $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$.

and the general solution is $x_1 = .7 + .3x_2 + .2x_3$, with x_2 and x_3 free. That is,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .7 + .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \underset{\substack{\uparrow \\ \mathbf{p}}}{\begin{bmatrix} .7 \\ 0 \\ 0 \end{bmatrix}} + x_2 \underset{\substack{\uparrow \\ x_2\mathbf{u}}}{\begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix}} + x_3 \underset{\substack{\uparrow \\ x_3\mathbf{v}}}{\begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix}}$$

The solution set of the nonhomogeneous equation $A\mathbf{x} = \mathbf{b}$ is the translated plane $\mathbf{p} + \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$, which passes through $\mathbf{p}$ and is parallel to the solution set of the homogeneous equation (see Figure 6.2).

$$10x_1 - 3x_2 - 2x_3 = 0$$

□

Example 25 *Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where A is row equivalent to the matrix*

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & -4 & -2 & 0 & 3 & -5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4 & -2 & 0 & 0 & 7 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcl} x_1 & - & 4x_2 & & & & 5x_6 & = & 0 \\ & & & x_3 & & - & x_6 & = & 0 \\ & & & & x_5 & - & 4x_6 & = & 0 \\ & & & & & & 0 & = & 0 \end{array}$$

Some students are not sure what to do with x_4. Some ignore it; others set it equal to zero. In fact, x_4 is free; there is no constraint on x_4, at all. The basic variables are x_1, x_3, and x_5. The remaining variables are free. So, $x_1 = 4x_2 - 5x_6$, $x_3 = x_6$, and $x_5 = 4x_6$, with x_2, x_4, and x_6 free.

In parametric vector form,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4x_2 - 5x_6 \\ x_2 \\ x_6 \\ x_4 \\ 4x_6 \\ x_6 \end{bmatrix} = x_2 \underset{\uparrow \atop \mathbf{u}}{\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}} + x_4 \underset{\uparrow \atop \mathbf{v}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + x_6 \underset{\uparrow \atop \mathbf{w}}{\begin{bmatrix} -5 \\ 0 \\ 1 \\ 0 \\ 4 \\ 1 \end{bmatrix}}$$

The solution set is the same as Span$\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. □

Study Tip: When solving a system, identify (and perhaps circle) the basic variables. All other variables are free.

Example 26 *Solve the following homogeneous system of linear equations by using Gauss–Jordan elimination.*

$$\left\{\begin{array}{rcrcrcrcrcl} 2x_1 & + & 2x_2 & - & x_3 & & & + & x_5 & = & 0 \\ -x_1 & - & x_2 & + & 2x_3 & - & 3x_4 & + & x_5 & = & 0 \\ x_1 & + & x_2 & - & 2x_3 & & & - & x_5 & = & 0 \\ & & & & x_3 & + & x_4 & + & x_5 & = & 0 \end{array}\right. \tag{6.1}$$

Solution: The augmented matrix for the system is

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Reducing this matrix to reduced row-echelon form, we obtain

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{array}{rcrcrcrcrcl} x_1 & + & x_2 & & & & & + & x_5 & = & 0 \\ & & & & x_3 & & & + & x_5 & = & 0 \\ & & & & & & x_4 & & & = & 0 \end{array} \tag{6.2}$$

Solving for the leading variables yields

$$\begin{array}{rcrcrcr} x_1 & = & - & x_1 & - & x_2 \\ x_3 & = & - & x_5 & & \\ x_4 & = & & 0 & & \end{array}$$

Thus, the general solution is

$$\begin{array}{rcl} x_1 & = & -s-t \\ x_2 & = & s \\ x_3 & = & -t \\ x_4 & = & 0 \\ x_5 & = & t \end{array}.$$

Note that the trivial solution is obtained when $s = t = 0$. □

Example 26 illustrates two important points about solving homogeneous systems of linear equations. First, none of the three elementary row operations alters the final column of zeros in the augmented matrix, so the system of equations corresponding to the reduced row-echelon form of the augmented matrix must also be a homogeneous system [see system 6.2]. Second, depending on whether the reduced row-echelon form of the augmented matrix has any zero rows, the number of equations in the reduced system is the same as or less than the number of equations in the original system [compare systems 6.1 and 6.2]. Thus, if the given homogeneous system has m equations in n unknowns with $m < n$, and if there are r nonzero rows in the reduced row-echelon form of the augmented matrix, we will have $r < n$. It follows that the system of equations corresponding to the reduced row-echelon form of the augmented matrix will have the form

$$\begin{array}{cccccccccc} \cdots & x_{k_1} & & & & + & \sum() & = & 0 \\ & \cdots & x_{k_2} & & & + & \sum() & = & 0 \\ & & & \cdots & & & \vdots & & \\ & & & & \cdots\ x_{k_r} & + & \sum() & = & 0 \end{array} \tag{6.3}$$

where $x_{k_1}, x_{k_2},...,x_{k_r}$ are the leading variables and denotes sums (possibly all different) that involve the free variables[compare system 6.3 with system 6.2 above]. Solving for the leading variables gives

$$\begin{array}{rcl} x_{k_1} & = & -\sum() \\ x_{k_2} & = & -\sum() \\ & \vdots & \\ x_{k_r} & = & -\sum() \end{array}$$

As in Example 26, we can assign arbitrary values to the free variables on the right-hand side and thus obtain infinitely many solutions to the system. In summary, we have the following important conclusion.

Conclusion 27 *A homogeneous system of linear equations with more unknowns than equations has infinitely many solutions.*

6.2 Exercises

Exercise 6.1 (True of False)

a) *A homogeneous equation is always consistent.*

b) *The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution if and only if the equation has at least one free variable.*

c) *The equation* $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ *describes a line through* $\mathbf{v}$ *parallel to* $\mathbf{p}$.

d) *The solution set of* $A\mathbf{x} = \mathbf{b}$ *is the set of all vectors of the form* $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, *where* $\mathbf{v}_h$ *is any solution of the equation* $A\mathbf{x} = \mathbf{0}$.

e) *A homogeneous system of equations can be inconsistent.*

f) *If* $\mathbf{x}$ *is a nontrivial solution of* $A\mathbf{x} = \mathbf{0}$, *then every entry in* $\mathbf{x}$ *is nonzero.*

g) *The effect of adding* $\mathbf{p}$ *to a vector is to move the vector in a direction parallel to* $\mathbf{p}$.

h) *The equation* $A\mathbf{x} = \mathbf{b}$ *is homogeneous if the zero vector is a solution.*

i) *If a linear system has more unknowns than equations, then it must have infinitely many solutions.*

Exercise 6.2 *If the linear system*

$$\begin{array}{rcrcrcr} a_1x & + & b_1y & + & c_1z & = & 0 \\ a_2x & - & b_2y & + & c_2z & = & 0 \\ a_3x & + & b_3y & - & c_3z & = & 0 \end{array}$$

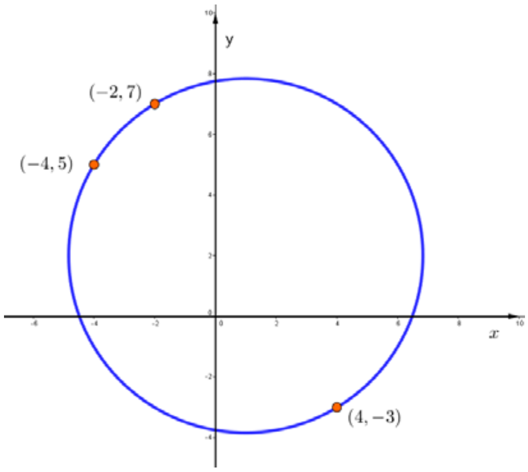
has only the trivial solution, what can be said about the solutions of the following system?

$$\begin{array}{rcrcrcr} a_1x & + & b_1y & + & c_1z & = & 3 \\ a_2x & - & b_2y & + & c_2z & = & 7 \\ a_3x & + & b_3y & - & c_3z & = & 11 \end{array}$$

Solution: The nonhomogeneous system will have exactly one solution.

Exercise 6.3 *Find the coefficients a, b, c, and d so that the curve shown in the accompanying figure is given by the equation*

$$ax^2 + ay^2 + bx + cy + d = 0$$



Exercise 6.4

a) *Prove that if* $ad - bc \neq 0$*, then the reduced row echelon form of*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \textit{is} \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

b) *Use the result in part a) to prove that if* $ad - bc \neq 0$*, then the linear system*

$$\begin{array}{ccccc} ax & + & by & = & k \\ cx & + & dy & = & l \end{array}$$

has exactly one solution.

Exercise 6.5 *Show that the following nonlinear system has* 18 *solutions if* $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq 2\pi$, *and* $0 \leq \gamma \leq 2\pi$.

$$\left\{ \begin{array}{ccccccc} \sin\alpha & + & 2\cos\beta & + & 3\tan\gamma & = & 0 \\ 2\sin\alpha & + & 5\cos\beta & + & 3\tan\gamma & = & 0 \\ -\sin\alpha & - & 5\cos\beta & + & 5\tan\gamma & = & 0 \end{array} \right.$$

HINT: Begin by making the substitutions $x = \sin\alpha$, $y = \cos\beta$, $z = \tan\gamma$.

7
Linear Independence (Exercises)

7.1 Practice Problems

Example 28 *Let*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

a) *Determine if the set* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is linearly independent.*

b) *If possible, find a linear dependence relation among* $\mathbf{v}_1, \mathbf{v}_2$ *and* $\mathbf{v}_3$

Solution:

a) We must determine if there is a nontrivial solution of equation

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \tag{7.1}$$

Row operations on the associated augmented matrix show that

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, x_1 and x_2 are basic variables, and x_3 is free. Each nonzero value of x_3 determines a nontrivial solution of (7.1). Hence $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent (and not linearly independent).

b) To find a linear dependence relation among $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$, completely row reduce the augmented matrix and write the new system:

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{array}{rcrcrcr} x_1 & & & - & 2x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 0 \\ & & & & 0 & = & 0 \end{array}$$

Thus $x_1 = 2x_3$, $x_2 = -x_3$, and x_3 is free. Choose any nonzero value for x_3—say, $x_3 = 5$. Then $x_1 = 10$ and $x_2 = -5$. Substitute these values into equation (7.1) and obtain

$$10\mathbf{v}_1 - 5\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}.$$

This is one (out of infinitely many) possible linear dependence relations among $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$.

Example 29 *Determine if the vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *are linearly independent, where*

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 9 \\ 4 \\ -8 \end{bmatrix}.$$

Justify each answer.

Solution: Use an augmented matrix to study the solution set of

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{0}, \tag{7.2}$$

where $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ are the three given vectors. Since

$$\begin{bmatrix} 5 & 7 & 9 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & -6 & -8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix},$$

there are no free variables. So the homogeneous equation (7.2) has only the trivial solution. The vectors are linearly independent. □

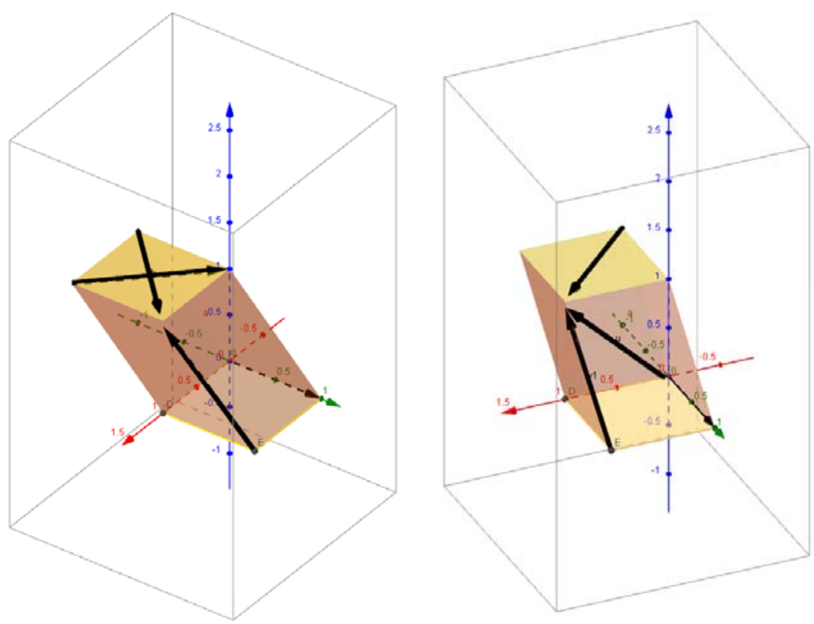
Warning: *Whenever you study a homogeneous equation, you may be tempted to omit the augmented column of zeros because it never changes under row operations. I urge you to keep the zeros, to avoid possibly misinterpreting your own calculations.*

Example 30 *Are the following vectors in* $\mathbb{R}^7$ *linearly independent?*

$$\mathbf{v}_1 = \begin{bmatrix} 7 \\ 0 \\ 4 \\ 0 \\ 1 \\ 9 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 6 \\ 0 \\ 7 \\ 1 \\ 4 \\ 8 \\ 0 \end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix} 5 \\ 0 \\ 6 \\ 2 \\ 3 \\ 1 \\ 7 \end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 3 \\ 2 \\ 2 \\ 4 \end{bmatrix}.$$

Solution: Let's look for "redundant" vectors (as far as the span is concerned) in this list. Vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are clearly nonredundant, since $\mathbf{v}_1$ is nonzero and $\mathbf{v}_2$ fails to be a scalar multiple of $\mathbf{v}_1$ (look at the fourth components). Looking at the last components, we realize that $\mathbf{v}_3$ cannot be a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, since any linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$ will have a $\mathbf{0}$ in the last component, while the last component of $\mathbf{v}_3$ is 7. Looking at the second components, we can see that $\mathbf{v}_4$ isn't a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$. Thus the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$ are linearly independent. □

