MATH 20100: Calculus II Lecture Notes

Created by Dr. Amanda Harsy

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Contents

1 Syllabus and Schedule

Thanks for taking Calculus II with me! It is my favorite class to teach and it is the best course of the calculus sequence (in my opinion). You may be asking yourself (or have asked yourself), "What is Calculus, and why do I have to take this class?" Calculus is, in my opinion, ultimately is the study of *change*. In particular, calculus gives us the tools to be able to understand how changing one or more linked variables reflects change in other variables [1]. In other words, Calculus is the study and modeling of dynamical systems[2]. In Calculus I, we learned about the derivative of a function and some of its applications. Recall, a derivative is a measure of *sensitivity* of change in one variable to change in the other -the instantaneous rate of change. When we learn about integration, we are measuring *accumulation* or the limit of a summation of smaller parts[2]. In Calculus II, we will build upon this idea that we can use integrals to calculate and model complex situations by accumulating the sums of simpler parts. We will also learn techniques used in calculating and approximating these integrals and discuss ways of modeling functions and infinite systems.

I hope you will enjoy this semester and learn a lot! Please make use of my office hours and plan to work hard in this class. My classes have a high work load (as all math classes usually do!), so make sure you **stay on top of your assignments and get help early**. Remember you can also email me questions if you can't make my office hours or make an appointment outside of office hours for help. When I am at Lewis, I usually keep the door open and feel free to pop in at any time. If I have something especially pressing, I may ask you to come back at a different time, but in general, I am usually available. The HW Assignments, and Practice Problems for Exams are at the end of this course packet. I have worked hard to create this course packet for you, but it is still a work in progress. Please be understanding of the typos I have not caught, and politely bring them to my attention so I can fix them for the next time I teach this course. I look forward to meeting you and guiding you through the magnificent course that is Calculus II.

Cheers,
Dr. H

Acknowledgments: No math teacher is who she is without a little help. I would like to thank my own undergraduate professors from Taylor University: Dr. Ken Constantine, Dr. Matt Delong, and Dr. Jeremy Case for their wonderful example and ideas for structuring excellent learning environments. I want to thank the members from both the MAA META Math the MathVote and Projects (https://www.maa.org/programs-and-communities/curriculum%20resources/meta-mathand http://mathquest.carroll.edu/) for sharing some of their clicker questions. I also want to thank Dr. Ryan Hooper and Dr. Danielle Champney their problem sequences from their own courses. *And finally, I would like to thank you and all the other students for making this job worthwhile and for all the suggestions and encouragement you have given me over the years to improve.*

[1] J. Epstein. 2013 The Calculus Concept Inventory -Measurement of the Effect of Teaching Methodology in Mathematics. *Notices of the AMS* **60** (8), 1018-1026

[2] Schumacher, etc. 2015 *2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences* **18**

2 Syllabus Crib Notes

The full syllabus is posted in Blackboard. Here are some highlights from the syllabus:

2.1 Office Hours

Please come to my office hours! Helping you with the material is the best part of my job! Normally I have 5 weekly office hours which I hold, but due to us being remote, I will be only holding 3 standing drop-in remote hours. I encourage you to instead make appointments for me to meet with you at a time that works for both of us! My office is in **AS-124-A**, but this semester I will hold my office hours in a Bb collaborate classroom. I will have a link for each of these posted in Bb. Remember if none of these times work, send me an email and we can schedule another time to meet. I can also answer questions through email! This semester my office hours will be:

Mondays: 3:00-4:00
Thursdays: 12:30-1:30
Fridays: 2:00-3:00
Or By Appointment!

Note: Sometimes I have meetings or class that goes right up to my office hours, so if I am not there, please wait a few minutes. Also sometimes I have unexpected meetings that get scheduled during my office hours. If this happens, I will do my best to let you know as soon as possible and I usually hold replacement office hours.

Help: Don't wait to get help. Visit me during my office hours, use the discussion forum in Blackboard, go to the Math Study Tables, find a study partner, get a tutor!

2.2 Grades

Category	Percentage
Quizzes	8
Homework Takehome	15
Webassign Homework	7
ICE	5
Mastery Exams	65

Exam #	Date
1	9/30
2	10/21
3	11/11
4	12/8

Final Exam: We will not have a formal final exam. Instead Finals week will be a final Testing Week (see description above under Master-Based Testing).

Homework: Almost every week, I will collect a homework assignment. I will post these homework assignments on Blackboard. You may work with others on the homework, but it must be your own work. If I catch you copying homework, you will get a 0. Please see the academic honesty section in the posted syllabus in Bb.

2.2.1 Exams

Mastery-Based Testing:
This course will use a testing method called, "Mastery-Based Testing." There will be four (4) paper-and-pencil, in-class Mastery Exams given periodically throughout the semester. In mastery-based testing, students receive credit only when they display "mastery", but they receive multiple attempts to do so. The primary source of extra attempts comes from the fact that test questions appear on every subsequent test. In this Calculus III course, Test 1 will have 5 questions. Test 2 will have 10 questions -a remixed version of the five from Test 1 and five new questions. Test 3 will have 15 questions- a remixed version of the ten from Test 2 and five new questions. Test 4 will have 18 questions- a remixed version of the fifteen from Test 3 and three new questions. We will also have four testing weeks. During these weeks, students can use doodle to sign up to retest concepts during (extended) office hours. Students are allowed to test any concept, but cannot retest that concept the rest of the week. So for example, a student can test concepts 2,3 and 5 on Monday and concept 6 on Tuesday, but would not be able to test concept 5 again. These testing weeks are tentatively set for 10/9-10/13, 11/6-11/10, 11/27-12/1, and during Finals Week.

Grading of Mastery-Based Tests:
The objectives of this course can be broken down into 18 sorts of problems. These 18 exam concepts fall into two categories: core concepts and those which lie outside the core. There will be 7 core questions. These core questions include topics that I feel like they really could not manage Calculus III without a firm grasp of these techniques. For each sort of problem on the exam, I identify three levels of performance: master level, journeyman level, and apprentice level. I will record how well the student does on each problem (an M for master level, a J for journeyman level, an N/0 for apprentice/novice level) on each exam. After the Final, I will make a record of the highest level of performance the student has made on each sort of problem or project and use this record to determine the student's total exam grade. If they have at some point during the semester displayed a mastery of each of the core problems and projects, the student will receive least a C. Each core concept is worth roughly 10% of your exam total. Non-core concepts are worth roughly 2.7%. So you can build your grade from mastering a combination of core and non-core concepts. So for example, if you master the 7 core concepts and 2 additional concepts, your exam grade will be a 75.4%. In borderline cases, the higher grade will be assigned to those students who have no novice level concepts at the end of the semester.

This particular way of arriving at the course grade is unusual. It has some advantages. Each of you will get several chances to display mastery of almost all the problems. Once you have displayed mastery of a problem there is no need to do problems like it on later exams. So it can certainly happen that if you do well on the midterms you might only have to do one or two problems on the Final. (A few students may not even have to take the final.) On the other hand, because earlier weak performances are not averaged in, students who come into the Final on shaky ground can still manage to get a respectable grade for the course. This method of grading also stresses working out the problems in a completely correct way, since accumulating a lot of journeyman level performances only results in a journeyman level performance. So it pays to do one problem carefully and correctly as opposed to trying to get four problems partially correctly. Finally, this method of grading allows you to see easily which parts of the course you are doing well with, and which parts deserve more attention. The primary disadvantage of this grading scheme is that it is complicated. At any time, if you are uncertain about how you are doing in the class I would be more than glad to clarify the matter.

2.3 Expectations

This is a college level Math class and is much different than one taught in high school. We cover a lot of (very different) material in a very limited class time. You cannot expect to be able to pass this class if you do not spend several hours every day reading the sections and working problems outside of class. Paying attention and taking notes only during class time will not be enough. After the problems are worked, find a common thread, idea, or technique.

Calculator Policy: No calculators or other forms of technology can be used on in-class, closed-books assessments (quizzes, tests, final).

Academic Integrity: Scholastic integrity lies at the heart of Lewis University. Plagiarism, collusion and other forms of cheating or scholastic dishonesty are incompatible with the principles of the University. This includes using "tutoring"' sites for homework, quizzes, and exams. Students engaging in such activities are subject to loss of credit and expulsion from the University. Cases involving academic dishonesty are initially considered and determined at the instructor level. If the student is not satisfied with the instructors explanation, the student may appeal at the department/program level. Appeal of the department /program decision must be made to the Dean of the college/school. The Dean reviews the appeal and makes the final decision in all cases except those in which suspension or expulsion is recommended, and in these cases the Provost makes the final decision.

Make-Ups: There will be no make-ups for any assignments. If you are late or miss class, your assignment will not be accepted and there will be no make-up offered, except in extenuating and unpredictable circumstances. If you will miss class for a justifiable & unavoidable reason, you can contact me **before** you miss class & it is possible you can have a make-up. If you do not contact me & explain your absence, you will not be allowed a make-up.

Dr. Harsy's web page: For information on undergraduate research opportunities, about the Lewis Math Major, or about the process to get a Dr. Harsy letter of recommendation, please visit my website: `http://www.cs.lewisu.edu/~harsyram`.

Blackboard: Check the Blackboard website regularly (at least twice a week) during the semester for important information, announcements, and resources. It is also where you will find the course discussion board. Also, check your Lewis email account every day. I will use email as my primary method of communication outside of office hours.

The full syllabus and schedule is subject to change and the most updated versions are posted in the Blackboard.