Example 31 *Are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ (all in black) in part (a) of the accompanying figure linearly independent? What about those in part (b)? Explain.*



a) *b)*

`Answer:`

Exercise 7.1 **a)** *They are linearly independent since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ do not lie in the same plane when they are placed with their initial points at the origin.*

b) *They are not linearly independent since $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ lie in the same plane when they are placed with their initial points at the origin.*

7.2 Exercises

Exercise 7.2 (True or False) **a)** *A set containing a single vector is linearly independent.*

b) *The set of vectors* $\{\mathbf{v}, k\mathbf{v}\}$*is linearly dependent for every scalar* k*.*

c) *Every linearly dependent set contains the zero vector.*

d) *If the set of vectors* $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ *is linearly independent, then* $\{k\mathbf{v}_1, k\mathbf{v}_2, k\mathbf{v}_3\}$ *is also linearly independent for every nonzero scalar* k*.*

e) *If* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ *are linearly dependent nonzero vectors, then at least one vector* $\mathbf{v}_k$ *is a unique linear combination of* $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_{k-1}$.

f) *The columns of a matrix* A *are linearly independent if the equation* $A\mathbf{x} = \mathbf{0}$ *has the trivial solution.*

g) *If* S *is a linearly dependent set, then each vector is a linear combination of the other vectors in* S*.*

h) *The columns of any* 5×6 *matrix are linearly dependent.*

i) *If* $\mathbf{x}$ *and* $\mathbf{y}$ *are linearly independent, and if* $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *is linearly dependent, then* $\mathbf{z}$ *is in* $\text{Span}\{\mathbf{x}, \mathbf{y}\}$.

j) *If three vectors in* $\mathbb{R}^3$ *lie in the same plane in* $\mathbb{R}^3$*, then they are linearly dependent.*

k) *If a set contains fewer vectors than there are entries in the vectors, then the set is linearly independent.*

l) *If a set in* $\mathbb{R}^n$ *is linearly dependent, then the set contains more than* n *vectors.*

m) *If* $\mathbf{v}_1, ..., \mathbf{v}_4$ *are in* $\mathbb{R}^4$ *and* $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$*, then* $\{\mathbf{v}_1, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v}_4\}$ *is linearly dependent.*

n *If* $\mathbf{v}_1, ..., \mathbf{v}_5$ *are in* $\mathbb{R}^5$ *and* $\mathbf{v}_3 = \mathbf{0}$*, then* $\{\mathbf{v}_1, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v}_4, \mathbf{v}_5\}$ *is linearly dependent.*

o) *If* $\mathbf{v}_1, ..., \mathbf{v}_4$ *are in* $\mathbb{R}^4$ *and* $\{\mathbf{v}_1, \mathbf{v_2}, \mathbf{v_3}\}$ *is linearly dependent, then* $\{\mathbf{v}_1, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v}_4\}$ *is also linearly dependent.*

p) *If* $\{\mathbf{v}_1, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v}_4\}$ *is a linearly independent set of vectors in*$\mathbb{R}^4$*, then* $\{\mathbf{v}_1, \mathbf{v_2}, \mathbf{v_3}\}$ *is also linearly independent. [HINT: Think about* $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 + 0\mathbf{v}_4 = \mathbf{0}$*.]*

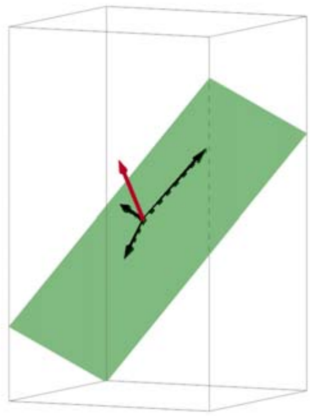


Fig. 7.1. Vector **z** (in red) is not a linear combination of **u**, **v**, and **w** (all in black)

Exercise 7.3 *Let*

$$\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -6 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ 2 \end{bmatrix}, \textit{ and } \quad \mathbf{z} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}.$$

a) *Are the sets* $\{\mathbf{u}, \mathbf{v}\}$*;* $\{\mathbf{u}, \mathbf{w}\}$*;* $\{\mathbf{u}, \mathbf{z}\}$*;* $\{\mathbf{v}, \mathbf{w}\}$*;* $\{\mathbf{v}, \mathbf{z}\}$*, and* $\{\mathbf{w}, \mathbf{z}\}$ *each linearly independent? Why or why not?*

b) *Does the answer to part a) imply that* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ *is linearly independent?*

c) *To determine if* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ *is linearly dependent, is it wise to check if, say,* **w** *is a linear combination of* **u***,* **v***, and* **z***?*

d) *Is* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z}\}$ *linearly dependent?*

Warning: When testing for linear independence, it is usually a poor idea to check if one selected vector is a linear combination of the others. It may happen that the selected vector is not a linear combination of the others and yet the whole set of vectors is linearly dependent. In the Exercise 7.3, **z** is not a linear combination of **u**, **v**, and **w** (see Figure 7.1).

Exercise 7.4 *Use as many columns of*

$$A = \begin{bmatrix} 3 & -4 & 10 & 7 & -4 \\ -5 & -3 & -7 & -11 & 15 \\ 4 & 3 & 5 & 2 & 1 \\ 8 & -7 & 23 & 4 & 15 \end{bmatrix}$$

as possible to construct a matrix B with the property that the equation $B\mathbf{x} = \mathbf{0}$ has only the trivial solution. Solve $B\mathbf{x} = \mathbf{0}$ to verify your work.

8

Introduction to Linear Transformations (Exercises)

8.1 Practice Problems

Definition 32 *A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a linear transformation if for all vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$ and for all scalars $c_1, c_2 \in \mathbb{R}$, T satisfies the linearity property $T(c_1\mathbf{v} + c_2\mathbf{v}) = c_1T(\mathbf{v}) + c_2T(\mathbf{v})$. This can also be expressed more geometrically by saying that T* preserves vector addition, *i.e.* $T(\mathbf{v}_1+\mathbf{v}_2) = T(\mathbf{v}_1) + T(\mathbf{v}_2)$ *, and T* preserves scalar multiplication, *i.e.* $T(cv) = cT(v)$ *.*

We call the input space $\mathbb{R}^n$ the **domain** (as expected), and we refer to the output space $\mathbb{R}^m$ as the **codomain**.

Example 33 *Prove that $T(\mathbf{x}) = A\mathbf{x}$ (for an $m \times n$ matrix A) is a linear transformation.*

Solution: If we write the matrix A in terms of its columns,

$$A = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ \downarrow & & \downarrow \end{bmatrix}$$

and let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

and let $\alpha, \beta \in \mathbb{R}$, then

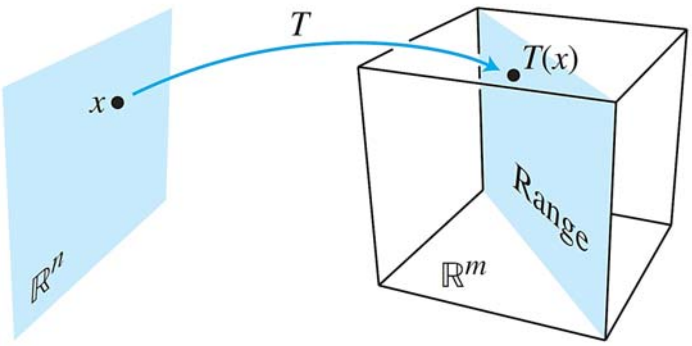
$$\alpha\mathbf{x}+\beta\mathbf{y} = \alpha \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \beta \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \vdots \\ \alpha x_n \end{bmatrix} + \begin{bmatrix} \beta y_1 \\ \vdots \\ \beta y_n \end{bmatrix} = \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_n + \beta y_n \end{bmatrix}$$

using basic facts about scaling and adding vectors. Using our definition of the product of a matrix and a vector, we have:

$$\begin{aligned} T(\alpha\mathbf{x}+\beta\mathbf{y}) &= A(\alpha\mathbf{x}+\beta\mathbf{y}) = \begin{bmatrix} \uparrow & & \uparrow \\ \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ \downarrow & & \downarrow \end{bmatrix} \begin{bmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_n + \beta y_n \end{bmatrix} \\ &= (\alpha x_1 + \beta y_1)\,\mathbf{a}_1 + \cdots + (\alpha x_n + \beta y_n)\,\mathbf{a}_n \\ &= \alpha x_1\mathbf{a}_1 + \beta y_1\mathbf{a}_1 + \cdots + \alpha x_n\mathbf{a}_n + \beta y_n\mathbf{a}_n \\ &= \alpha\,(x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n) + \beta\,(y_1\mathbf{a}_1 + \cdots + y_n\mathbf{a}_n) \\ &= \alpha A\mathbf{x}+\beta A\mathbf{y} = \alpha T\,(\mathbf{x}) + \beta T\,(\mathbf{y})\,. \end{aligned}$$

As you can see, the linearity property ultimately flows from the distributive law for vector addition. □

Definition 34 *The set of all images $T(\mathbf{x})$ is called the **range** of T.*



Domain, codomain, and range of $T : \mathbb{R}^n \to \mathbb{R}^m$.

Example 35 *Let* $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, *and define a transformation* $T : \mathbb{R}^2 \to \mathbb{R}^3$ *by* $T(\mathbf{x}) = A\mathbf{x}$, *so that*

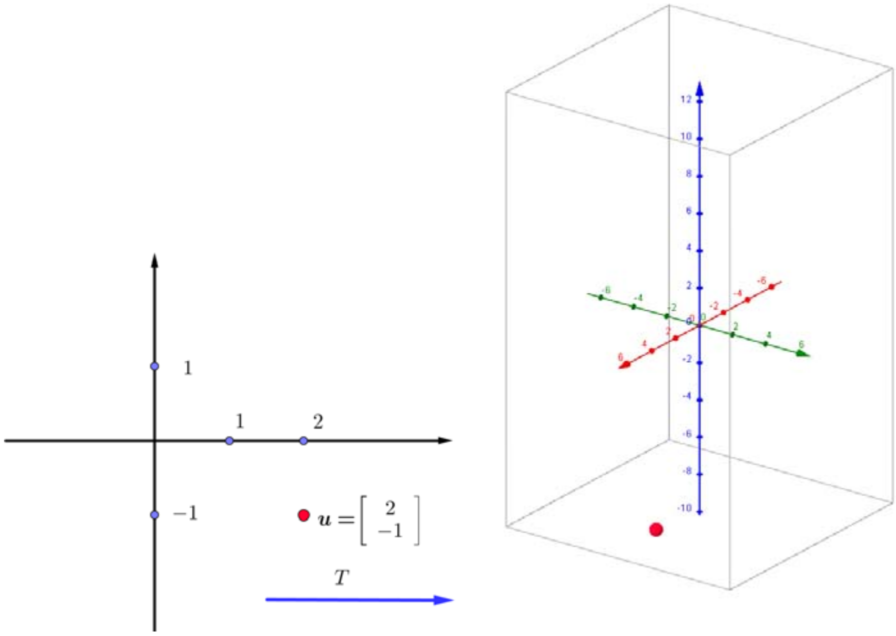
$$T(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ 7x_2 - x_1 \end{bmatrix}.$$

a) *Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation T.*

b) *Find an* $\mathbf{x}$ *in* $\mathbb{R}^2$ *whose image under* T *is* $\mathbf{b}$.

c) *Is there more than one* $\mathbf{x}$ *whose image under* T *is* $\mathbf{b}$*?*

d) *Determine if* c *is in the range of the transformation* T.



Solution:

a) Compute

$$T(\mathbf{u}) = A\mathbf{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}. \qquad (8.1)$$

b) Solve $T(\mathbf{x}) = \mathbf{b}$ for $\mathbf{x}$. That is, solve $A\mathbf{x} = \mathbf{b}$, or

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \qquad (8.2)$$

Using the Gauss-Jordan method, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence $x_1 = 1.5$, $x_2 = -.5$, and $\mathbf{x} = \begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}$. The image of this $\mathbf{x}$ under T is the given vector $\mathbf{b}$.

c) Any $\mathbf{x}$ whose image under T is $\mathbf{b}$ must satisfy equation (8.1). From (8.2), it is clear that equation (8.1) has a unique solution. So there is exactly one $\mathbf{x}$ whose image is $\mathbf{b}$.

d) The vector $\mathbf{c}$ is in the range of T if $\mathbf{c}$ is the image of some $\mathbf{x}$ in $\mathbb{R}^2$, that is, if $\mathbf{c} = T(\mathbf{x})$ for some $\mathbf{x}$. This is just another way of asking if the system $A\mathbf{x} = \mathbf{c}$ is consistent. To find the answer, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

The third equation, $0 = -35$, shows that the system is inconsistent. So $\mathbf{c}$ is not in the range of T . □

The next example is important, because it will help you to connect the concepts of linear dependence and linear transformation.

Example 36 *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a linearly dependent set in $\mathbb{R}^n$. Explain why the set $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent.*

Solution: To construct the proof, first write in mathematical terms what is given.

Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent, there exist scalars c_1, c_2, c_3, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}. \tag{8.3}$$

Apply to both sides of (8.3) and use linearity of T, obtaining

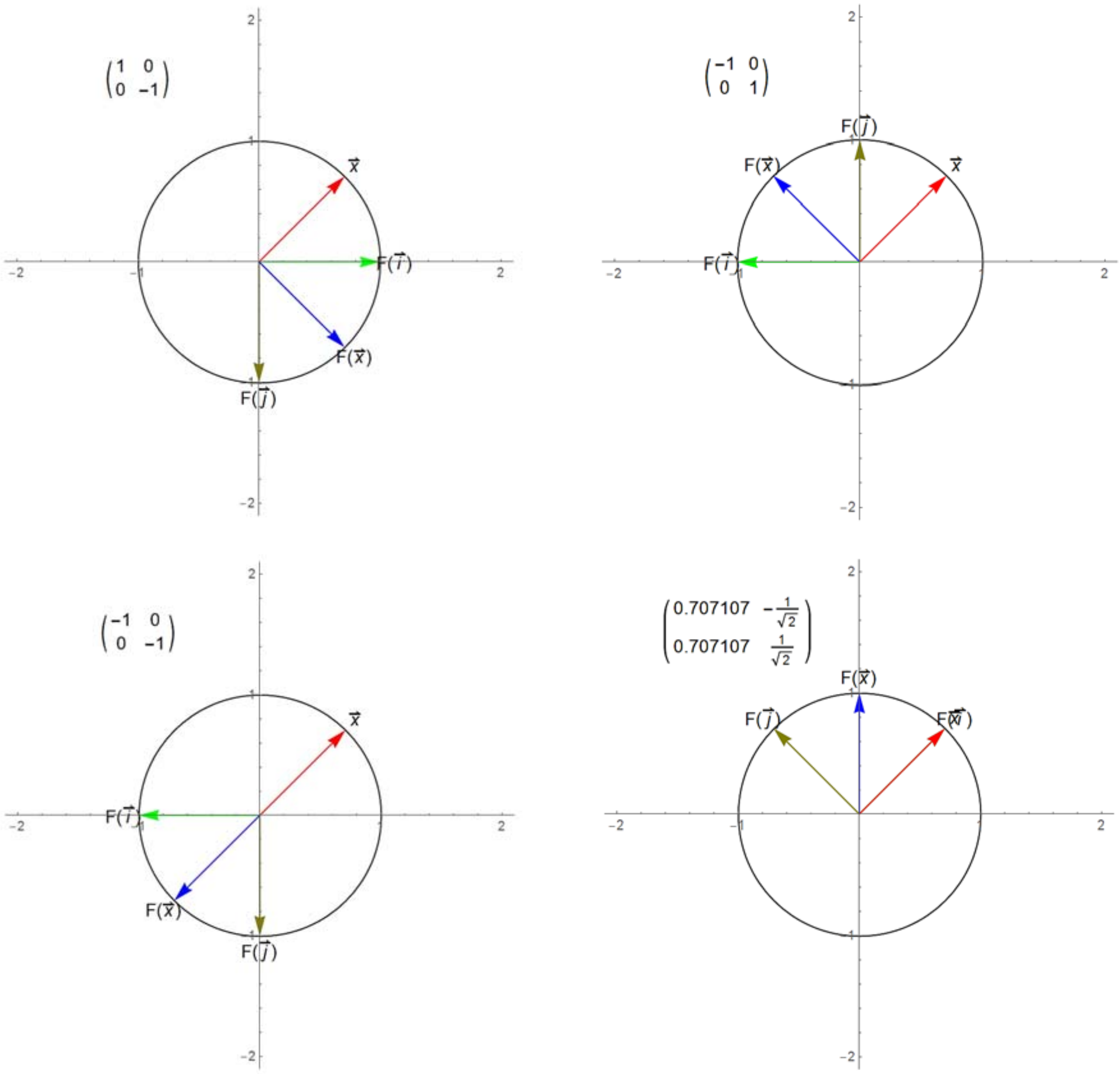
$$T(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3) = T(\mathbf{0})$$

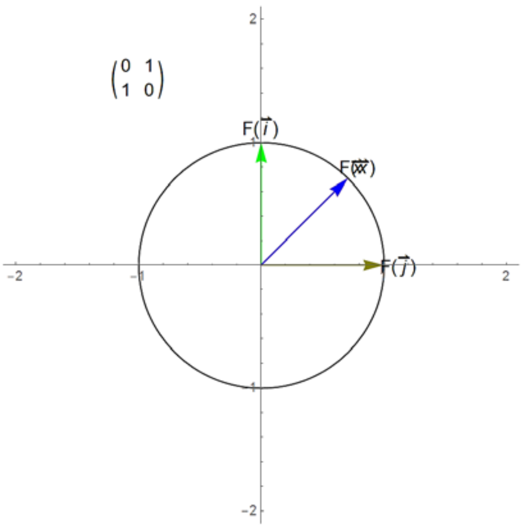
and

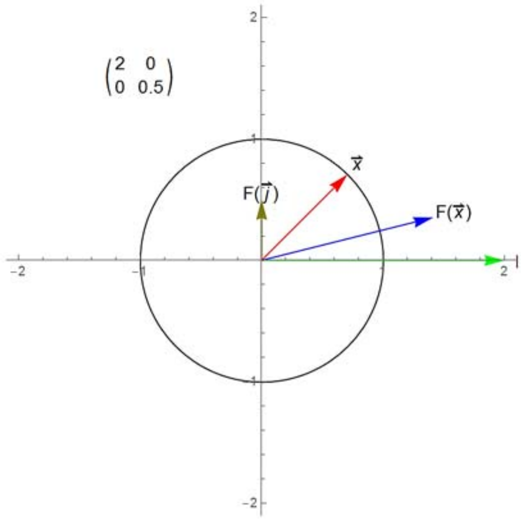
$$c_1T(\mathbf{v}_1) + c_2T(\mathbf{v}_2) + c_3T(\mathbf{v}_3) = \mathbf{0}$$

Since not all the weights are zero, $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent set. □

Example 37 *A linear transformation is completely determined by the images of a set of basis vectors. In the case of a linear transformation* $F:\mathbb{R}^2\to\mathbb{R}^2$ *,* $F(\mathbf{x})=A\mathbf{x}$*, where the columns of the matrix* A *are the vectors* $F(\mathbf{i})$ *and* $F(\mathbf{j})$*.(Here* $\mathbf{i}=\mathbf{e}_1,\ \mathbf{j}=\mathbf{e}_2$*). Look for examples:*







8.2 Exercises

Exercise 8.1 (True or False)

a) *If A is a 3×5 matrix and T is a transformation defined by $T(\mathbf{x}) = A\mathbf{x}$, then the domain of T is $\mathbb{R}^3$.*

b) *The range of the transformation $\mathbf{x} \to A\mathbf{x}$ is the set of all linear combinations of the columns of A.*

c) *A linear transformation preserves the operations of vector addition and scalar multiplication.*

d) *A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ always maps the origin of $\mathbb{R}^n$ to the origin of $\mathbb{R}^m$*

Exercise 8.2 *Suppose $T : \mathbb{R}^5 \to \mathbb{R}^2$ and $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A and for each $\mathbf{x}$ in $\mathbb{R}^5$. How many rows and columns does A have?*

Exercise 8.3 *Let $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Give a geometric description of the transformation $\mathbf{x} \to A\mathbf{x}$.*

Exercise 8.4 *The line segment from $\mathbf{0}$ to a vector $\mathbf{u}$ is the set of points of the form $t\mathbf{u}$, where $0 \leq t \leq 1$. Show that a linear transformation T maps this segment into the segment between 0 and $T(\mathbf{u})$.*

Exercise 8.5 *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be linearly independent vectors in* $\mathbb{R}^3$*, and let* P *be the plane through* $\mathbf{u}$*,* $\mathbf{v}$*, and* $\mathbf{0}$*. The parametric equation of* P *is* $x = s\mathbf{u} + t\mathbf{v}$ *(with* $s; t$ *in* $\mathbb{R}$*). Show that a linear transformation* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *maps* P *onto a plane through* $\mathbf{0}$*, or onto a line through* $\mathbf{0}$*, or onto just the origin in* $\mathbb{R}^3$*. What must be true about* $T(\mathbf{u})$ *and* $T(\mathbf{v})$ *in order for the image of the plane* P *to be a plane?*

Exercise 8.6 *Define* $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = mx + b$*.*

a) *Show that* f *is a linear transformation when* $b = 0$*.*

b) *Find a property of a linear transformation that is violated when* $b \neq 0$*.*

c) *Why is* f *called a linear function?*

Exercise 8.7 *An affine transformation* $T : \mathbb{R}^n \to \mathbb{R}^m$ *has the form* $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$*, with* A *an* $m \times n$ *matrix and* $\mathbf{b}$ *in* $\mathbb{R}^m$*. Show that* T *is not a linear transformation when* $\mathbf{b} \neq 0$*. (Affine transformations are important in computer graphics.)*

Exercise 8.8 *The conversion formula* $C = \frac{5}{9}(F - 32)$ *from Fahrenheit to Celsius (as measures of temperature) is nonlinear, in the sense of linear algebra (why?). Still, there is a technique that allows us to use a matrix to represent this conversion.*

Exercise 8.9 *In the financial pages of a newspaper, one can sometimes find a table (or matrix) listing the exchange rates between currencies. In this exercise we will consider a miniature version of such a table, involving only the Canadian dollar (C\$) and the South African Rand (ZAR). Consider the matrix*

$$\begin{array}{cc} \begin{bmatrix} 1 & 1/8 \\ 8 & 1 \end{bmatrix} & \begin{array}{c} C\$ \\ ZAR \end{array} \\ \begin{array}{cc} C\$ & ZAR \end{array} & \end{array}$$

representing the fact that $C\$\ 1$ *is worth* $ZAR\ 8$ *(as of June* 2008*). After a trip you have* $C\$\ \ 100$ *and* $ZAR\ 1,600$ *in your pocket. We represent these two values in the vector*

$$\mathbf{x} = \begin{bmatrix} 100 \\ 1600 \end{bmatrix}.$$

Compute $A\mathbf{x}$*. What is the practical significance of the two components of the vector* $A\mathbf{x}$*?*

9

The Matrix of a Linear Transformation (Exercises)

9.1 Practice Problems

Example 38 *Let*

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is called a ***shear*** *transformation. It can be shown that if T acts on each point in the 2×2 square shown in Figure below, then the set of images forms the green parallelogram. The key idea is to show that T maps line segments onto line segments (as shown in Exercises of the previous chapter) and then to check that the corners of the square map onto the vertices of the parallelogram. For instance, the image of the point $\mathbf{u} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is*

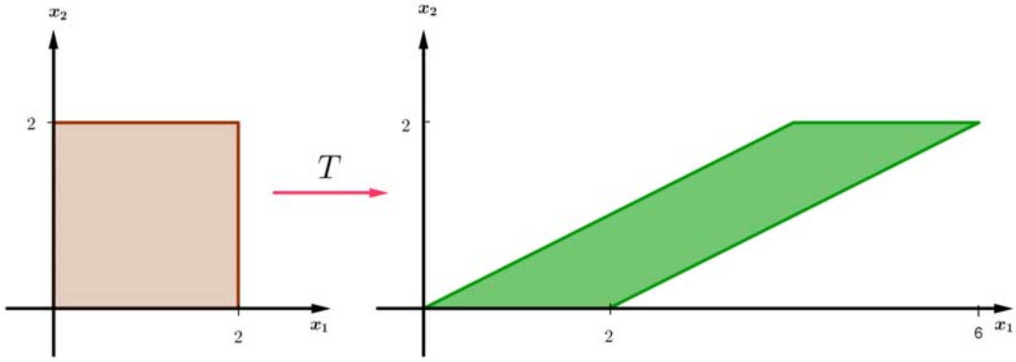
$$T(\mathbf{u}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix},$$

and the image of $\mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is

$$T(\mathbf{v}) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix},$$

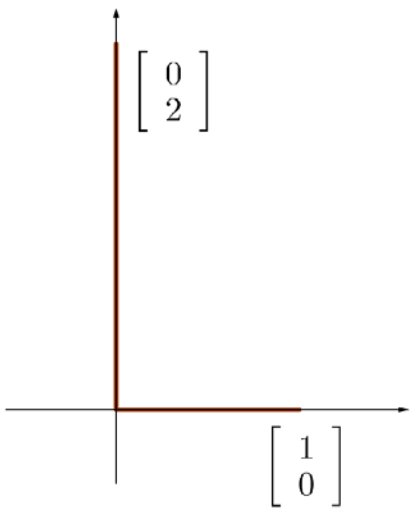
T deforms the square as if the top of the square were pushed to the right while the base is held fixed. Shear transformations appear in physics, geology, and

crystallography.



Example 39 *Consider the letter L in Figure below, made up of the vectors* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *and* $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$. *Show the effect of the linear transformation*

$$T(\mathbf{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{x}$$



The letter L.

on this letter, and describe the transformation in words.

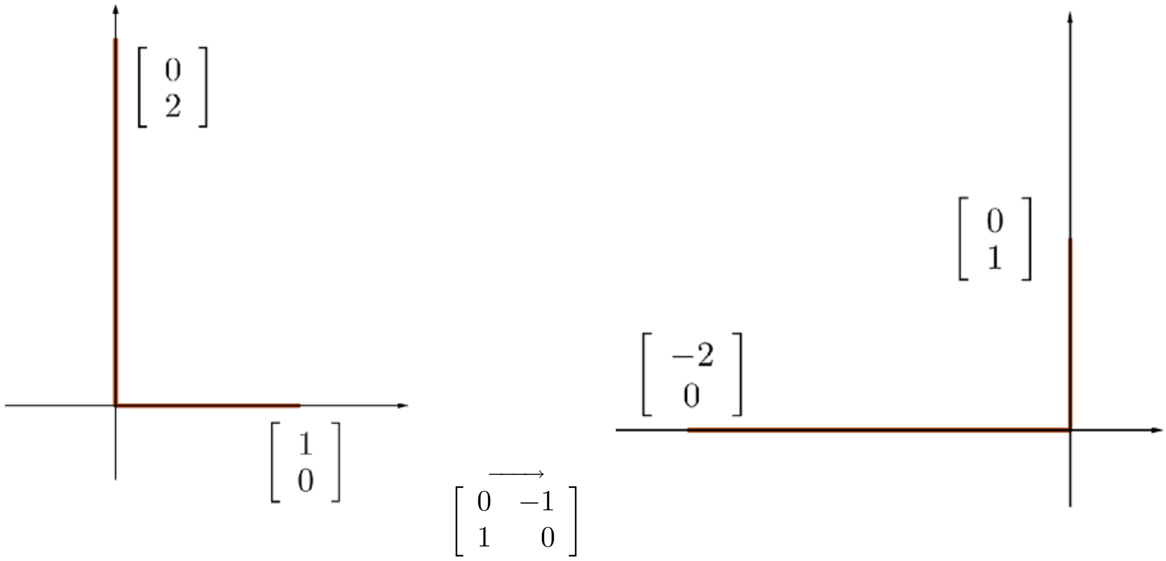
Solution: We have

$$T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$T\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

as shown below.