Calculus II Schedule Fall 2020

The table below outlines the tentative topics to be covered each day and quiz and homework due dates.
You also have WebAssign HW due most Tuesdays and Thursdays. This schedule is subject to change.

Monday	Tuesday	Wednesday	Friday
8/31 Intro, Review	9/1 One to one Fns	9/2 One to one Fns cont.	9/4 Exponential Fns **HW 1 Due**
9/7 **Labor Day, No Class**	9/8 Exponential Fns cont.	9/9 Logarithmic Fns	9/11 Derivatives of Log Functions **HW 2 Due**
9/14 Exponential Growth	9/15 Intro to Differential Equations.	9/16 Differential Equations cont.	9/18 Slope Fields **Quiz 1**
9/21 Inverse Trig Functions.	9/22 Inverse Trig Functions cont.	9/23 Hyperbolic Functions	9/25 L'Hoptial's Rule. (not on Exam 1) **HW 3 Due**
9/28 Integration By Parts	9/29 Flex/Review	9/30 **Exam 1**	**10/2** Integration By Parts cont.
10/5 Trig Integration	10/6 Trig Integration cont.	10/7 Inverse Trig Sub **Quiz 2**	10/9 Inverse Trig Sub Cont.
10/12 Partial Fractions **(Testing Week)**	10/13 Partial Fractions Cont. **(Testing Week)**	10/14 Numerical Integration **(Testing Week)**	10/16 Numerical Integration cont. **(Testing Week)** **HW 4 Due**
10/19 Arc Length (not on Exam 2)	10/20 Strategy for Integration and Review	10/21 **Exam 2**	**10/23** Surface Area (not on Exam 2)
10/26 Work	10/27 Work cont.	10/28 Improper Integrals	10/30 Improper Integrals Cont. **Quiz 3**
11/2 11.1: Sequences **(Testing Week)**	**11/3** Sequences cont. **(Testing Week)**	**11/4** Series (Not on Exam 3) **(Testing Week)**	11/6 Series Cont. (Not on Exam 3) **(Testing Week)** **HW 5 Due**
11/9 Flex/Review	11/10 Integral Test	11/11 **Exam 3**	11/13 Integral Test cont.
11/16 Comparison Tests **(Testing Week)**	11/17 Alternating Series **(Testing Week)**	11/18 Ratio & Root Test **(Testing Week)**	11/20 Power Series **(Testing Week)** **HW 6 Due**
11/23 Functions as Series	11/24 Flex Day	**11/25** **Thanksgiving Break, No Class**	**11/27** **Thanksgiving Break, No Class**
11/30 Taylor Series	12/1 Taylor Series Cont.	12/2 Taylor Series Modeling	**12/4** Strategy for Series **Quiz 4**
12/7 **HW 7 Due**	**12/8** **Exam 4**	12/9 The Coolest Thing **Cumulative Quiz 6**	12/11 **HW 8 Due & Bonus Quiz**
12/14 **(Final Testing Week)**	12/15 **(Final Testing Week)**	12/16 **(Final Testing Week)**	**12/18** **(Final Testing Week)**

Assessment and Mapping of Student Learning Objectives:

Baccalaureate Characteristics:

BC 1. The baccalaureate graduate of Lewis University will read, write, speak, calculate, and use technology at a demonstrated level of proficiency.

Measurable Student Learning Outcome:

Advocate for a cause or idea, presenting facts and arguments, in an organized and accurate manner using some form of technology. Include qualitative and quantitative reasoning.

Course Student Learning Objectives	Baccalaureate Characteristic	Demonstrated by
1. Differentiate exponential, logarithmic, and inverse trigonometric functions.	1-Reinforced	Homework, Exams, or Quizzes
2. Integrate exponential, logarithmic, and inverse trigonometric functions.	1-Reinforced	Homework, Exams, or Quizzes
3. Recognize integrands for which integration by parts is appropriate.	1-Reinforced	Homework, Exams, or Quizzes
4. Perform integration by parts.	1-Reinforced	Homework, Exams, or Quizzes
5. Apply techniques for integrals of products and higher powers of sines and cosines.	1-Reinforced	Homework, Exams, or Quizzes
6. Apply techniques for integrals of secants and tangents, and for cosecants and cotangents.	1-Reinforced	Homework, Exams, or Quizzes
7. Apply techniques of trigonometric substitution to integrate various forms of integrands.	1-Reinforced	Homework, Exams, or Quizzes
8. Apply the technique of partial fraction decomposition to reduce an integrand to a more easily integrated form.	1-Reinforced	Homework, Exams, or Quizzes
9. Given a random integration problem, choose the proper method and proceed with integration.	1-Reinforced	Homework, Exams, or Quizzes
10. Identify indeterminate limit forms.	1-Reinforced	Homework, Exams, or Quizzes
11. Evaluate limits using L'Hôpital's Rule.	1-Reinforced	Homework, Exams, or Quizzes
12. Approximate definite integrals using numerical integration techniques and solve related problems.	1-Reinforced	Homework, Exams, or Quizzes
13. Apply slicing and integration techniques to calculate volumes, work, hydrostatic force, arc length, and surface area.	1-Reinforced	Homework, Exams, or Quizzes
14. Recognize improper integrals and put them in proper form for determination.	1-Reinforced	Homework, Exams, or Quizzes
15. Determine if an improper integral diverges or converges (and if so, to what?).	1-Reinforced	Homework, Exams, or Quizzes
16. Identify and compare different types of sequences.	1-Reinforced	Homework, Exams, or Quizzes
17. Determine if a sequence diverges or converges (and if so, to what?).	1-Reinforced	Homework, Exams, or Quizzes
18. Apply infinite series tests for convergence and divergence.	1-Reinforced	Homework, Exams, or Quizzes
19. Determine the error associated with a partial sum of an alternating series.	1-Reinforced	Homework or Quizzes
20. Find the interval of convergence and radius of convergence for a given power series.	1-Reinforced	Homework, Exams, or Quizzes
21. Generate power series representations of some functions from a geometric series perspective.	1-Reinforced	Homework or Quizzes
22. Generate power series representations of some functions from a Taylor Series perspective.	1-Reinforced	Homework, Exams, or Quizzes
23. Recognize and manipulate important Maclaurin Series using differentiation, integration, and substitution.	1-Reinforced	Homework
24. Find the n^{th} degree Taylor Polynomial of a function f at a point a and determine the error associated with the estimate.	1-Reinforced	Homework

3 Review Calculus I

Recall the following derivatives:

Function	$\sin x$	$\cos x$	$\tan x$	$\csc x$	$\sec x$	$\cot x$
Derivative						

1. The Math Cat Posse: Eva, Archer, Gilbert, and Habenero are trying to remember their Calculus I as they work to differentiate the following functions. With a partner or in small groups, describe why each cat's solution is problematic (if at all) and find the correct derivative.

   (a) $y = \sec(x^2)$ **Correct Solution:**

   Archer's Solution:
   $y' = \sec(x^2)\tan(x^2)$

   Gilbert's Solution:
   $y' = 2x \sec\tan(x^2)$

   (b) $f(x) = x^3 \tan(x)$ **Correct Solution:**
   Habenero's Solution:
   $f'(x) = 3x \sec^2(x)$

(c) $y = \dfrac{1}{x^5} - \sqrt[4]{x - x^3} + x - \pi^2$ **Correct Solution:**

Gilbert's Solution:

$y' = \dfrac{1}{x^4} - (1 - 3x)\sqrt[3]{x - x^3} + 1 - 2\pi$

(d) $k(x) = \dfrac{x^3}{\cos(x) + 1}$ **Correct Solution:**

Archer's Solution:

$k'(x) = \frac{3x^2}{-\sin(x)}$

Eva's Solution:

$k'(x) = \frac{3x^2}{\cos x + 1} + \frac{x^3 \sin x}{(\cos x + 1)^2}$

2. Integrate the following functions:

a) $\int (\frac{1}{x^3} - \sqrt[5]{x} + x - 1)dx$

c) $\int x^5 \sec^2(x^6)dx$

b) $\int_1^5 \frac{x^7 - x^2}{x}dx$

d) $\int \sec^2 x \tan x dx$

3. What is differentiation used for?

4. What does the derivative represent?

5. What does the integral represent?

6. Differentiate: $y^2 = x\sin(y) - x^7$

7. If the position function of a particle is given by $p(t) = t^3 - 4t + \cos(t)$, find the acceleration function.

4 Inverse Functions

4.1 One-To-One Functions

A __________ is a rule that assigns every value (usually a number) in its domain to a __________ number in its range.
The __________ is a set of values that "make sense" to use in a function.
The __________ is a set of values that occur over the domain of the function.

Definition: A function is **one-to-one** if each x-value has __________ __________ y value **AND** every y-value has __________ __________ x value.

Recall, we can test if a graph is a function using the ____________________ line test.

Vertical Line Test: f is a function iff any vertical line only intersects the graph of f at exactly one point.

Now, we can test if a function is one-to-one by using the ____________________ line test.
A one-to-one function must pass **both** tests!

Horizontal Line Test: A function f is __________ iff any horizontal line only intersects the graph of f at exactly one point.

Example 4.1. *Determine whether the following functions are one-to-one:*

(a) $\{(-1,6),(0,3),(2,0),(5,1)\}$

(b) $\{(-8,6),(-4,3),(0,6),(5,10)\}$

(c) $f(w)$ is the cost of mailing a letter weighing w ounces.

(d) $f(t)$ is the number of customers in Macy's at t minutes past noon on November 7, 1984.

(e) $f(x) = (x-1)^2$

(f) $g(x) = 2x - 3$

Great, why do we care about whether a function is one-to-one?

4.2 Inverse Functions

________ ________ are a pair of functions that "undo" each other. That is, $g(x)$ is the inverse function of $f(x)$ if $(f \circ g)(x) = (g \circ f)(x) =$ ________. We denote $g(x)$ as $f^{-1}(x)$.
To write a formula for an inverse function, switch _____ and _____. Then solve for y.

Formal Definition: The inverse of a one-to-one function f, denoted f^{-1}, is the set of all ordered pairs of the form (y, x), where (x, y) belongs to f. In other words, $f^{-1}(y) = x \iff$ ________ for any y in the range of f

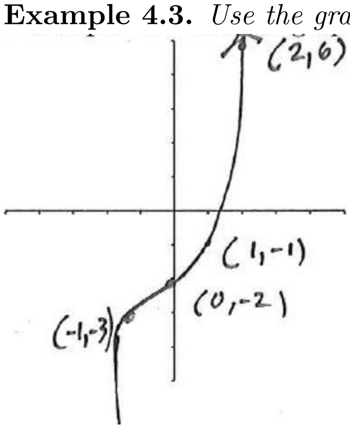
Notice because the inverse is formed by interchanging x & y, the domain of f becomes the ________ of $f^{-1}(x)$ and the range of f becomes the ________ of $f^{-1}(x)$.

Warning: $f^{-1}(x)$ means *"f-inverse"*. It **DOES NOT** mean $\dfrac{1}{f(x)} = [f(x)]^{-1}$

Example 4.2. *What is the inverse function of $f(x) = 2x + 1$?*

Another way to find f^{-1} is by using the graph of f. By our definition of f^{-1}, if (x, y) is a point on $f(x)$, then ________ is a point on the graph of $f^{-1}(x)$.

Example 4.3. *Use the graph of $f(x) = x^3 - 2$ to graph f^{-1} on the same axis.*



Example 4.4. *If $g(x) = x^2$, what is the inverse of $g(x)$?*

The problem is that $y = x^2$ is a 2:1 map; __________=__________=__________
(ie it fails the horizontal line test!)
We didn't have a problem with $f(x) = 2x + 1$ because $f(x)$ is a 1:1 map, that is, a 1-1 function.

Example 4.5. *Is $f(x) = \sin x$ a one-to-one function?*

4.3 Restricted Domains (Sneak Peak at Inverse Trig Functions)

But wait a minute, we do have an inverse function for $y = \sin x$. It is __________ __________. How can this be?

And wait a minute, in Example 4.5, we couldn't find an inverse function for $y = x^2$, but aren't "square roots" and "squaring" considered "opposites" of each other. So the inverse function should be $y = \sqrt{x}$? Why do we have this for only sometimes?

Solution: Notice $y = x^2$ and $y = \sin(x)$ are both one-to-one on part of their domains. The solution

for $y = x^2$ is to restrict the domain $y = x^2$ by cutting out $\{x < 0\}$.
And then we can find an inverse function on the restricted domain which would be
We can impose a similar domain restriction on $y = \sin x$ to get the inverse of $\sin x$. More on this later in when we learn about inverse trig functions.

Example 4.6. *a) What are some methods for showing a function is one-to-one?*

b) Would a function that is always strictly increasing (decreasing) be a 1-to-1 function? How do we show a function is increasing/decreasing?

c) Show that $f(x) = x^5 + x^3 + x$ is one-to-one using a reasonable method.

d) Find $f(f^{-1})(2)$

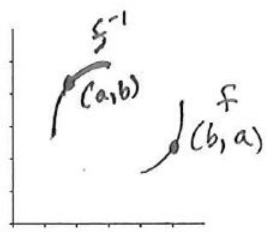
e) Find $f^{-1}(3)$

Example 4.7. *Let $f(x) = \cos(x)$.*
a) Over what intervals does f have an inverse?

b) What is the range and domain of f^{-1} if we take the restricted domain of f to be $[0, \pi]$?

4.4 Calculating Derivatives of Inverse Functions

Theorem: If f is one-to-one and continuous on an interval, then f^{-1} is as well.



Theorem: If f is one-to-one and differentiable on an interval and $f(f^{-1})(a) \neq 0$. Then $f^{-1}(x)$ is differentiable at a and $(f^{-1})'(a) =$

Pf. Omitted see textbook.

Why is this theorem useful?

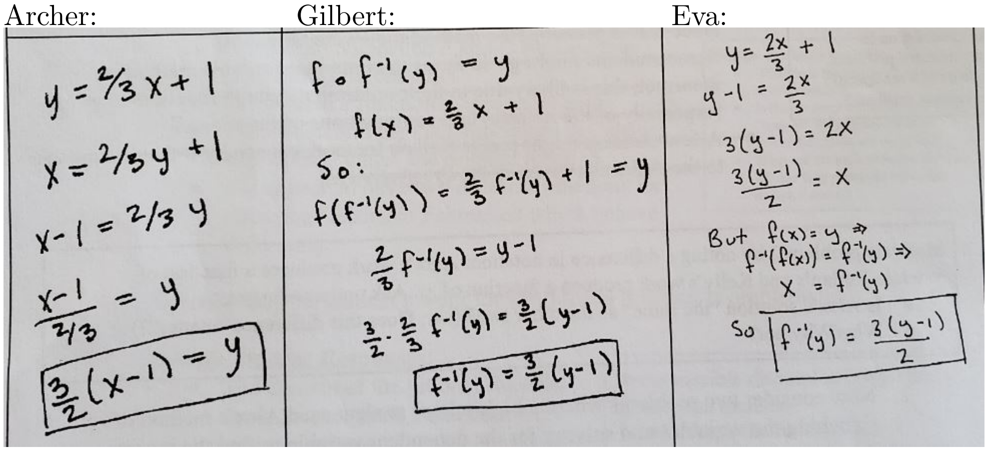
Example 4.8. *Find* $(f^{-1})'(0)$ *if* $f(x) = \dfrac{x-1}{2x+3}$ *and* $f'(x) = \dfrac{-1}{(2x+3)^2}$

Example 4.9. *Find* $(f^{-1})'(0)$ *if* $f(x) = \displaystyle\int_4^x \sqrt{2+t^3}\,dt.$

4.5 ICE –Inverse Functions

1. Compare how the math kitties Archer, Gilbert, and Eva found the inverse function of $f(x) = \frac{2}{3}x + 1$.

Archer: Gilbert: Eva:



Archer:

$y = \frac{2}{3}x + 1$

$x = \frac{2}{3}y + 1$

$x - 1 = \frac{2}{3}y$

$\frac{x-1}{2/3} = y$

$\boxed{\frac{3}{2}(x-1) = y}$

Gilbert:

$f \circ f^{-1}(y) = y$

$f(x) = \frac{2}{3}x + 1$

So:

$f(f^{-1}(y)) = \frac{2}{3}f^{-1}(y) + 1 = y$

$\frac{2}{3}f^{-1}(y) = y - 1$

$\frac{3}{2}\cdot\frac{2}{3}f^{-1}(y) = \frac{3}{2}(y-1)$

$\boxed{f^{-1}(y) = \frac{3}{2}(y-1)}$

Eva:

$y = \frac{2x}{3} + 1$

$y - 1 = \frac{2x}{3}$

$3(y-1) = 2x$

$\frac{3(y-1)}{2} = x$

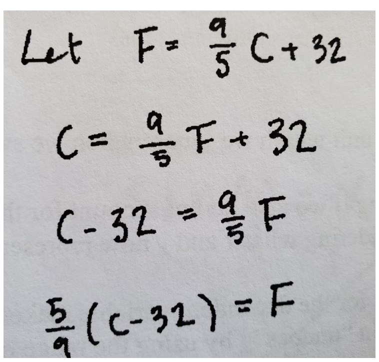
But $f(x) = y \Rightarrow$

$f^{-1}(f(x)) = f^{-1}(y) \Rightarrow$

$x = f^{-1}(y)$

So $\boxed{f^{-1}(y) = \frac{3(y-1)}{2}}$

2. Now consider how Archer used his method of switching the variables and solving for the dependent variable to find the inverse function of $T(C) = \frac{9}{5}C + 32$ where C is the temperature in Celsius and $F = T(C)$ gives the temperature in Fahrenheit. Describe why his work is problematic.



Let $F = \frac{9}{5}C + 32$

$C = \frac{9}{5}F + 32$

$C - 32 = \frac{9}{5}F$

$\frac{5}{9}(C-32) = F$

3. Find $(g^{-1})'(1)$ for $g(x) = \sqrt{5 - 2x}$.

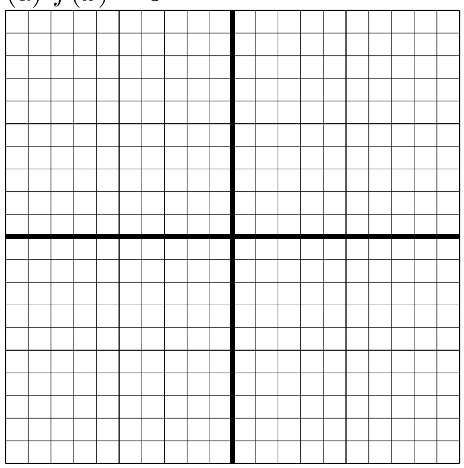
4. Determine whether the following functions are one-to-one. If so, find a formula for the inverse function and its domain and range. Assume the domain of $f(x)$ is all real numbers unless stated otherwise.

a) $g(x) = \dfrac{1}{x - 2}$

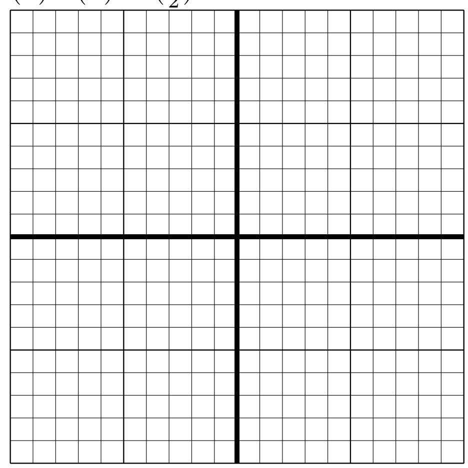
b) $k(x) = x^4 + 2$ for $0 \le x \le 10$

5. Graph the following functions. You may want to make a table of values to graph some points. Then sketch the graphs of their inverse functions using the graph of the functions.