The L is rotated through an angle of $90°$ in the counterclockwise direction. □

Example 40 *Find the standard matrix A for the dilation transformation $T(\mathbf{x}) = 3\mathbf{x}$, for $\mathbf{x}$ in $\mathbb{R}^2$.*

Solution:

$$T(\mathbf{e}_1) = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \quad \text{and} \quad T(\mathbf{e}_2) = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

So

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

□

Example 41 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that first performs a horizontal shear that maps $\mathbf{e}_2$ into $\mathbf{e}_2 - .5\mathbf{e}_1$ (but leaves $\mathbf{e}_1$ unchanged) and then reflects the result through the x_2-axis. Assuming that T is linear, find its standard matrix. [HINT:: Determine the final location of the images of $\mathbf{e}_1$ and $\mathbf{e}_2$.]*

Solution: Follow what happens to $\mathbf{e}_1$ and $\mathbf{e}_2$. First, $\mathbf{e}_1$ is unaffected by the shear and then is reflected into $-\mathbf{e}_1$. So $T(\mathbf{e}_1) = -\mathbf{e}_1$. Second, $\mathbf{e}_2$ goes to $\mathbf{e}_2$ $-.5\mathbf{e}_1$ by the shear transformation. Since reflection through the x_2-axis changes $\mathbf{e}_1$ into -$\mathbf{e}_1$ and leaves $\mathbf{e}_2$ unchanged, the vector $\mathbf{e}_2 - .5\mathbf{e}_1$ goes to $\mathbf{e}_2$ $+.5\mathbf{e}_1$. So $T(\mathbf{e}_1) = \mathbf{e}_2$ $+.5\mathbf{e}_1$. Thus the standard matrix of T is

$$\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} -\mathbf{e}_1 & \mathbf{e}_2 + .5\mathbf{e}_1 \end{bmatrix} = \begin{bmatrix} -1 & .5 \\ 0 & 1 \end{bmatrix}.$$

□

9.2 Exercises

Exercise 9.1 (True or False) a) *A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is completely determined by its effect on the columns of the $n \times n$ identity matrix.*

b) *If $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates vectors about the origin through an angle ϕ, then T is a linear transformation.*

Exercise 9.2 *$T : \mathbb{R}^2 \to \mathbb{R}^2$ first rotates points through $-3\pi/4$ radians (clockwise) and then reflects points through the horizontal x_1-axis. Find the standard matrix of T .*

Exercise 9.3 *$T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the horizontal x_1- axis and then reflects points through the line $x_2 = x_1$. Find the standard matrix of T . Show that T is merely a rotation about the origin. What is the angle of the rotation?*

Exercise 9.4 *A linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points through the x_1- axis and then reflects points through the x_2- axis. Find the standard matrix of T . Show that T can also be described as a linear transformation that rotates points about the origin. What is the angle of that rotation?*

Fill in the missing entries of the matrix, assuming that the equation holds for all values of the variables.

$$\begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_3 \\ x_1 - x_3 \\ -x_2 - x_3 \end{bmatrix}.$$

10
Matrix Operations (Exercises)

10.1 Diagonal Matrices

A square matrix is an $n \times n$ matrix; that is, a matrix with the same number of rows as columns. The diagonal of a square matrix is the list of entries $a_{11}, a_{22}, ..., a_{nn}$. A *diagonal matrix* is a square matrix with all the entries which are not on the diagonal equal to 0. So A is diagonal if it is $n \times n$ and $a_{ij} = 0$ if $i \neq j$. Then A looks as follows

$$A = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}.$$

Notice, that some of the diagonal elements in this diagonal matrix could be equal to zero.

Example 42 *Which of these matrices are diagonal?*

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

Answer: *Only the second matrix is diagonal.* □

Definition 43 *Two matrices are* equal *if they are the same shape and if corresponding entries are equal. That is, if $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then*

$$A = B \Leftrightarrow a_{ij} = b_{ij} \qquad 1 \leq i \leq m, \ 1 \leq j \leq n.$$

10.2 Matrix addition and scalar multiplication

From now we will restrict our attention (unless it is explicitly stated) to the most common class of matrices i.e. to the $\mathbb{R}$-valued matrices (real matrices).

If A and B are two real matrices, then provided they are the same shape we can add them together to form a new matrix $A+B$. We define $A+B$ to be the matrix whose entries are the sums of the corresponding entries in A and B.

Definition 44 *If $A = [a_{ij}]$ and $B = [b_{ij}]$ are both $m \times n$ matrices, then*

$$A + B = [a_{ij} + b_{ij}] \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Example 45 *Let*

$$A = \begin{bmatrix} 1 & -4 & \frac{1}{2} & 3 \\ 0 & \frac{1}{4} & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} \frac{1}{4} & 3 & 1 & 0 \\ 1 & 2 & 1 & \frac{1}{8} \\ 1 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} \frac{5}{4} & -1 & \frac{3}{2} & 3 \\ 1 & \frac{9}{4} & 2 & \frac{9}{8} \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

□

We can also multiply any matrix by a real number, referred to as a *scalar* in this context. If λ is a scalar and A is a matrix, then λA is the real matrix whose entries are λ times each of the entries of A.

Definition 46 (Scalar multiplication) *If $A = [a_{ij}]$ is an $m \times n$ real matrix and $\lambda \in \mathbb{R}$, then*

$$\lambda A = [\lambda a_{ij}] \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Example 47 *Let*

$$A = \begin{pmatrix} 1 & -4 & \frac{1}{2} & 3 \\ 0 & \frac{1}{4} & 1 & 1 \end{pmatrix}.$$

Then

$$-3A = \begin{bmatrix} -3 & 12 & -\frac{3}{2} & -9 \\ 0 & -\frac{3}{4} & -3 & -3 \end{bmatrix}.$$

□

10.3 Matrix multiplication

Is there a way to multiply two matrices together in the meaningful way? The answer is sometimes, depending on the shapes of the matrices. If A and B are

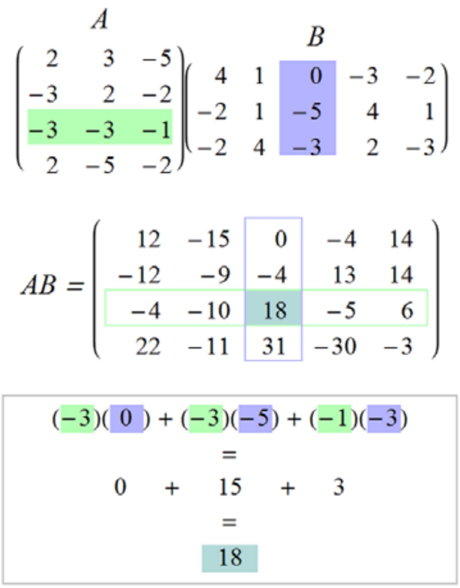


Fig. 10.1. Element in row 3 and column 3 of the product is obtained.

matrices such that the number of columns of A is equal to the number of rows of B, then we can define a matrix C which is the product of A and B. We do this[1] by saying what the entry c_{ij} of the product matrix AB should be.

Definition 48 (Matrix multiplication) *If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then the product is the matrix $AB = C = [c_{ij}]$ with*

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}.$$

Although this formula looks daunting, it is quite easy to use in practice. What it says is that the element in row i and column j of the product is obtained by taking each entry of row i of A and multiplying it by the corresponding entry of column j of B, then adding these n products together (see Fig. 10.1 and Fig. 10.2).

Be sure you understand that for the product of two matrices to be defined, the number of columns of the first matrix must equal the number of rows of

[1] We shall see in later chapters that this definition of matrix multiplicationis exactly what is needed for applying matrices in our study of linear algebra.

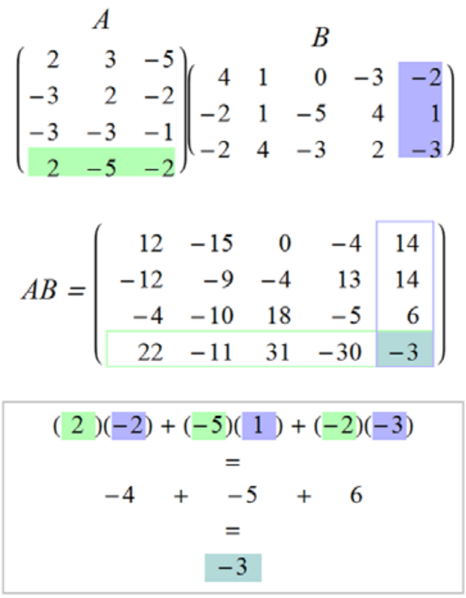


Fig. 10.2. Element in row 4 and column 5 of the product AB is obtained.

the second matrix. That is,

$$\begin{array}{ccccc} A & * & B & = & AB \\ \mathrm{m{\times}n} & & \mathrm{n{\times}p} & & \mathrm{m{\times}p} \\ \uparrow & \overleftrightarrow{\textit{equal}} & \uparrow & & \\ \Uparrow & \overleftrightarrow{\textit{shape of } AB} & \Uparrow & & \end{array}$$

What shape is $C = AB$? The matrix C must be $m \times p$ since it will have one entry for each of the m rows of A and each of the p columns of B. It is easy to check, that approximately $m \times p \times n$ of floating point operations (called *flops*) are required to execute just described method of computing AB on computer.

Example 49 *if*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times 3} \quad \textit{and} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}_{3\times 4}$$

then inside ones match

$$\begin{array}{ccccccc} (& 2 & \times & \mathbf{3} &) & & \\ & (& & \mathbf{3} & \times & 4 &), \end{array}$$

so the product AB exists and has shape 2×4. We need 24 flops to get the result. $\square$

It is an important consequence of our definition (48) that:

- $AB \neq BA$ in general. That is, matrix multiplication is *not 'commutative'*.

To see just how non-commutative matrix multiplication is, let's look at some examples, starting with the two matrices A and B in the example (49) above. The product AB is defined, but the product BA is not even defined. Since A is 2×3 and B is 3×4, it is not possible to multiply the matrices in the order BA.

Now consider the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 4 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}$$

Both products AB and BA are defined, but they are different shapes, so they cannot be equal. What shapes are they?

Even if both products are defined and the same shape, it is still generally true that $AB \neq BA$.

Example 50 *Let*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & 4 \\ 1 & 3 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 2 & 10 \\ 4 & 24 \end{bmatrix} \neq \begin{bmatrix} 12 & 16 \\ 10 & 14 \end{bmatrix} = BA.$$

$\square$

The fragment (written in MATLAB)

```
C = zeros(m,n);
for j=1:n
  for i=1:m
    for k=1:r
      C(i,j) = C(i,j) + A(i,k)*B(k,j);
    end
  end
```

```
end
```

computes the product AB as it was described above and assigns the result to C.[2] However, there are a number of different ways to look at matrix multiplication, and we shall present several distinct versions later.

Notice, that matrix products that involve diagonal factors are especially easy to compute. For example,

$$\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{11}b_{13} & a_{11}b_{14} \\ a_{22}b_{21} & a_{22}b_{22} & a_{22}b_{23} & a_{22}b_{24} \\ a_{33}b_{31} & a_{33}b_{32} & a_{33}b_{33} & a_{33}b_{34} \end{bmatrix}$$

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

:

$$= \begin{bmatrix} a_{11}b_{11} & a_{22}b_{12} & a_{33}b_{13} \\ a_{11}b_{21} & a_{22}b_{22} & a_{33}b_{23} \\ a_{11}b_{31} & a_{22}b_{32} & a_{33}b_{33} \\ a_{11}b_{41} & a_{22}b_{42} & a_{33}b_{43} \end{bmatrix}.$$

In words, *to multiply a matrix B on the left by a diagonal matrix A, one can multiply successive rows of B by the successive diagonal entries of A, and to multiply B on the right by A, one can multiply successive columns of B by the successive diagonal entries of A.*

10.4 Why do it this way

If you were given the task of formulating a definition for composing two matrices A and B in some sort of "natural" multiplicative fashion, your first attempt would probably be to compose A and B by multiplying corresponding entries—much the same way matrix addition is defined. Asked then to defend the usefulness of such a definition, you might be hard pressed to provide a truly satisfying response. Unless a person is in the right frame of mind,

[2] MATLAB supports matrix-matrix multiplication, and so this can be implemented with the one-liner $C = A * B$

the issue of deciding how to best define matrix multiplication is not at all transparent, especially if it is insisted that the definition be both "natural" and "useful." The world had to wait for Arthur Cayley to come to this proper frame of mind. Matrix algebra appeared late in the game. Manipulation on arrays and the theory of determinants existed long before modern theory of matrices (in ancient China for example). Perhaps this can be attributed to the fact that the "correct" way to multiply two matrices eluded discovery for such a long time. Around 1855, Cayley became interested in composing linear functions. Typical examples of two such functions are

$$f(\mathbf{x}) = f\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix}$$

and

$$g(\mathbf{x}) = g\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix}.$$

Consider, as Cayley did, composing f and g to create another linear function

$$h(x) = f\left(g(x)\right) = f\begin{bmatrix} Ax_1 + Bx_2 \\ Cx_1 + Dx_2 \end{bmatrix} = \begin{bmatrix} (aA + bC)x_1 + (aB + bD)x_2 \\ (cA + dC)x_1 + (cB + dD)x_2 \end{bmatrix}.$$

It was Cayley's idea to use matrices of coefficients to represent these linear functions. That is, f, g, and h are represented by

$$F = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad G = \begin{bmatrix} A & B \\ C & D \end{bmatrix}, \quad \text{and} \quad H = \begin{bmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{bmatrix}$$

respectively. After making this association, it was only natural for Cayley to call H the *composition* (or product) of F and G, and to write

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} aA + bC & aB + bD \\ cA + dC & cB + dD \end{bmatrix}. \tag{10.1}$$

In other words, the product of two matrices represents the composition of the two associated linear functions. By means of this observation, Cayley brought to life the subjects of matrix analysis and linear algebra.