(a) $f(x) = 3^x$



(b) $h(x) = (\frac{1}{2})^x$



6. Find $(f^{-1})'(3)$ for $f(x) = x^3 - 2\sin(x) + 3\cos(2x)$

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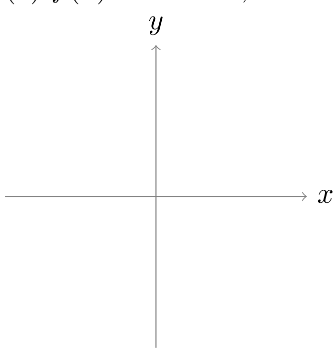
5 Exponential Functions

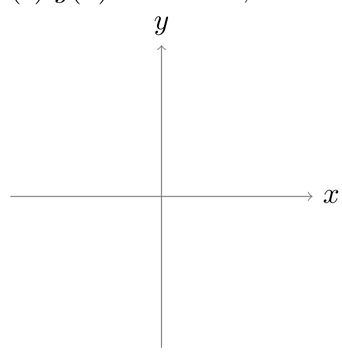
5.1 Review of General Exponential Functions

An ____________ function is a function of the form $f(x) = P_0 a^x + b$ for $a > 0$, $a \neq 1$. It is an exponential function because the variable is in the exponent. What does P_0 represent?
Why do we need the restriction $a \neq 1$?
Why do we need the restriction $a > 0$?

(a) $f(x) = 2a^x + 3$, $a > 1$



(c) $g(x) = 3a^x + 1$, $0 < a < 1$



Example 5.1. *How do we determine* $5^{\sqrt{2}}$*?*

Theorem $a^x = \lim_{r \to x} a^r$ for...

Properties of Exponents Review
$a^x a^y =$
$\frac{a^x}{a^y} =$
$(a^x)^y =$
$(a \cdot b)^x =$

Properties of Exponential functions: Given $f(x) = P_0 a^x + b$

1. $f(x)$ is continuous (and differentiable)
2. The graph looks like a J if __________. This is called exponential __________.
3. The graph looks like a L if __________. This is called exponential __________.
4. The horizontal asymptote is at __________.
5. If $a > 1$, $\lim_{x\to\infty} f(x) =$ __________ $\implies \lim_{x\to\infty} a^x =$ __________
6. If $a > 1$, $\lim_{x\to-\infty} f(x) =$ __________ $\implies \lim_{x\to-\infty} a^x =$ __________
7. If $0 < a < 1$, $\lim_{x\to\infty} f(x) =$ __________ $\implies \lim_{x\to\infty} a^x =$ __________
8. If $0 < a < 1$, $\lim_{x\to-\infty} f(x) =$ __________ $\implies \lim_{x\to-\infty} a^x =$ __________
9. The domain of a^x is ____________________ and the range is ____________________.

Example 5.2. *Determine* $\lim_{x\to2^-} 4^{\frac{1}{2-x}}$.

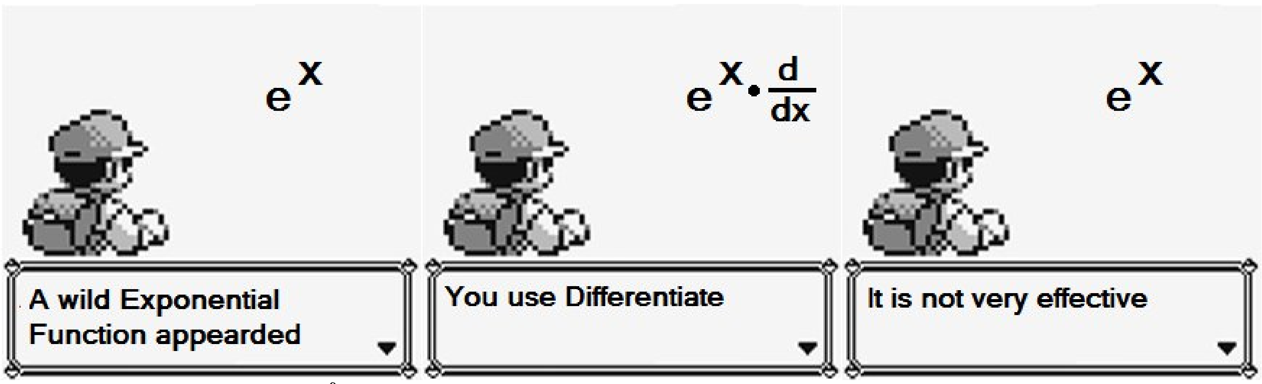
5.2 The "Natural" Exponential

Most of us have encountered the irrational number e^1. It is a number approximately equal to 2.7182818. There are many ways to characterize the exponential function e^x. Right now, we will define it using a calculus characterization.

Definition of e^x: The natural exponential function $f(x) = e^x$ is a function which is its own derivative. That is

How did we come up with this? How do we know $e^1 \approx 2.7182818$? This number was discovered in the early 18th century by the mathematician, Leonard Euler. He was exploring continuous compound interest which we will talk about in a few days. Feel free to read the proof in the textbook in the meantime.

Derivative of e^x: $\dfrac{d}{dx}(e^x) =$



[1]

Antiderivative of e^x: $\displaystyle\int e^x dx =$

Example 5.3. *Differentiate* $y = 3xe^{\sqrt{x}}$

Example 5.4. *Find the equation of the line tangent to* $f(x) = \cos(e^{3x} - 1)$ *at* $x = 0$.

[1]Meme from https://knowyourmeme.com/photos/156115-a-wild-x-appears-wild-x-appeared

Example 5.5. *Evaluate* $\int e^x(2-e^x)^4 dx$.

Example 5.6. *Evaluate* $\lim_{x\to\infty} \frac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$ *[Hint: What if it were a rational function?].*

5.3 ICE – Exponential Functions

2 sides!

1. Find the derivative of the following functions.

   a) $g(x) = e^{\sin(x^3)}$

   b) $e^y = x$ [Use Implicit Differentiation!]

2. Evaluate $\displaystyle\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$.

3. Evaluate $\displaystyle\int e^x e^{e^x}\,dx$.

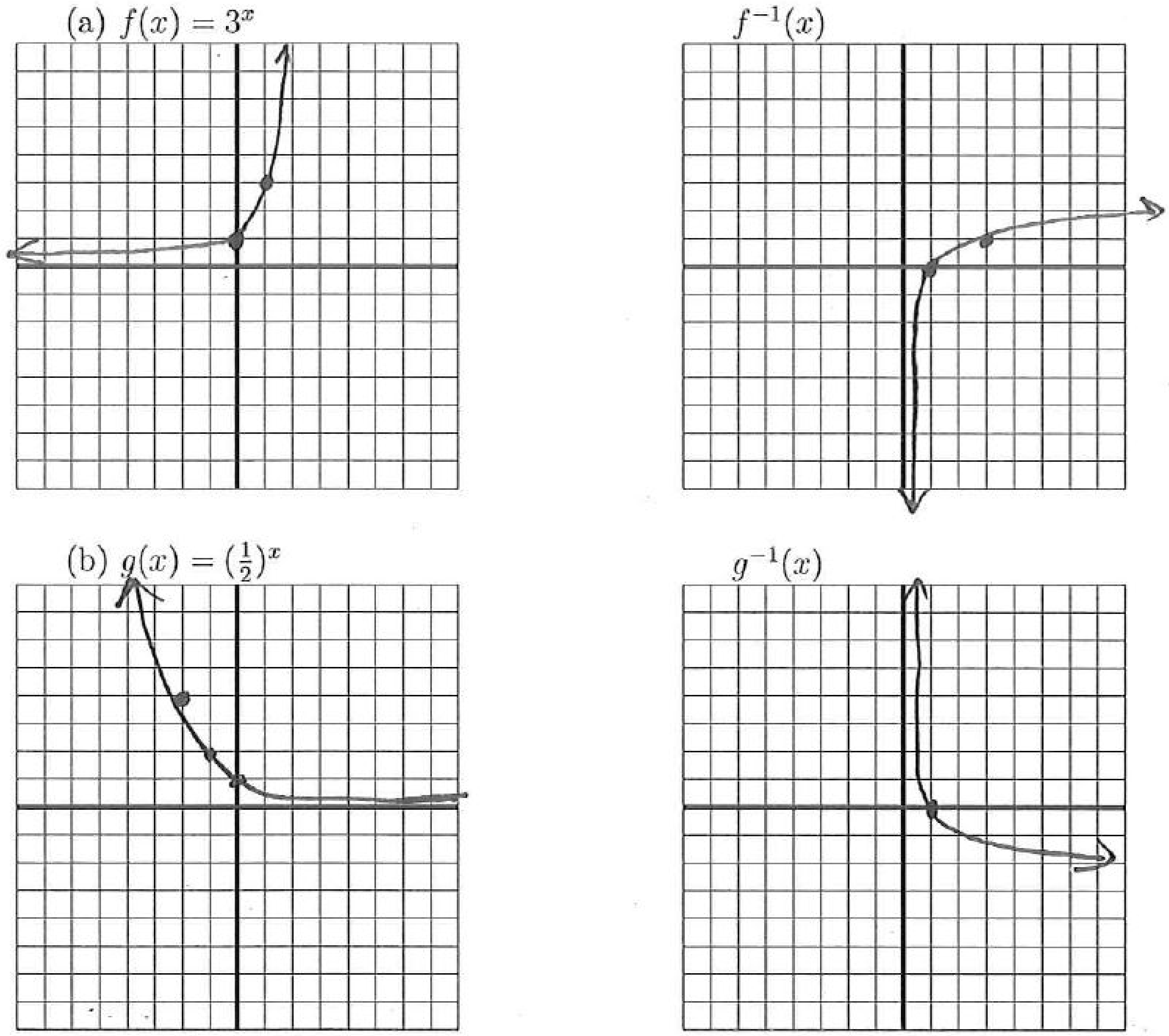
4. Differentiate $y - x^2 = e^{x^2y^3}$

5. Evaluate $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$.

6 Logarithmic Functions

6.1 Review of Logarithmic Functions

During ICE-Inverse, we sketched the graphs of the inverse functions of $f(x) = 3^x$ and $g(x) = (\frac{1}{2})^x$.



The inverse of an exponential function is a ____________________.
The fundamental rule that governs logarithmic functions is $y = \log_b(x) \Leftrightarrow x = b^y$

Remember, $\log_b(x)$ is just an exponent. It is the special exponent on b that gives you x!
Now look again at the graph of $f^{-1}(x)$ and $g^{-1}(x)$. We can write $f^{-1}(x) = \log_3 x$ and $g^{-1}(x) = \log_{\frac{1}{2}} x$

Example 6.1. *Use the definition of a logarithm to simplify the following expressions by changing it to exponential notation.*

(a) $\log_{10} 100$ (b) $\log_{16} 4$ (c) $\log_{10} 0$

(d) $\log_5 \frac{1}{5}$ (e) $\log_9 1$ (f) $\log_2(-1)$

The ____________ ____________ (__________) is the base 10 logarithm. Example: $\log 100 =$

The ____________ ____________ (__________) is the base e logarithm
Remember e is an irrational number that is about 2.7183
$\ln(x)$ is not defined for $x < 0$... why?
What is the range and domain of $\ln(x)$?

Properties of Logarithms

1. $\log_b(AB) =$
2. $\log_b(A^p) =$
3. $\log_b(b^x) =$
4. $b^{\log_b(x)} =$
5. $\log_b(1) =$

Objective 2: ***Using Logs to Solve Exponential Equations***
Sometimes it is useful to express logarithms in terms of natural log (or another base) for this we have the...
Change of Basis Formula: For any $a > 0$, $a \neq 1$, we have $\log_a x = \dfrac{\ln x}{\ln a}$.

Proof: See Text.

We can use the fact that logarithms are the inverses of exponential functions and the properties of logarithms to solve exponential equations. Some of you may use the "change of basis" rule to solve these as well.

To solve an exponential equation...
(a) Get the ____________ part by itself.
(b) Take the ____________ of both sides.

Example 6.2. *Solve the following equations.*

(a) $2^{x-3} = 8$

(c) $4 + 5e^{-x} = 9$

(b) $3^{m+3} = 4$

(d) $\ln e^{x/4} = \sqrt{3}$

To solve a logarithmic equation...
(a) Write as a single ______________ using the properties of logarithms if needed.
(b) Change to ______________ notation and solve the resulting exponential equation. Or just take the appropriate exponential of both sides.
(c) Remember to check if the answer is in the domain of the logarithmic function.

Example 6.3. *Solve the following equations.*

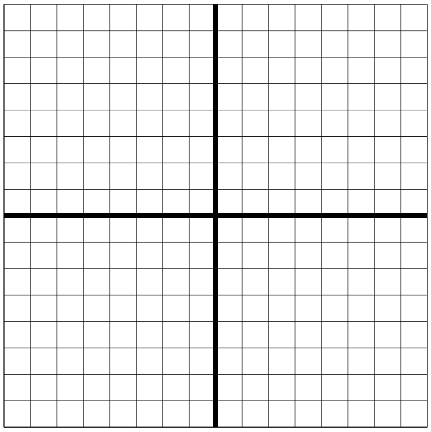
(a) $\log_2(x + 10) = 2$

(c) $\log_4(x + 2) + \log_4(x - 7) = \log_4 10$

(b) $\ln(x - 6) - \ln(x + 3) = \ln 2$

Objective 4: *Graphs and Growth of Natural Log*

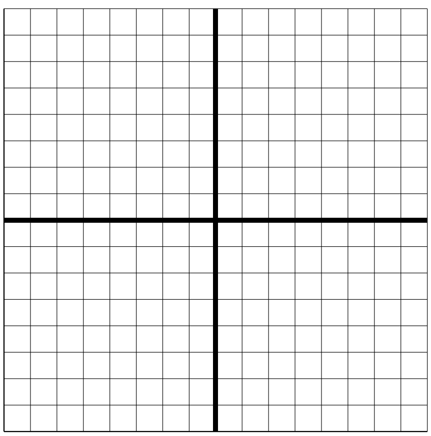
We have already graphed a few logarithmic functions. Let's graph the natural logarithm (base e):



$$\lim_{x\to\infty} \ln x =$$

$$\lim_{x\to 0^+} \ln x =$$

Example 6.4. *Now use properties of transformations to graph* $f(x) = \ln(x+2) - 3$.



Example 6.5. *Find the limit* $\lim_{x\to\infty} [\ln(3+x) - \ln(x+4)]$.

6.2 Logarithmic Differentiation

Objective 1: ***Derivatives of Natural Logarithm***

Derivative of Natural Log: $\dfrac{d}{dx}(\ln x) = \dfrac{1}{x}$

Why? Let $y = \ln x \implies$ __________ Then we can use our solution from ICE 2!

Example 6.6. *Differentiate the following functions.*

(a) $f(x) = \ln(5x^2 + 1)$

(b) $g(x) = \dfrac{x}{\ln x}$

Example 6.7. *Differentiate* $g(x) = \ln(2x \sin x)$ *two different ways.*

Example 6.8. *Differentiate* $f(x) = \ln |x|$

So $\dfrac{d}{dx}\ln|x| =$
Thus...
Antiderivative of $\dfrac{1}{x}$: $\displaystyle\int \frac{1}{x}dx =$

Antiderivative of $\tan x$: $\displaystyle\int \tan x dx =$

Pf: $\displaystyle\int \tan x dx$

Objective 2: ***General Logarithmic and Exponential Equations***

Recall the Change of Basis Formula: $\log_a x = \dfrac{\ln x}{\ln a}$.

Example 6.9. *Use the change of basis formula to differentiate $f(x) = \log_a x$.*

Derivative of $\log_a x$: $\dfrac{d}{dx}(\log_a x) =$

Derivative of a^x: $\dfrac{d}{dx}(a^x) =$

Pf.

Thus
Antiderivative of a^x: For $a \neq 1$, $\displaystyle\int a^x dx =$

Example 6.10. *What is the value of* $\int_0^2 4^x dx$*?*

Objective 3: ***Logarithmic Differentiation***

Consider differentiating the function $y = \dfrac{(x+2)^4 \sin^2 x}{\sqrt[3]{x}}$. This would kinda suck right?

Steps to Use Logarithmic Differentiation:

Step 1: Take the natural log of both sides and use your properties to simplify.
Step 2: Use implicit differentiation to find $\dfrac{dy}{dx}$ [or y'].
Step 3: Substitute back in for y.
[Note: If $f(x) < 0$, we can write $|y| = |f(x)|$.]

Example 6.11. *Differentiate* $y = \dfrac{(x+2)^4 \sin^2 x}{\sqrt[3]{x}}$.

Example 6.12. *Find the derivative of* $y = (\sin x)^x$.

6.3 ICE – Logarithmic Differentiation

1. Use logarithmic differentiation to find the derivative of $y = \dfrac{x\sqrt{x^2-2}}{(x+9)^{\frac{1}{3}}}$.

2. Differentiate $y = (\ln x)^{3^x}$.

3. Evaluate the integrals a) $\displaystyle\int \frac{\pi^{\ln x}}{x} dx.$ b) $\displaystyle\int \frac{x+1}{x^2+4x+3} dx.$

c) $\displaystyle\int \frac{x+1}{x^2+x+3} dx$ d) $\displaystyle\int \frac{x+1}{x^2+2x+3} dx$

4. Use logarithmic differentiation to prove $\dfrac{d}{dx}(x^n) = nx^{n-1}$ [Hint: First set $|y| = |x|^n$]

5. Determine $\lim\limits_{x\to\infty}[\ln(2+x) - \ln(1+x^2)]$

7 Differential Equations

7.1 Brief Introduction to Differential Equations

Suppose I know that $f'(x) = 2x$ or $y' = 2x$. In other words, we know the derivative of a function is equal to $2x$. Can you determine what y is? What did you do?

The example above is called a separable differentiable equation. It is called a differential equation because it has an unknown function and its derivative. It is separable because you can solve the equation by separating the variables and integrating.