10.5 Matrix algebra

Matrices are useful because they provide a compact notation and we can perform algebra with them. For example, given a matrix equation such as

$$3A + 2B = 2(B - A + C), \tag{10.2}$$

we can solve this for the matrix C using the rules of algebra. You must always bear in mind that to perform the operations they must be defined. In this equation, it is understood that all the matrices A, B and C are the same shape, say $m \times n$.

We list the rules of algebra satisfied by the operations of addition, scalar multiplication and matrix multiplication. The shapes of the matrices are dictated by the operations being defined. The first rule is that addition is 'commutative':

$$A + B = B + A. \tag{10.3}$$

This is easily shown to be true. The matrices A and B must be of the same shape, say $m \times n$, for the operation to be defined, so both $A + B$ and $B + A$ are $m \times n$ matrices for some m and n. They also have the same entries. The (i, j) entry of $A + B$ is $a_{ij} + b_{ij}$ and the (i, j) entry of $B + A$ is $b_{ij} + a_{ij}$, but $a_{ij} + b_{ij} = b_{ij} + a_{ij}$ by the properties of real numbers. So the matrices $A + B$ and $B + A$ are equal.

On the other hand, as we have seen, matrix multiplication is not commutative:

$$AB \neq BA$$

in general.

We have the following 'associative' laws:

$$\begin{aligned} (A + B) + C &= A + (B + C), \\ \lambda(AB) &= (\lambda A)B = A(\lambda B), \\ (AB)C &= A(BC). \end{aligned} \tag{10.4}$$

These rules allow us to remove brackets. For example, the last rule says that we will get the same result if we first multiply AB and then multiply by C on the right as we will if we first multiply BC and then multiply by A on the left, so the choice is ours.

We can show that all these rules follow from the definitions of the operations, just as we showed the commutativity of addition. We need to know that the matrices on the left and on the right of the equals sign have the same shape and that corresponding entries are equal. Only the associativity of multiplication presents any complications, but you just need to carefully write down the (i, j) entry of each side and show that,by rearranging terms, they are equal.

Similarly, we have three 'distributive' laws:

$$\begin{aligned} A(B+C) &= AB+AC, \\ (B+C)A &= BA+CA, \\ \lambda(A+B) &= \lambda A+\lambda B. \end{aligned} \tag{10.5}$$

Why do we need both of the first two rules (which state that matrix multiplication distributes through addition)? Well, since matrix multiplication is not commutative,we cannot conclude the second distributive rule from the first; we have to prove it is true separately. These statements can be proved from the definitions of the operations, as above, but we will not take the time to do this here. If A is an $m \times n$ matrix, what is the result of $A - A$? We obtain an $m \times n$ matrix all of whose entries are 0. This is an '*additive identity*'; that is, it plays the same role for matrices as the number 0 does for numbers, in the sense that

$$A + 0 = 0 + A = A.$$

There is a zero matrix of any shape $m \times n$.

Definition 51 (Zero matrix) *A* zero matrix*, denoted* 0*, is an* $m \times n$ *matrix with all entries zero:*

$$\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{m \times n}$$

Then:

- $A + 0 = A$,
- $A - A = 0$,
- $0A = 0, \quad A0 = 0$,

where the shapes of the zero matrices above must be compatible with the shape of the matrix A. We also have a '*multiplicative identity*',which acts like the number 1 does for multiplication of numbers.

Definition 52 (Identity matrix) *The $n \times n$ identity matrix, denoted I_n or simply I, is the diagonal matrix with $a_{ii} = 1$,*

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

If A is any $m \times n$ matrix, then:

$$AI = A \quad \text{and} \quad IA = A, \tag{10.6}$$

where it is understood that the identity matrices are the appropriate shape for the products to be defined. Particularly, if A is $m \times n$ and $IA = A$ then I is $m \times m$.

Example 53 *We can apply these rules to solve the equation 10.2 i.e.*

$$3A + 2B = 2(B - A + C)$$

for C. We will pedantically apply each rule so that you can see how it is being used. In practice, you don't need to put in all these steps, just implicitly use the rules of algebra. We begin by removing the brackets using the distributive rule.

$3A + 2B = 2B - 2A + 2C$	*(distributive rule)*
$3A + 2B - 2B$	*(add $-2B$ to both sides)*
$= 2B - 2A + 2C - 2B$	
	(commutativity, associativity
$3A + (2B - 2B)$	*of addition)*
$= -2A + 2C + (2B - 2B)$	*(additive inverse)*
$3A + 0 = -2A + 2C + 0$	*(additive identity)*
$3A + 2A = -2A + 2C + 2A$	*(add $2A$ to both sides)*
$5A = 2C$	*(commutativity, associativity of addition, additive identity)*
$C = \frac{5}{2}A$	*(scalar multiplication).*

If $AB = AC$, can we conclude that $B = C$? The answer is 'no' (in general), as the following example shows.

Example 54 *If*

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 6 & 4 \\ -5 & -1 \end{bmatrix}$$

then the matrices B and C are not equal, but

$$AB = AC = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}.$$

□

Example 55 *Let*

$$A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$$

Construct a 2×2 matrix B such that AB is the zero matrix. Use two different nonzero columns for B.

Solution: One possible matrix B is like that: $\begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$.

WARNINGS

Properties above are analogous to properties of real numbers. But **NOT ALL** real number properties correspond to matrix properties.

1. It is not the case that AB always equal BA. (see Example 50).
2. Even if $AB = AC$, then B may not equal C. (see Example 56).
3. It is possible for $AB = 0$ even if $A \neq 0$ and $B \neq 0$. (see Example).

10.6 Exercises

Exercise 10.1 (True or False)

a) *The matrix* $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ *has no main diagonal.*

b) *An matrix $m \times n$ has m column vectors and n row vectors.*

c) *If A and B are 2×2 matrices, then $AB = BA$.*

d) *For every matrix A, it is true that $\left(A^T\right)^T = A$.*

e) *If A and B are square matrices of the same order, then $(AB)^T = A^T B^T$.*

f) *If A, B, and C are matrices of the same size such that $A - C = B - C$, then $A = B$.*

g) *If $AB + BA$ is defined, then A and B are square matrices of the same size.*

h) $(ABC)^T = C^T B^T A^T$,

i) *If* A *and* B *are* 3×3 *matrices and* $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix}$, *then* $AB = \begin{bmatrix} A\mathbf{b}_1 + A\mathbf{b}_2 + A\mathbf{b}_3 \end{bmatrix}$

j) *If* A *is an* $n \times n$ *matrix, then* $(A^T)^2 = (A^2)^T$.

Exercise 10.2 *What size is the identity matrix if* A *is* $m \times n$ *matrix and* $IA = A$*?*

Exercise 10.3 *If a matrix* A *is* 5×3 *and the product* AB *is* 5×7 *what is the size of* B?

Exercise 10.4 *Suppose the third column of* B *is all zeros. What can be said about the third column of* AB*?*

Exercise 10.5 *Show that if* AB *and* BA *are both defined, then* AB *and* BA *are square matrices.*

Exercise 10.6 *Suppose the third column of* B *is the sum of the first two columns. What can be said about the third column of* AB*? Why?*

Exercise 10.7 *Given the matrices:*

$$A = \begin{bmatrix} 1 & 3 & 4 \\ -1 & 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 1 \\ 3 & 2 \\ -1 & 4 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

which of the following matrix expression are defined? Compute those which are defined.

a) $A\mathbf{d}$,

b) $AB + C$,

c) $A + C^T$,

d) $C^T C$,

e) BC,

f) $\mathbf{d}^T B$,

g) $C\mathbf{d}$,

h) $\mathbf{d}^T\mathbf{d}$,

i) $\mathbf{d}\mathbf{d}^T$.

Exercise 10.8 *Given the matrices:*

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \\ 0 & 3 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & -1 \\ 4 & 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 \\ 2 & 5 \\ 6 & 3 \end{bmatrix}$$

which of the following matrix expression are defined? Compute those which are defined.

a) $A\mathbf{b}$,

b) CA,

c) $A + C\mathbf{b}$,

d) $A + D$,

e) $\mathbf{b}^T D$,

f) $DA^T + C$,

g) $\mathbf{b}^T\mathbf{b}$,

h) $\mathbf{b}\mathbf{b}^T$,

i) $C\mathbf{b}$.

Exercise 10.9 *Find, if possible, a matrix A and a constant x such that*

$$\begin{bmatrix} 1 & 7 \\ 5 & 0 \\ 9 & 3 \end{bmatrix} A = \begin{bmatrix} -4 & 14 \\ 15 & 0 \\ 24 & x \end{bmatrix}.$$

Exercise 10.10 *In each part, find a matrix $[a_{ij}]$ that satisfies the stated condition. Make your answers as general as possibly using letters rather than specific numbers for the nonzero entries.*

a) $a_{ij} = 0$ *if* $i \neq j$,

b) $a_{ij} = 0$ *if* $i > j$,

c) $a_{ij} = 0$ *if* $i < j$,

d) $a_{ij} = 0$ *if* $|i - j| > 1$,

Exercise 10.11 *If* $A = \begin{bmatrix} 1 & -3 \\ -3 & 5 \end{bmatrix}$ *and* $AB = \begin{bmatrix} -3 & -11 \\ 1 & 17 \end{bmatrix}$, *determine the first and second columns of* B.

11

The Inverse of a Matrix (Exercises)

11.1 Practice Problems

If $AB = AC$, can we conclude that $B = C$? The answer is 'no' (in general), as the following example shows.

Example 56 *If*

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 6 & 4 \\ -5 & -1 \end{pmatrix}$$

then the matrices B and C are not equal, but

$$AB = AC = \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix}.$$

□

On the other hand, if $A + 5B = A + 5C$, then we can conclude that $B = C$ because the operations of addition and scalar multiplication have inverses. If we have a matrix A, then the matrix$-A = (-1)A$ is an additive inverse because it satisfies $A + (-A) = 0$. If we multiply a matrix A by a non-zero scalar c, we can 'undo' this by multiplying cA by $1/c$.

What about matrix multiplication? Is there a multiplicative inverse? The answer is '*sometimes*'.

Definition 57 (Inverse matrix) *The $n \times n$ matrix A is invertible if there is a matrix B such that*

$$AB = BA = I,$$

where I is the $n \times n$ identity matrix. The matrix B is called the inverse of A and is denoted by A^{-1}.

Notice that the matrix A must be square, and that both I and $B = A^{-1}$ must also be square $n \times n$ matrices, for the products to be defined.

Example 58 *Let*

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$$

Then with

$$B = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{4}{5} & -\frac{1}{5} \end{pmatrix}$$

we have $AB = BA = I$, $B = A^{-1}$. □

You might have noticed that we have said that B is *the* inverse of A. This is because an invertible matrix has only one inverse. We will prove this.

Example 59 *Prove that if* A *is an* $n \times n$ *invertible matrix, then the matrix* A^{-1} *is unique.*

Solution: Assume the matrix A has two inverses, B and C, so that $AB = BA = I$ and $AC = CA = I$. We will show that B and C must actually be the same matrix; that is, that they are equal. Consider the product CAB. Since matrix multiplication is associative and $AB = I$, we have

$$CAB = C(AB) = CI = C.$$

On the other hand, again by associativity,

$$CAB = (CA)B = IB = B$$

since $CA = I$. We conclude that $C = B$, so there is only one inverse matrix of A. □

Not all square matrices will have an inverse. We say that A is *invertible* or *non-singular* if it has an inverse. We say that A is *non-invertible* or *singular* if it has no inverse.

Example 60 *Show that the matrix*

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix},$$

is not invertible

Solution: It is not possible for a matrix to satisfy

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ a+c & b+d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

since the $(1,1)$ entry of the product is 0 and $0 \neq 1$. □

On the other hand we have:

Example 61 *If*

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{where} \quad ad - bc \neq 0$$

then A has the inverse

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{11.1}$$

Indeed

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix} &= \begin{pmatrix} a\frac{d}{ad-bc} - b\frac{c}{ad-bc} & 0 \\ 0 & a\frac{d}{ad-bc} - b\frac{c}{ad-bc} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

This tells us how to find the inverse of any 2×2 invertible matrix. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the scalar $ad - bc$ is called the determinant *of the 2×2 matrix A, denoted* $\det(A)$. *It is called determinant, since its value being zero or nonzero determines whether A has an inverse. So if*

$$\det(A) = ad - bc \neq 0,$$

then to construct A^{-1} we take the matrix A, switch the main diagonal entries and put minus signs in front of the other two entries, then multiply by the scalar $1/\det(A)$.