Some differentiable equations are not as easy to solve by hand, like $y'' + y' = \frac{y}{x} + 1$, and if you take Differentiable Equations, you will learn techniques to solve or approximate solutions to these equation.

Differentiable equations are often used in modeling. For example, we can represent the population of foxes as a function of time, say $f(t)$.
What does $f'(t)$ represent?

Often $f'(t)$ is proportional to the population at time t, $f(t)$[2].
That is, $f'(t) =$

We could also write this as $y' = ky$ or $\frac{dy}{dx} = ky$. In words, this means we are looking for a function whose derivative is a constant multiple of itself. What function(s) satisfy this property?

It turns out that $\frac{dy}{dx} = ky$ is separable and we can solve it.

[2]Recall, y is proportional to x if $y = kx$.

7.1.1 Law of Natural Growth/Decay

Our work above gives us this important result.

Law of Natural Growth/Decay: If y is the value of a quantity at time t and if the rate of change of y with respect to t is proportional to the size of y at time t, then $\frac{dy}{dt} =$
k is the **relative growth/decay rate**.
If $k > 0$, this change is natural ________
If $k < 0$, this change is natural ________

Theorem: The only solutions to the differential equation $\frac{dy}{dt} = ky$ or $f'(t) = kf(t)$ are exponential functions with the form $y(t) = y(0)e^{kt}$ or $f(t) = P_0e^{kt}$ That is, if a population has a constant relative growth (decay) rate, it must grow (decay) exponentially.

Example 7.1. *A bacteria culture has a constant relative growth rate. (That is, it has a constant growth rate that is proportional to its size.) After 2 hours, the bacteria count was 600 and after 8 hours the count was 75,000. When will there be 200,000 bacteria?*

Example 7.2. *Tritium-3 has a half-life of 50000 years. Find its constant relative decay rate.*

7.1.2 Newton's Law of Cooling

Newton's Law of Cooling states that the rate of cooling of an object is proportional to the temperature difference between the object and its surroundings. That is if T is the temperature of the object and T_s is the temperature of its surroundings, then $\frac{dT}{dt} =$

Newton's Law of Cooling: $T(t) - T_s = (T(0) - T_s)e^{kt}$ Where $T(t)$ is the temperature of the object at time t and $T(0)$ is the initial temperature.
Alternative form: $T(t) = (T(0) - T_s)e^{kt} + T_s$

Example 7.3. *A root beer is taken from the fridge and its temperature is* 5° *C. After 25 minutes, the root beer is* 10° *C. The temperature in your house is* 20° *C. What is the temperature of the root beer after 50 minutes?*

7.1.3 Continuous Compounded Interest

Compounded interest is the result of leaving money in an investment and earning interest on top of old interest.
If a principal of P_0 dollars is deposited at an annual rate of interest r compounded n times per year, the account will have $A = P_0(1 + \frac{r}{n})^{nt}$ dollars after t years.

Continuous Compound interest is the result of interest accumulating instantaneously.
If a principal of P_0 dollars is deposited at an interest rate of r compounded continuously for t years, the account will have $A = P_0e^{rt}$.

If this is written as a differential equation, we get: $\frac{dA}{dt} =$__________ with initial condition $A(0) = P_0$

Example 7.4. *If \$500 is borrowed at 14% interest. What is owed after 2 years if interest is compounded continuously?*

7.2 General Differential Equations

From looking at the last few examples, we can see that there are many applications to just one type of differential equations. In fact we have only scratched the surface of some of the applications. Other applications include modeling first-order chemical reaction rates, heat transfer, population dynamics, Dielectric breakdown, Computer processing power (Moores law), and more!

Remember a differential equation is an equation with an unknown function y and its derivative. Beware, y is not a number and represents a function often in terms of time or other variables $y = f(t)$.

Examples:

A) $\dfrac{dy}{dx} - 2y = \sin(x)$ B) $y'(t) = 3t$ C) $e^y \dfrac{dy}{dx} = 7t$

D) $\dfrac{d^2y}{dx^2} + 4y = 0$ E) $y' = x^2 - y$ F) $\dfrac{dP}{dt} = 0.1P(1 - \dfrac{P}{200})$

Definition 7.1. *The __________ of a differential equation is the order of the highest-order derivative that appears in the equation.*

Example: Equations involving only the first derivative of a function (ie no higher order derivatives) are called __________-ordered Differential Equations and have the form: ____________________

Definition 7.2. *A differential equation is **linear** if the unknown function (ie $y, f(t)$, ...) and its derivative appear only to the first power and are not composed with other functions.*

Example: A First-ordered Linear Differential Equation is an equation which involves only linear terms of the function and its derivative.

The standard form of a 1^{st}-order Linear Differential Equation:

Note: $P(x)$ and $Q(x)$ are functions of x that need NOT be linear!

Example 7.5. *Go back to the example above and identify which differential equations are first order linear equations.*

Definition 7.3. *The __________ solution of a D.E. is the most generic possible solution to the D.E. and contains all possible solutions.*

Definition 7.4. *The __________ solution of a D.E. is a specific solution to the D.E and satisfies a given initial value or condition of the equation.*

Definition 7.5. *A First-ordered Differential Equation, $y'(x) = f(x, y)$ is* __________ *if f can be expressed as a product of a function of x and y.*

These are set up to be solved through integration!
Standard Form:

Examples:

Example 7.6. *Show that $y = Ae^{\frac{x^3}{3}}$ is a solution to the differential equation $\frac{dy}{dx} = x^2y$.*

Example 7.7. *a) Find the general solution to* $e^y \frac{dy}{dt} = 2t$.

b) Find the particular solution which satisfies the the initial condition, $y(2) = 0$, $t > \sqrt{3}$.

7.3 Solving First-Order Linear Differential Equations

Recall a First-Order Linear Differential Equation can be written in the form:

$$\frac{dy}{dx} + P(x)y = Q(x).$$

Suppose $P(x)$ and $Q(x)$ are constants, say $P(x) = p$ and $Q(x) = q$ where p and q are constants. Then the first-order linear differential equation becomes

$$\frac{dy}{dx} + py = q.$$

We already know how to solve this type! Consider $y'(t) - 7y(t) = 1$.

Conclusion: The general solution to a first-order linear differential equation of the form $y'(t) + py(t) = q$ where p and q are constant is newpage

Now how do we solve $\frac{dy}{dx} + P(x)y = Q(x)$ when $P(x)$ and $Q(x)$ are not constant?

Consider the differential equation. $y' + \frac{1}{x}y = 2$.

How do we find our integrating factor $I(x)$ in general?

- Recall $\frac{dv(x)y}{dx} = v\frac{dy}{dx} + vP(x)y$, differentiating (using implicit differentiation and the product rule to differentiate $v(x)y$), we get:
  $\frac{dv}{dx}y + v\frac{dy}{dx} = v\frac{dy}{dx} + vP(x)y$
- So $\frac{dv}{dx}y = vP(x)y$
- Solving the separable differential equation, we get
  $\int \frac{dv}{v} = \displaystyle\int P(x)dx$
  Thus $\ln|v| = \ln(v) = \int P(x)dx$ (recall v is assumed to be positive)
  Thus $v = e^{\int P(x)dx}$

Example 7.8. *Which of the following are first order differential equations? Write each first order differential equation in standard form, and and identify $P(x)$ and $Q(x)$.*

1. $xy' = x^2y - 1$
2. $x\frac{dy}{dx} - x^2y = \frac{-1}{x}$
3. $yy' = y^2x - 2x$
4. $-y' + 4\sin(x)y + x\ln(x) = 1$

2. To Solve a First Order Linear Differential Equation:

1. Write your differential equation in standard form: $\dfrac{dy}{dx} + P(x)y = Q(x)$.
2. Find your integrating factor: $I(x) = e^{\int P(x)dx}$
3. Multiply both sides of your differential equation by $I(x)$.
4. Recognize your equation as $(I(x)y)' = I(x)Q(x)$.
5. Integrate both sides to solve for y, so $y = \dfrac{1}{I(x)}\left(\int I(x)Q(x)dx + C\right)$.
6. Apply initial conditions to find particular solutions if needed.

Example 7.9. *Find the solution to* $x\dfrac{dy}{dx} + y - 2\sqrt{x} = 0$ *for* $x > 0$.

7.4 ICE -Differential Equations

1. What is the order of $y'' = 10 - 9y$?

2. True or False: $y''(t) = 10 - 9y(t)$ is linear.

3. Find the general (or particular) solution to the following Differential Equations.

   (a) $x^3 \frac{dw}{dx} = \sqrt{w}(2x - 1)$

   (c) $2y' - y = e^{\frac{x}{4}}$

   (b) $xy' + 3y = 1 - \frac{1}{x}$

   (d) $\dfrac{dy}{dx} = e^{x-y}$ where $y(0) = \ln(2)$

4. Find the general (or particular) solution to the following Differential Equations.

(a) $y' = \sqrt{xy}$

(b) $\dfrac{dy}{dx} = 2 - y$, $y(1) = 2$

5. Give an example of a non-linear differential equation.

6. Give an example of a separable differential equation.

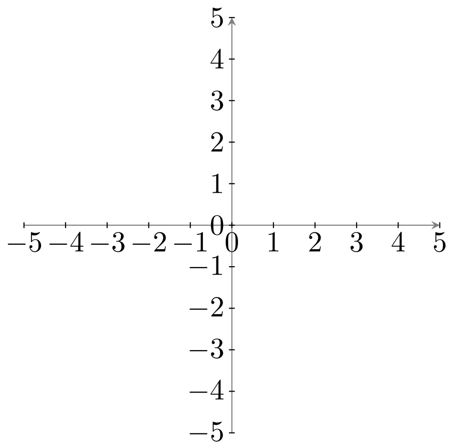
7. Give an example of a 3rd-order differential equation.

7.5 Slope Fields

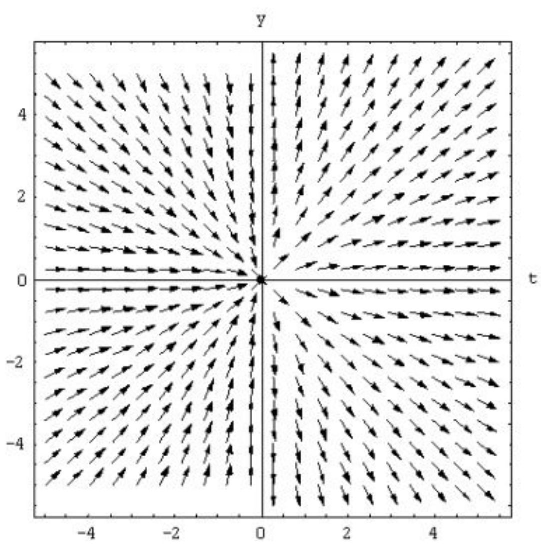
Since general solutions to differential equations are families of functions, sometimes it is helpful to visualize these solutions. As we mentioned before, solving differential equations is often pretty difficult, so sometimes we can "plot" our differential equations so we can visualize and approximate solutions. This graphical way to represent solutions to differential equations are called __________ fields or directional fields.

Method: For $\frac{dy}{dt} = f(t, y)$ [Remember y is a function so you can also represent this by $\frac{dy(t)}{dt} = f(t, y(t))$]. A solution to this equation has the property that at each point (t, y) of the solution curve the slope of the curve is $F(t, y)$. We visualize this by plotting the slop of the solution at selected points of the ty-plane. (sometimes instead of t, you could have x)

Example 7.10. *Plot the Slope Field for $y'(x) = 2x$*



Example 7.11. *The slope field below represents which of the following differential equations?*



a) $y' = yt$

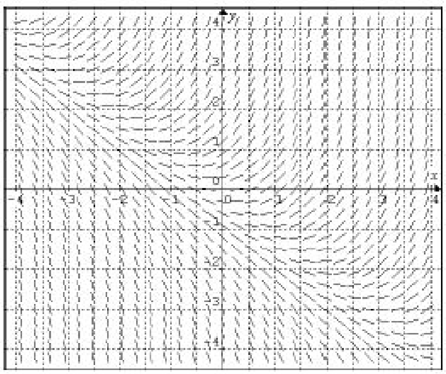
b) $y' = \frac{y}{t}$

c) $y' = -yt$

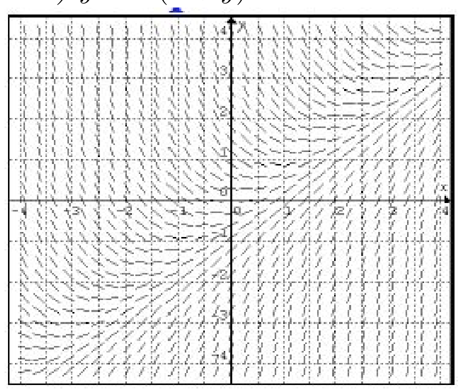
d) $y' = \frac{-y}{t}$

Example 7.12. *Match the following Differential Equations with their Slope Fields.*

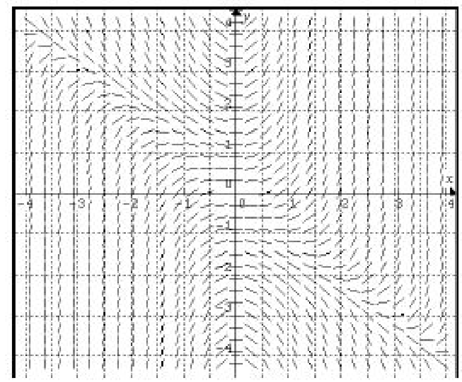
1) $y' = x - y$ *2)* $y' = x(x + y)$ *3)* $y' = x + y$



A

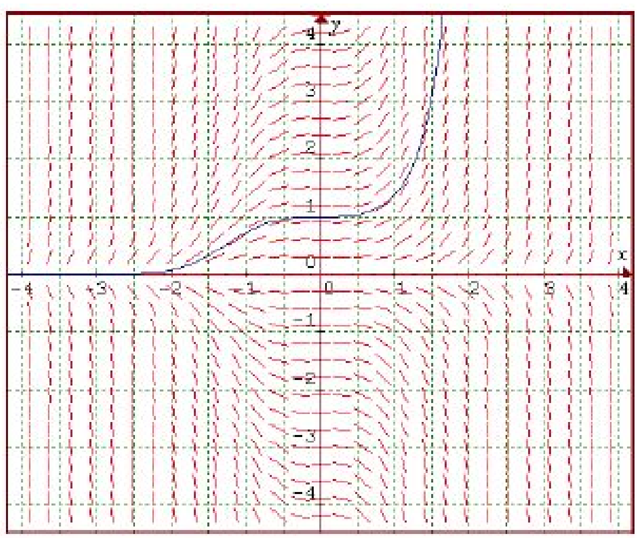


B



C

Example 7.13. *Recall in Example 7.6, we verified that* $y = Ae^{\frac{x^3}{3}}$ *is a solution to the differential equation* $\frac{dy}{dx} = x^2y$*. Below is the Slope field for* $\frac{dy}{dx} = x^2y$*. Use it to approximate the rate of change of* $y(x)$ *at* $(1, 2)$*.*



Note: Slope Fields are also the basis for many computer-based methods (Like Euler) for approximating solutions of differential equations.

There are many plotting tools online for plotting slope fields.
Here is one: `https://www.desmos.com/calculator/p7vd3cdmei`

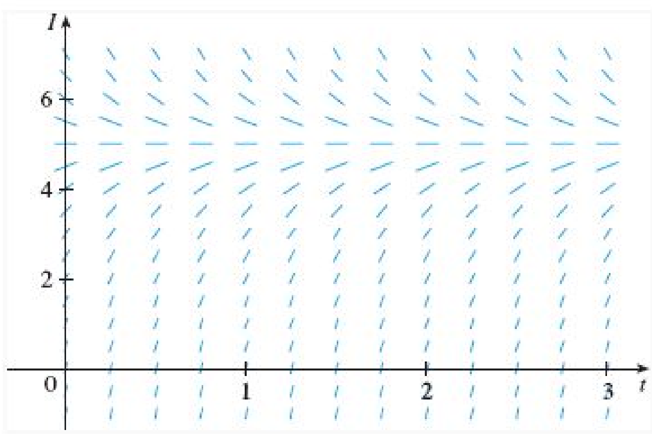
Example 7.14. *Kirchoff's Law states that a simple circuit containing a resistor of R ohms and an inductor of L Henrys in series with a source of electromotive force that supplies a voltage $V(t)$ volts at time t satisfies $V(t) = RI + L\frac{dI}{dt}$.*

a) Suppose the resistance is 12 ohms, the inductance is 4 Henrys, and a batery gives a constant voltage of 60V. Write $\frac{dI}{dt}$ in terms of these values.

b) Below is the slope field for this D.E. What can you say about the limiting value of the current?

(c)Identify any equilibrium solutions using the slope field below.

(d) Note if the switch is closed initially, the current starts with $I(0) = 0$. Use the direction field to sketch the solution curve.

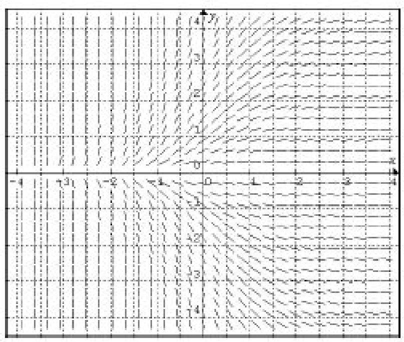


e) Find the solution to this differential equation with these values (you could also do this in general!).