Example 62 *Let*

$$A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}.$$

Then determinant: $\det(A) = -5$ *and*

$$A^{-1} = -\frac{1}{5} \begin{pmatrix} 3 & -2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{3}{5} & \frac{2}{5} \\ \frac{4}{5} & -\frac{1}{5} \end{pmatrix}$$

which is consistent with the above example 58 .

Example 63 *If $AB = AC$, and A is invertible, can we conclude that $B = C$? (Compare with Example 56). This time the answer is 'yes', because we can multiply each side of the equation on the left by A^{-1}:*

$$A^{-1}AB = A^{-1}AC \Longrightarrow IB = IC \Longrightarrow B = C.$$

Warning: Be careful! If $AB = CA$, then we cannot conclude that $B = C$, only that

$$B = A^{-1}CA.$$

It is not possible to 'divide' by a matrix. We can only multiply on the right or left by the inverse matrix.

Example 64 *The matrix*

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 5 & 0 \\ 3 & 6 & 0 \end{bmatrix}$$

is singular. To see why, let

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

be any matrix. The third column of AB is

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus

$$BA \neq I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

□

11.1.1 Properties of the inverse

If A is an invertible matrix, then, by definition, A^{-1} exists and $AA^{-1} = A^{-1}A = I$. This statement also says that the matrix A is the inverse of A^{-1}; that is,

$$(A^{-1})^{-1} = A. \tag{11.2}$$

It is important to understand the definition of an inverse matrix and be able to use it. Essentially, if we can find a matrix that satisfies the definition, then that matrix is the inverse, and the matrix is invertible. For example, if A is an invertible$n \times n$ matrix, then:

$$(\lambda A) - 1 = \frac{1}{\lambda} A^{-1} \qquad \text{provided} \quad \lambda \neq 0. \tag{11.3}$$

This statement says that the matrix λA is invertible, and its inverse is given by the matrix $C = (1/\lambda)A^{-1}$. To prove this is true, we just need to show that the matrix C satisfies $(\lambda A)C = C(\lambda A) = I$. This is straightforward using matrix algebra:

$$(\lambda A)\left(\frac{1}{\lambda}A^{-1}\right) = \lambda \frac{1}{\lambda} AA^{-1} = I$$

and

$$\left(\frac{1}{\lambda}A^{-1}\right)(\lambda A) = \lambda \frac{1}{\lambda} AA^{-1} = I.$$

If A and B are invertible $n \times n$ matrices, then using the definition of the inverse you can show the following important fact:

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{11.4}$$

This last statement says that if A and B are invertible matrices of the same shape $n \times n$, then the product AB is invertible and its inverse is the product of the inverses in the *reverse order*. The proof of this statement is quite easy. By definition, to prove that the matrix AB is invertible you have to show that there exists a matrix, C, such that

$$(AB)C = C(AB) = I.$$

You are given that $C = B^{-1}A^{-1}$. Since both A and B are invertible matrices, you know that both A^{-1} and B^{-1} exist and both are $n \times n$, so the matrix product $B^{-1}A^{-1}$ is defined. So all you need to do is to show that if you multiply AB on the left or on the right by the matrix $B^{-1}A^{-1}$, then you will obtain the identity matrix, I.

$$\begin{aligned}
(AB)\left(B^{-1}A^{-1}\right) &= A(BB^{-1})A^{-1} && \text{(matrix multiplication is associative)} \\
&= AIA^{-1} && \text{(by the definition of } B^{-1}\text{)} \\
&= AA^{-1} && \text{(since } AI = A \text{ for any matrix } A\text{)} \\
&= I && \text{(by the definition of } A^{-1}\text{).}
\end{aligned}$$

In the same way,

$$(B^{-1}A^{-1})(AB) = (B^{-1})(A^{-1}A)(B) = B^{-1}IB = B^{-1}B = I.$$

Hence $B^{-1}A^{-1}$ is the inverse of the matrix AB.

Notice, that there is no formula to relate $(A + B)^{-1}$ to the sum of inverses.

Example 65

11.1.2 Inverses and Powers of Diagonal Matrices

Let A be a diagonal matrix

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix} \tag{11.5}$$

Such a diagonal matrix is invertible if and only if all of its diagonal entries are nonzero; in this case the inverse of (11.5) is

$$A^{-1} = \begin{pmatrix} \frac{1}{a_{11}} & 0 & \cdots & 0 \\ 0 & \frac{1}{a_{22}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{a_{nn}} \end{pmatrix}$$

The reader should verify that indeed

$$AA^{-1} = A^{-1}A = I.$$

11.1.3 An Algorithm for finding A^{-1}

Example 66 *Find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}$$

So:

$$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}.$$

□

Example 67 *Find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & -2 & -1 \\ -1 & 5 & 6 \\ 5 & -4 & 5 \end{bmatrix}.$$

Solution:

$$\begin{aligned} \begin{bmatrix} A & I \end{bmatrix} &= \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix} \\ &\sim \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix} \end{aligned}$$

So $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to a matrix of the form $\begin{bmatrix} B & D \end{bmatrix}$, where B is square and has a row of zeros. Further row operations will not transform B into I, so we stop. A does not have an inverse. □

11.2 Exercises

Exercise 11.1 (True or False)

a) *In order for a matrix B to be the inverse of A, the equations $AB = I$ and $BA = I$ must both be true.*

b) *If A and B are $n \times n$ and invertible, then $A^{-1}B^{-1}$ is the inverse of AB.*

c) *If A is an invertible n n matrix, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for each $\mathbf{b}$ in $\mathbb{R}^n$.*

d) *If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ab - cd \neq 0$, then A is invertible.*

e) *If A is invertible, then the inverse of A^{-1} is A itself.*

f) *Each elementary matrix is invertible.*

g) *If A can be row reduced to the identity matrix, then A must be invertible.*

Exercise 11.2 *Compute the inverses of the following matrices.*

a) $\begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$,

b $\begin{bmatrix} 2 & -3 \\ 4 & 4 \end{bmatrix}$,

c) $\begin{bmatrix} 6 & 4 \\ -2 & -1 \end{bmatrix}$,

d) $\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$.

Exercise 11.3 *Use the matrices A and B in Exercise 71 to verify that*

a) $\left(A^{-1}\right)^{-1} = A$,

b) $\left(B^T\right)^{-1} = \left(B^{-1}\right)^T$.

Exercise 11.4 *Use the matrices A, B, and C in Exercise 71 to verify that*

a) $(AB)^{-1} = B^{-1}A^{-1}$,

b) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

Exercise 11.5 *Find the inverse of*

$$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Exercise 11.6 *Let A be the matrix*

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Determine whether A is invertible, and if so, find its inverse. `HINT`*: Solve $AX = I$ by equating corresponding entries on the two sides.*

Exercise 11.7 *Find 2×2 matrices A and B such that $(A+B)^{-1} \neq A^{-1}+B^{-1}$.*

Exercise 11.8 *Use the algorithm from this section to find the inverses of*

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exercise 11.9 *Repeat the strategy of Exercise 11.8 to guess the inverse B of*

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & & \vdots \\ 3 & 3 & 3 & & \\ \vdots & & & \ddots & \vdots \\ n & n & n & n & n \end{bmatrix}.$$

Show that $AB = I$.

Exercise 11.10 *Let*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$$

Find the third column of A^{-1} without computing the other columns.

12

Characterizations of Invertible Matrices (Exercises)

12.1 Practice Problems

Example 68 *Determine whether*

$$B = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} \qquad or \qquad C = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix}$$

are inverses for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution: B is an inverse if and only if $AB = BA = I$; C is an inverse if and only if $AC = CA = I$. Here,

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} & 1 \\ \frac{13}{3} & \frac{5}{2} \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

while

$$AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & \frac{-1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = CA$$

Thus, B is not an inverse for A, but C is.We may write $A^{-1} = C$. □

Example 69 *Use determinants to determine which of the following matrices are invertible.*

$$a) \begin{bmatrix} 4 & -8 \\ 2 & 4 \end{bmatrix} \qquad b) \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \qquad c) \begin{bmatrix} 4 & -2 \\ -8 & 4 \end{bmatrix},$$

Solution:

a) $\det \begin{bmatrix} 4 & -8 \\ 2 & 4 \end{bmatrix} = 32$. The determinant is nonzero, so The determinant is nonzero, so

b) $\det \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = 6$. The matrix is invertible.

c) $\det \begin{bmatrix} 4 & -2 \\ -8 & 4 \end{bmatrix} = 0$. The matrix is not invertible. □

Example 70 *Find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix},$$

if it exists.

Solution:

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 2 & -2 & 0 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{4} & \frac{3}{4} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

So

$$A^{-1} = \begin{bmatrix} -\frac{3}{4} & \frac{3}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

□

Example 71 *Find the inverses of*

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Let A be the corresponding $n \times n$ matrix, and let B be its inverse. Guess the form of B and then show that $AB = I$.

Solution: Let

$$B = \begin{bmatrix} 1 & 0 & 0 & & \dots & 0 \\ -1 & 1 & 0 & & \dots & 0 \\ 0 & -1 & 1 & & & 0 \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ 0 & 0 & \dots & & -1 & 1 \end{bmatrix},$$

and for $j = 1, ..., n$, let $\mathbf{a}_j$, $\mathbf{b}_j$, and $\mathbf{e}_j$, denote the j-th columns of A, B, and I, respectively. Note that for $j = 1, ..., n-1$, $\mathbf{a}_j - \mathbf{a}_{j+1} = \mathbf{e}_j$, (because $\mathbf{a}_j$ and $\mathbf{a}_{j+1}$ have the same entries except for the j-th row), $\mathbf{b}_j = \mathbf{e}_j - \mathbf{e}_{j+1}$ and $\mathbf{a}_n = \mathbf{b}_n = \mathbf{e}_n$.

12.2 Exercises

Exercise 12.1 *Find the inverses of the following matrices:*

a) $\begin{bmatrix} 1 & -4 \\ 1 & 2 \end{bmatrix}$,

b) $\begin{bmatrix} 1 & -4 \\ 0 & 2 \end{bmatrix}$,

c) $\begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix}$,

d) $\begin{bmatrix} 1 & -4 \\ 1 & -2 \end{bmatrix}$.

Exercise 12.2 *For which values of the constants a and b is the matrix*

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

invertible? What is the inverse in this case?

Exercise 12.3 *Show that the inverse for a diagonal matrix D having only nonzero elements on its main diagonal is also a diagonal matrix whose diagonal elements are the reciprocals of the corresponding diagonal elements of D. That is, if*

$$D = \begin{bmatrix} \lambda_1 & & & \mathbf{0} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \lambda_n \end{bmatrix}$$

then

$$D^{-1} = \begin{bmatrix} 1/\lambda_1 & & & \mathbf{0} \\ & 1/\lambda_2 & & \\ & & \ddots & \\ \mathbf{0} & & & 1/\lambda_n \end{bmatrix}$$

Exercise 12.4 *When you represent a three-dimensional object graphically in the plane (on paper, the blackboard, or a computer screen), you have to transform spatial coordinates,*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

into plane coordinates, $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. *The simplest choice is a linear transformation, for example, the one given by the matrix*

$$A = \begin{bmatrix} -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{bmatrix}.$$

a) *Use this transformation to represent the unit cube with comer points*

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Include the images of the x_1 , x_2, *and* x_3 *axes in your sketch.*

b) *Represent the image of the point* $\begin{bmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ *in your figure in part (a). Explain.*

c) *Find all the points* $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ *that are transformed to* $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. *Explain.*

Exercise 12.5 *Repeat the strategy of Example 71 to guess the inverse* B *of*

$$A = \begin{bmatrix} 1 & 0 & 0 & & \cdots & 0 \\ 2 & 2 & 0 & & \cdots & 0 \\ 3 & 3 & 3 & & & 0 \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ n & n & n & & n & n \end{bmatrix}.$$

Show that $AB = I$.

Exercise 12.6 *Prove that the inverse of the lower (upper) triangular matrix (if exists) is lower (upper) triangular.*

Exercise 12.7 *Suppose* A *and* B *are* $n \times n$ *matrices,* B *is invertible, and* AB *is invertible. Show that* A *is invertible. [HINT: Let* $C = AB$*, and solve this equation for* A*.]*

Exercise 12.8 *Suppose* A *is an* $n \times n$ *matrix with the property that the equation* $A\mathbf{x} = \mathbf{b}$ *has at least one solution for each* $\mathbf{b}$ *in* $\mathbb{R}^n$. *Explain why each equation* $A\mathbf{x} = \mathbf{b}$ *has in fact exactly one solution.*

Exercise 12.9 *The color of light can be represented in a vector*

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

where $R =$ *amount of red,* $G =$ *amount of green, and* $B =$ *amount of blue. The human eye and the brain transform the incoming signal into the signal*

$$\begin{bmatrix} I \\ L \\ S \end{bmatrix}$$

where

intensity	$I = \dfrac{R+G+B}{3}$
long-wave signal	$L = R - G$
short-wave signal	$S = B - \dfrac{R+G}{2}$

Example 72

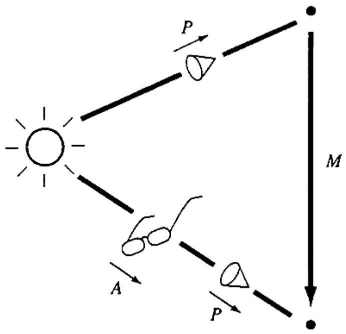
a) *Find the matrix* P *representing the transformation from*

$$\begin{bmatrix} R \\ G \\ B \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} I \\ L \\ S \end{bmatrix}.$$

b) *Consider a pair of yellow sunglasses for water sports that cuts out all blue light and passes all red and green light. Find the* 3×3 *matrix* A *that represents the transformation incoming light undergoes as it passes through the sunglasses.*

c) *Find the matrix for the composite transformation that light undergoes as it first passes through the sunglasses and then the eye.*

d) *As you put on the sunglasses, the signal you receive (intensity, long- and short-wave signals) undergoes a transformation. Find the matrix* M *of this transformation.*



Example 73 *Matrix inversion can be used to encode and decode sensitive messages for transmission. Initially, each letter in the alphabet is assigned a unique positive integer, with the simplest correspondence being*

$$\begin{array}{cccccccccccccc} A & B & C & D & E & F & G & H & I & J & K & L & M & N \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \end{array}$$

$$\begin{array}{cccccccccccc} O & P & Q & R & S & T & U & V & W & X & Y & Z \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 \end{array}$$

Zeros are used to separate words. Thus, the message

SHE IS A SEER

is encoded

$$19 \quad 8 \quad 5 \quad 0 \quad 9 \quad 19 \quad 0 \quad 1 \quad 0 \quad 19 \quad 5 \quad 5 \quad 18 \quad 0.$$

This scheme is too easy to decipher, however, so a scrambling effect is added prior to transmission. One scheme is to package the coded string as a set of 2-tuples, multiply each 2-tuple by a 2×2 invertible matrix, and then transmit the new string. For example, using the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

the coded message above would be scrambled into

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 19 \\ 9 \end{bmatrix} = \begin{bmatrix} 37 \\ 65 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 9 \\ 19 \end{bmatrix} = \begin{bmatrix} 47 \\ 75 \end{bmatrix}$$

etc. and the scrambled message becomes

$$35 \quad 62 \quad 5 \quad 10 \quad 47 \quad 75....$$

Note an immediate benefit from the scrambling: the letter S*, which was originally always coded as* 19 *in each of its three occurrences, is now coded as a* 35 *the first time and as* 75 *the second time. Continue with the scrambling, and determine the final code for transmitting the above message.*

13

Introduction to Determinants (Exercises)

13.1 Practice Problems

Determinant and volume

If A is a $n \times n$ matrix, then $\det(A)$ is the volume of the n-dimensional parallelepiped E_n spanned by the n column vectors $\mathbf{a}_j$ of A (see Fig.13.1).

Orientation

Determinants allow to define the orientation of n vectors in n-dimensional space. This is "handy" because there is no "right hand rule" in hyperspace... To do so, define the matrix A with column vectors $\mathbf{a}_j$ and define the orientation as the sign of $\det(A)$. In three dimensions, this agrees with the right hand rule: if $\mathbf{a}_1$ is the thumb, $\mathbf{a}_2$ is the pointing finger and $\mathbf{a}_3$ is the middle finger, then their orientation is positive.

Why do we care about determinants?

- check invertibility of matrices,
- have geometric interpretation as volume,
- explicit algebraic expressions for inverting a matrix,
- as a natural functional on matrices it appears in formulas in particle or statistical physics,
- allow to define orientation in any dimensions,
- appear in change of variable formulas in higher dimensional integration,
- proposed alternative concepts are unnatural, hard to teach and harder to understand,
- determinants are fun.

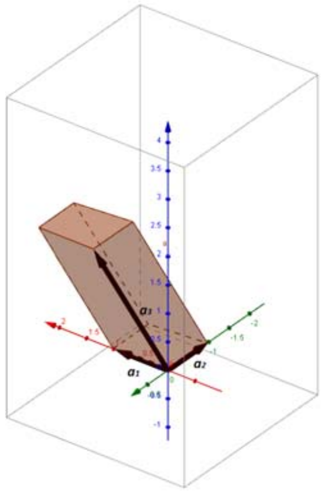


Fig. 13.1. The n-dimensional parallelepiped E_n spanned by the n column vectors $\mathbf{a}_j$ of A.

The art of calculating determinants.
When confronted with a matrix, it is good to go through a checklist of methods to crack the determinant. Often, there are different possibilities to solve the problem, in many cases the solution is particularly simple using one method.

- Is it a 2×2 or 3×3 matrix?
- Is it a upper or lower triangular matrix?
- Is it a trick: like A^{1000}?
- Does geometry imply noninvertibility?
- Do you see duplicated columns or rows?
- Can you row reduce to a triangular case?
- Are there only a few nonzero patters?
- Laplace expansion with some row or column?
- Later: Can you see the eigenvalues of A?

Example 74 *Use a determinant to decide if* $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ *are linearly independent, when*

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ -7 \\ 9 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -3 \\ 3 \\ -5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ -7 \\ 5 \end{bmatrix}.$$

Solution:

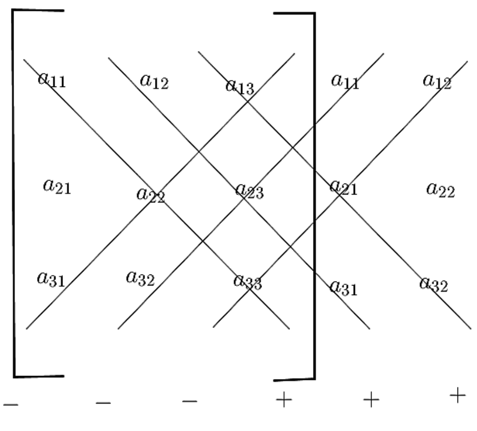
$$\det \begin{bmatrix} 5 & -3 & 2 \\ -7 & 3 & -7 \\ 9 & -5 & 5 \end{bmatrix} = \det \begin{bmatrix} 5 & -3 & 2 \\ -2 & 0 & -5 \\ 9 & -5 & 5 \end{bmatrix}$$

(row 1 added to row 2).Next, taking cofactors of column 2 we have

$$\begin{aligned} &= \det \begin{bmatrix} 5 & -3 & 2 \\ -2 & 0 & -5 \\ 9 & -5 & 5 \end{bmatrix} \\ &= -(-3)\det \begin{bmatrix} -2 & -5 \\ 9 & 5 \end{bmatrix} - (-5)\det \begin{bmatrix} 5 & 2 \\ -2 & -5 \end{bmatrix} = 0 \end{aligned}$$

So, the matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]$ is not invertible. The columns are linearly dependent. □

Here is a memory aid (called Sarrus's Rule) for the determinant of a 3×3 matrix: To find the determinant of a 3×3 matrix A, write the first two columns of A to the right of A. Then multiply the entries along the six diagonals shown below.



Add or subtract these diagonal products, as shown in the diagram:

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}.$$

Example 75 (Sarrus's Rule) *Find the determinant of*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

Solution: By Sarrus's rule, $\det(A) = 1 \cdot 5 \cdot 10 + 2 \cdot 6 \cdot 7 + 4 \cdot 8 \cdot 3 - 3 \cdot 5 \cdot 7 - 2 \cdot 4 \cdot 10 - 6 \cdot 8 \cdot 1 = -3$. Matrix A is invertible.

Example 76 *Compute* $\det A$*, where*

$$A = \begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 & 10 \end{bmatrix}.$$

Solution: The cofactor expansion down the first column of A has all terms equal to zero except the first. Thus

$$\det(A) = 3 \cdot \det\left(\begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 4 & 5 & 6 \\ 0 & 7 & 8 & 10 \end{bmatrix}\right)$$

Next, expand this 4×4 determinant down the first column, in order to take advantage of the zeros there. We have

$$\det(A) = 3 \cdot 2 \cdot \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}\right).$$

This 3×3 determinant was computed in Example 75 and found to equal -3. Hence

$$\det(A) = 3 \cdot 2 \cdot (-3) = -18$$

□

Example 77 *Using the Inverse Formula find* A^{-1} *when*

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}.$$

Solution:

$$C_{11} = +\det \begin{bmatrix} -1 & 1 \\ 4 & -2 \end{bmatrix} = -2 \quad C_{12} = -\det \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = 3 \quad C_{13} = +\det \begin{bmatrix} 1 & -1 \\ 1 & 4 \end{bmatrix} = 5$$
$$C_{21} = -\det \begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix} = 14 \quad C_{22} = +\det \begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} = -7 \quad C_{23} = -\det \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = -7$$
$$C_{31} = +\det \begin{bmatrix} 1 & 3 \\ -1 & 1 \end{bmatrix} = 4 \quad C_{32} = -\det \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} = 1 \quad C_{33} = +\det \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = -3$$

The adjugate matrix is the transpose of the matrix of cofactors. [For instance, C_{12} goes in the$(2,1)$ position.] Thus

$$\operatorname{adj} A = \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix}.$$

We could compute $\det(A)$ directly, but the following computation provides a check on the calculations above and produces $\det(A)$

$$\begin{aligned} (\operatorname{adj} A) \cdot A &= \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix} = 14I \end{aligned}$$

Since $(\operatorname{adj} A) \cdot A = 14I$, we conclude

$$A^{-1} = \frac{1}{14} \begin{bmatrix} -2 & 14 & 4 \\ 3 & -7 & 1 \\ 5 & -7 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{7} & 1 & \frac{2}{7} \\ \frac{3}{14} & -\frac{1}{2} & \frac{1}{14} \\ \frac{5}{14} & -\frac{1}{2} & -\frac{3}{14} \end{bmatrix}.$$

□

13.2 Application to Engineering

A number of important engineering problems, particularly in electrical engineering and control theory, can be analyzed by *Laplace transforms.* This approach converts an appropriate system of linear differential equations into a system of linear algebraic equations whose coefficients involve a parameter. The next example illustrates the type of algebraic system that may arise.

Example 78 *Consider the following system in which s is an unspecified parameter. Determine the values of s for which the system has a unique solution, and use Cramer's rule to describe the solution.*

$$\begin{array}{rcrcl} 3sx_1 & - & 2x_2 & = & 4 \\ -6x_1 & + & sx_2 & = & 1 \end{array}$$

Solution 79 *View the system as $A\mathbf{x} = \mathbf{b}$. Then*

$$A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}, \quad A_1(\mathbf{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}, \quad A_2(\mathbf{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix}.$$

Since

$$\det(A) = 3s^2 - 12 = 3(s+2)(s-2)$$

the system has a unique solution precisely when $s \neq \pm 2$. For such an s, the solution is

$$x_1 = \frac{\det(A_1(\mathbf{b}))}{\det(A)} = \frac{4s+2}{3(s+2)(s-2)}$$

$$x_2 = \frac{\det(A_2(\mathbf{b}))}{\det(A)} = \frac{3s+24}{3(s+2)(s-2)} = \frac{s+8}{(s+2)(s-2)}.$$

□

13.3 Exercises

Exercise 13.1 *Find* $\det(A)$ *when:*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 6 & 8 & 10 \\ 5 & 5 & 5 & 5 & 4 \\ 1 & 3 & 2 & 7 & 4 \\ 3 & 2 & 8 & 4 & 9 \end{bmatrix}$$

`HINT:` *Try row reduction.*

Exercise 13.2 *Find* $\det(A)$ *when:*

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

`HINT:` *Laplace expansion.*

Exercise 13.3 *Find* $\det(A)$ *when:*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 & 2 \end{bmatrix}$$

`HINT`: *Partitioned matrix.*

Exercise 13.4 *Find* $\det(A)$ *when:*

$$A = \begin{bmatrix} 1 & 6 & 10 & 1 & 15 \\ 2 & 8 & 17 & 1 & 29 \\ 0 & 0 & 3 & 8 & 12 \\ 0 & 0 & 0 & 4 & 9 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

`HINT`: *Make it tridiagonal.*

Exercise 13.5 *Test Cramer's rule for a random* 4×4 *matrix* A *and a random* 4×1 *vector* $\mathbf{b}$. *Compute each entry in the solution of* $A\mathbf{x} = b$, *and compare these entries with the entries in* $A^{-1}\mathbf{b}$.

Exercise 13.6 *Use Cramer's rule to compute the solution of the following system:*

$$\begin{cases} 2x_1 + x_2 = 7 \\ -3x_1 + x_3 = -8 \\ x_2 + 2x_3 = -3 \end{cases}.$$

Exercise 13.7 *Use Cramer's rule to compute the solution of the following system:*

$$\begin{cases} 2x_1 + x_2 + x_3 = 4 \\ -x_1 + 2x_3 = 2 \\ 3x_1 + x_2 + 3x_3 = -2 \end{cases}.$$

Exercise 13.8 *Find the volume of the parallelepiped with one vertex at the origin and adjacent vertices at* $(1, 0, -2)$, $(1, 2, 4)$, *and* $(.7; 1; 0)$.

Exercise 13.9 *Let* $x_1, ..., x_n$ *be fixed but mutually distinct numbers. The matrix below, called a Vandermonde matrix, occurs in applications such as signal* *processing, error-correcting codes, and polynomial interpolation.*

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}.$$

Find $\det(V)$.

14

Eigenvectors and Eigenvalues (Exercises)

14.1 Practice Problems

Definition 80 *An* ***eigenvector*** *of an $n \times n$ matrix A is a nonzero vector x such that $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar (real or complex) λ. A scalar λ is called an* ***eigenvalue*** *of A if there is a nontrivial solution $\mathbf{x}$ of $A\mathbf{x} = \lambda\mathbf{x}$; such an $\mathbf{x}$ is called an* ***eigenvector corresponding to*** *λ.*

Note that an eigenvector must be nonzero, by definition, but an eigenvalue may be zero.

It is easy to determine if a given vector is an eigenvector of a matrix. It is also easy to decide if a specified scalar is an eigenvalue.

Example 81 *Let* $A = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, *and* $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. *Are* $\mathbf{u}$ *and* $\mathbf{v}$ *eigenvectors of A?*

Solution:

$$A\mathbf{u} = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} = 5 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 5\mathbf{u}.$$

$$A\mathbf{v} = \begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Thus $\mathbf{u}$ is an eigenvector corresponding to an eigenvalue (5), but $\mathbf{v}$ is not an eigenvector of A, because $A\mathbf{v}$ is not a multiple of $\mathbf{v}$. $\square$

Example 82 *Show that 1 is an eigenvalue of matrix A in Example 81, and find the corresponding eigenvectors.*

Solution: The scalar 1 is an eigenvalue of A if and only if the equation

$$A\mathbf{x} = 1 \cdot \mathbf{x} \tag{14.1}$$

has a nontrivial solution. But (14.1) is equivalent to $A\mathbf{x} - \mathbf{x} = \mathbf{0}$, or

$$(A - I)\,\mathbf{x} = \mathbf{0}. \tag{14.2}$$

To solve this homogeneous equation, form the matrix

$$(A-I)=\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}.$$

The columns of $A-I$ are obviously linearly dependent, so (14.2) has nontrivial solutions. Thus 1 is an eigenvalue of A. To find the corresponding eigenvectors, use row operations:

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The general solution has the form $x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Each vector of this form with $x_2 \neq 0$ is an eigenvector corresponding to $\lambda = 1$. □

Example 83 *Let*

$$A=\begin{bmatrix} 1 & -4 & -4 \\ 8 & -11 & -8 \\ -8 & 8 & 5 \end{bmatrix}$$

An eigenvalue of A is -3. Find a basis for the corresponding eigenspace.

Solution: Form

$$A+3I=\begin{bmatrix} 4 & -4 & -4 \\ 8 & -8 & -8 \\ -8 & 8 & 8 \end{bmatrix}$$

and row reduce the augmented matrix for $(A+3I)\,\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix} 4 & -4 & -4 & 0 \\ 8 & -8 & -8 & 0 \\ -8 & 8 & 8 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

At this point, it is clear that -3 is indeed an eigenvalue of A because the equation $(A+3I)\,\mathbf{x}=\mathbf{0}$ has free variables. The general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=x_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad x_2 \text{ and } x_3 \text{ are free.}$$

The eigenspace is a two-dimensional subspace of $\mathbb{R}^3$. A basis is

$$\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

□

Theorem 84 *The eigenvalues of a triangular matrix are the entries on its main diagonal.*

Example 85 *Let*

$$A = \begin{bmatrix} -9 & 0 & 0 \\ -8 & -6 & 0 \\ 20 & 15 & 9 \end{bmatrix} \quad \textit{and} \quad B = \begin{bmatrix} 0 & -3 & -1 \\ 0 & 8 & 5 \\ 0 & 0 & 8 \end{bmatrix}.$$

Solution: The eigenvalues of A are $-9, -6$ and 9. The eigenvalues of B are 0 and 8. □

Example 86 *If* $\mathbf{x}$ *is an eigenvector of* A *corresponding to* λ, *what is* $A^2\mathbf{x}$?

Solution: If $\mathbf{x}$ is an eigenvector of A corresponding to , then $A\mathbf{x} = \boldsymbol{\lambda}\mathbf{x}$ and so

$$A^2\mathbf{x} = \lambda\mathbf{Ax} = A\mathbf{x} = \lambda^2\mathbf{x}.$$

The general pattern, $A^k\mathbf{x} = \lambda^k\mathbf{x}$ is proved by induction. □

14.2 Exercises

Exercise 14.1 (True or False) *Here A is an $n \times n$ matrix. Mark each statement True or False. Justify each answer*

a) If λ is an eigenvalue of A, and corresponding eigenvector is $\mathbf{x}$ then every nonzero scalar multiple of $\mathbf{x}$ is also an eigenvector of A.

b) Matrix A can have only one eigenvalue.

c) If λ is an eigenvalue of A, then the set of all eigenvectors of λ is a subspace of $\mathbb{R}^n$.

d) If λ is not an eigenvalue of A, then the linear system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ has only the trivial solution.

e) If $\lambda = 0$ is an eigenvalue of A, then A^2 is singular.

f) To find the eigenvalues of A, reduce A to echelon form.

g) The eigenvalues of a matrix are on its main diagonal.

Exercise 14.2 *Find the dimension of the eigenspace corresponding to eigenvalue $\lambda = 3$ when*

a) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$,

b) $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$,

c) $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$,

d) $A = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

Exercise 14.3 *Is* $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$ *an eigenvector of* $A = \begin{bmatrix} -4 & 3 & 3 \\ 2 & -3 & -2 \\ -1 & 0 & -2 \end{bmatrix}$? *If so, find the eigenvalue.*

Exercise 14.4 *Determine which of the following are eigenvectors for*

$$A = \begin{bmatrix} -9 & -6 & -2 & -4 \\ -8 & -6 & -3 & -1 \\ 20 & 15 & 8 & 5 \\ 32 & 21 & 7 & 12 \end{bmatrix}$$

and for those which are eigenvectors, identify the associated eigenvalue.

$$a) \quad \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad b) \quad \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad c) \quad \begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \quad d) \quad \begin{bmatrix} 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}.$$

Exercise 14.5 *Explain why a 2×2 matrix can have at most two distincteigenvalues. Explain why an $n \times n$ matrix can have at most n distinct eigenvalues.*

15
The Characteristic Equation (Exercises)

15.1 Practice Problems

To find the eigenvalues,we solve the characteristic equation $\det(A - \lambda I) = 0$. Let us illustrate with a 2×2 example

Example 87 *Let*

$$A = \begin{bmatrix} 7 & -15 \\ 2 & -4 \end{bmatrix}.$$

Then

$$A - \lambda I = \begin{bmatrix} 7 & -15 \\ 2 & -4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7-\lambda & -15 \\ 2 & -4-\lambda \end{bmatrix}$$

and the characteristic polynomial is

$$(7-\lambda)(-4-\lambda) + 30 = \lambda^2 - 3\lambda + 2.$$

So the eigenvalues are the solutions of $\lambda^2 - 3\lambda + 2 = 0$. To solve this for λ, we could use either the formula for the solutions to a quadratic equation, or simply observe that the characteristic polynomial factorises. We have$(\lambda-1)(\lambda-2) = 0$ with solutions $\lambda = 1$ and $\lambda = 2$. Hence the eigenvalues of A are 1 and 2, and these are the only eigenvalues of A.

Example 88 *Find the eigenvalues of*

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix}.$$

Solution: The characteristic polynomial of A is

$$\det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 4 & -17 & 8-\lambda \end{bmatrix} = -\lambda^3 + 8\lambda^2 - 17\lambda + 4$$

The eigenvalues of A must therefore satisfy the cubic equation

$$\lambda^3 - 8\lambda^2 + 17\lambda - 4 = 0. \tag{15.1}$$

To solve this equation, we shall begin by searching for integer solutions. This task can be greatly simplified by exploiting the fact that all integer solutions (if there are any) to a polynomial equation with integer coefficients

$$\lambda^n + c_1\lambda^{n-1} + \cdots + c_n = 0$$

must be divisors of the constant term, c_n. Thus, the only possible integer solutions of 2 are the divisors of -4, that is, $\pm1, \pm2, \pm4$. Successively substituting these values in 15.1 shows that $\lambda = 4$ is an integer solution. As a consequence, $\lambda - 4$ must be a factor of the left side of 15.1. Dividing $\lambda - 4$ into $\lambda^3 - 8\lambda^2 + 17\lambda - 4$ shows that 15.1 can be rewritten as

$$(\lambda - 4)\left(\lambda^2 - 4\lambda + 1\right) = 0$$

Thus the remaining solutions of 15.1 satisfy the quadratic equation

$$\lambda^2 - 4\lambda + 1 = 0$$

which can be solved by the quadratic formula. Thus the eigenvalues of A are

$$\lambda_1 = 4, \quad \lambda_2 = \sqrt{3} + 2, \quad \lambda_3 = 2 - \sqrt{3}.$$

□

Remark 89 *In practical problems, the matrix A is usually so large that computing the characteristic equation is not practical. As a result, other methods are used to obtain eigenvalues.*

Example 90 (Complex eigenvalues) *It is possible for the characteristic equation of a matrix with real entries to have complex solutions. In fact, because the eigenvalues of an $n\times n$ matrix are the roots of a polynomial of precise degree n, every matrix has exactly n eigenvalues if we count them as we count the roots of a polynomial (meaning that they may be repeated, and may occur in complex conjugate pairs). For example, the characteristic polynomial of the matrix*

$$A = \begin{bmatrix} 2 & 1 \\ -5 & -2 \end{bmatrix}$$

is

$$\det \begin{bmatrix} 2-\lambda & 1 \\ -5 & -2-\lambda \end{bmatrix} = \lambda^2 + 1$$

so the characteristic equation is

$$\lambda^2 + 1 = 0,$$

the solutions of which are the imaginary numbers and $\lambda_1 = -i, \quad \lambda_2 = i$. *Thus we are forced to consider complex eigenvalues, even for real matrices. This, in turn, leads us to consider the possibility of complex vector spaces—that is, vector spaces in which scalars are allowed to have complex values. Such vector spaces will be considered in more advanced courese of linear algebra. For now, we will allow complex eigenvalues, but we will limit our discussion of eigenvectors to the case of real eigenvalues.* □

Example 91 *We know that square matrix* A *is invertible if and only if* $\lambda = 0$ *is not an eigenvalue of* A. *The matrix* A *in Example 88 is invertible since it has eigenvalues*

$$\lambda_1 = 4, \quad \lambda_2 = \sqrt{3} + 2, \quad \lambda_3 = 2 - \sqrt{3}$$

neither of which is zero. □

Example 92 *Find the characteristic equation of*

$$A = \begin{bmatrix} 3 & -2 & 6 & 1 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

Solution: Form $A - \lambda I$, and use of tiangularity of A we have:

$$\begin{aligned} \det(A - \lambda I) &= \det \begin{bmatrix} 3-\lambda & -2 & 6 & 1 \\ 0 & 4-\lambda & 3 & 2 \\ 0 & 0 & 1-\lambda & -1 \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix} \\ &= (3-\lambda)(1-\lambda)(4-\lambda)^2. \end{aligned}$$

The characteristic equation is

$$(3-\lambda)(1-\lambda)(4-\lambda)^2 = 0.$$

The matrix A has eigenvalues

$$\lambda_1 = 3, \quad \lambda_2 = 3, \quad \lambda_3 = 4, \quad \lambda_4 = 4.$$

15.2 Exercises

Exercise 15.1 *Suppose that the characteristic polynomial of some matrix* A *is found to be*

$$(\lambda - 1)(\lambda - 2)^2 (\lambda - 3)^3,$$

In each part, answer the question and explain your reasoning.

a) *What is the size of A?*

b) *Is A invertible?*

c) *How many eigenspaces does A have?*

Exercise 15.2 *The eigenvectors for $\lambda = 0$ are the non-zero solutions of $Ax = 0$. To find these, row reduce the coefficient matrix*

$$A = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 4 & 4 \\ 4 & 4 & 8 \end{bmatrix}.$$

Similarly, you should find the eigenvectors for $\lambda = 12$.

Exercise 15.3 *We know that if λ is an eigenvalue of A, then λ^k is an eigenvalue of A^k. Give a 2×2 matrix A and an integer k that provide a counterexample to the Give a 2×2 matrix A and an integer k that provide a counterexample to the converse.*

Exercise 15.4 *Find the eigenvalues of*

$$A = \begin{bmatrix} 100 & 1 & 1 & 1 \\ 1 & 100 & 1 & 1 \\ 1 & 1 & 100 & 1 \\ 1 & 1 & 1 & 100 \end{bmatrix}.$$

Exercise 15.5 *Find the eigenvalues of A^{10} for*

$$A = \begin{bmatrix} \frac{1}{2} & 7 & 1 & 2 \\ 0 & 4 & -1 & -1 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix}.$$

Exercise 15.6 *Prove that the coefficient of λ^n in the characteristic polynomial of an $n \times n$ matrix A is $(-1)^n$.*

Exercise 15.7 *Show that the characteristic equation of a 2×2 matrix A can be expressed as*

$$\lambda^2 - \mathrm{tr}(A)\lambda + \det(A) = 0$$

where $\mathrm{tr}(A)$ is the sum of two (not necessarily distinct) eigenvalues of A.

Exercise 15.8 *Prove that if A is a square matrix, then A and A^T have the same eigenvalues.* `HINT:` *Look at the characteristic equation $\det(A - \lambda I) = 0$.*

Appendixes

A1. Greek letters used in mathematics, science, and engineering

The Greek letter forms used in mathematics are often different from those used in Greek-language text: they are designed to be used in isolation, not connected to other letters, and some use variant forms which are not normally used in current Greek typography. The table below shows Greek letters rendered in TEX

Table 15.1. Greek letters used in mathematics

α		alpha	ν		nu
β		beta	ξ	Ξ	xi
γ	Γ	gamma	π	Π	pi
δ	Δ	delta	ρ		rho
ϵ		epsilon	σ	Σ	sigma
ζ		zeta	τ		tau
η		eta	υ		upsilon
θ	Θ	theta	ϕ	Φ	phi
ι		iota	χ		chi
κ		kappa	ψ	Ψ	psi
λ	Λ	lambda	ω	Ω	omega
μ		mu	$\dagger$		dagger

TEXis a typesetting system designed and mostly written by Donald Knuth at Stanford and released in 1978.

Together with the Metafont language for font description and the Computer Modern family of typefaces, TeX was designed with two main goals in mind: to allow anybody to produce high-quality books using a reasonably minimal amount of effort, and to provide a system that would give exactly the same results on all computers, now and in the future.

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