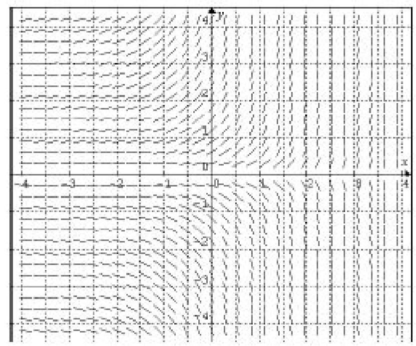
7.5.1 ICE -Slope Fields

1. Match the following Differential Equations with their Slope Fields.

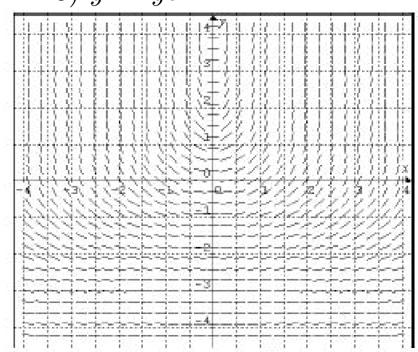
1) $y' = ye^{x}$ 2) $y' = xe^{y}$ 3) $y' = ye^{-x}$



A

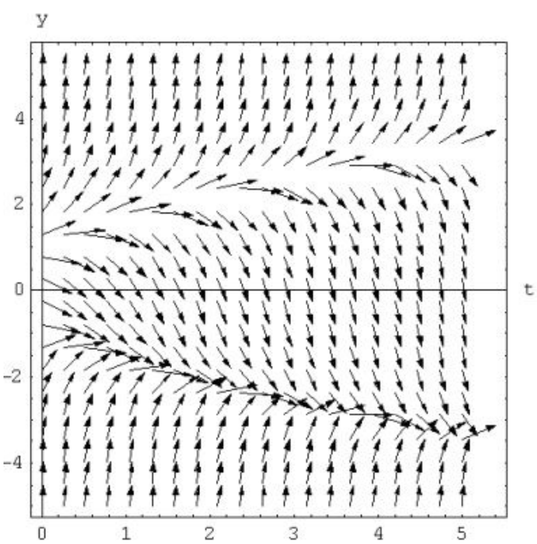


B

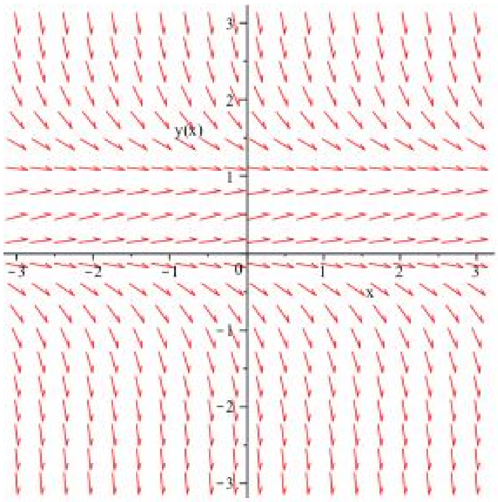


C

2. The arrows in the slope field below have slopes that match the derivative y' for a range of values of the function y and the independent variable t. Suppose that $y(0) = 0$. What would you predict for $y(5)$?

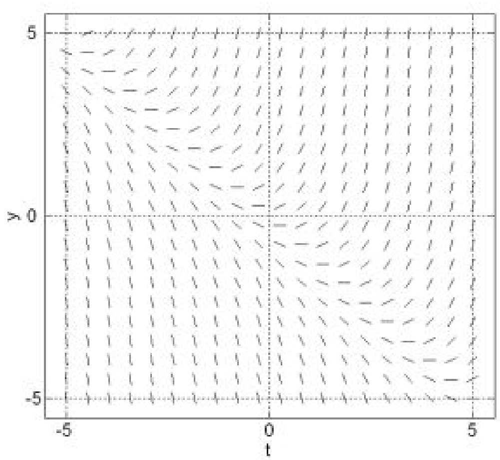


3. Below is the slope field for $\frac{dy}{dx} = y(1-y)$. As $x \to \infty$, the solution to the differential equation that satisfies the initial condition $y(0) = 2$ will...



   (a) Increase asymptotically to $y = 1$
   (b) Decrease asymptotically to $y = 1$
   (c) Increase without bound
   (d) Decrease without bound
   (e) Start and remain horizontal

4. The slope field below indicates that the differential equation has which form?



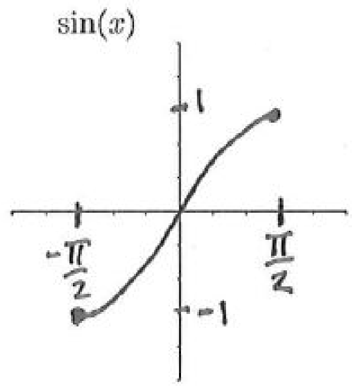
   (a) $y' = f(y)$
   (b) $y' = f(t)$
   (c) $y' = f(y, t)$
   (d) None of the above.

5. Suppose $\dfrac{dx}{dt} = 0.5x$ and $x(0) = 8$. Then the value of $x(2)$ is approximately...
   (a) 4 (b) 8 (c) 9 (d) 12 (e) 16

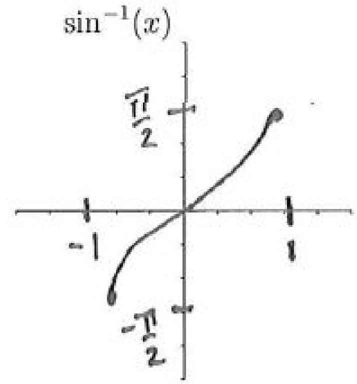
8 Inverse Trigonometric Functions

8.1 Review of Inverse Trigonometric Functions

Recall: What are the only types of functions that have inverses?
Is $y = \sin x$ one-to-one?
But $y = \sin x$ still has an inverse. What do we have to do?



D=
R=



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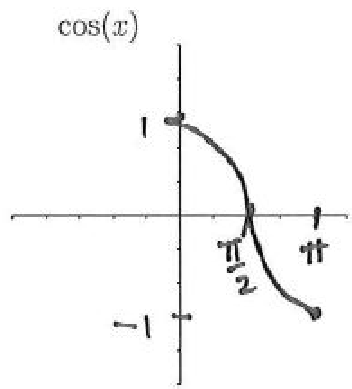

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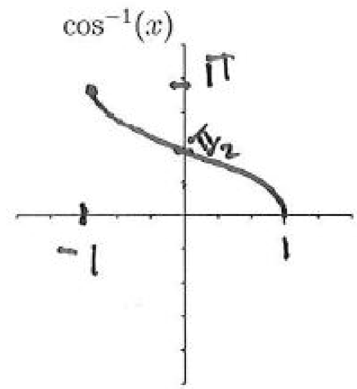

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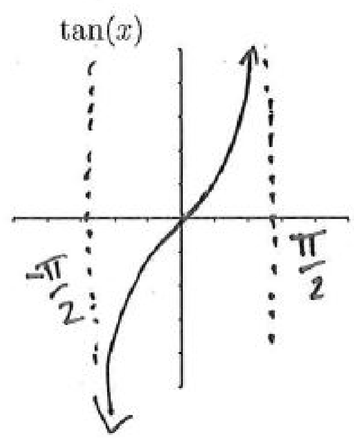

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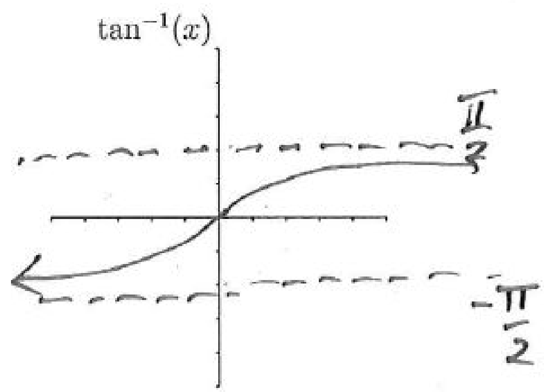
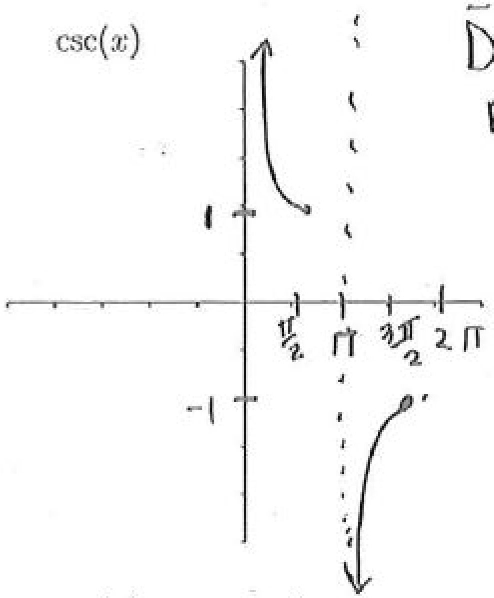


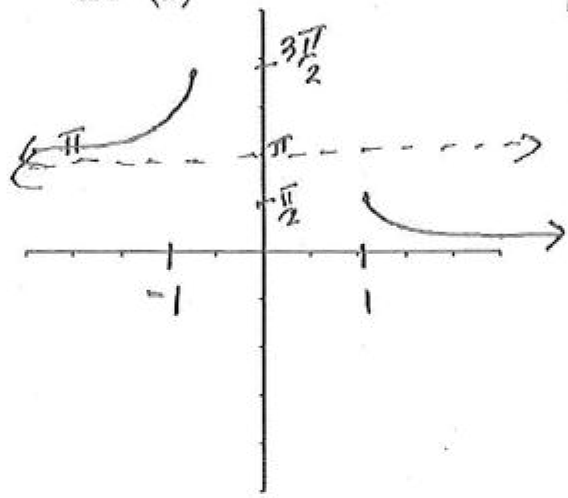
csc(x)



D =

R =

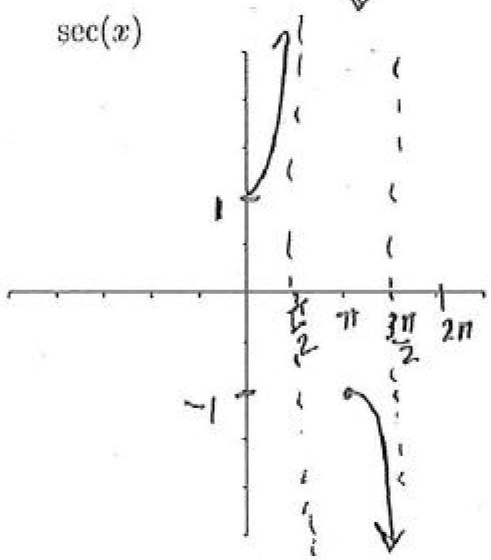
$\csc^{-1}(x)$



D =

R =

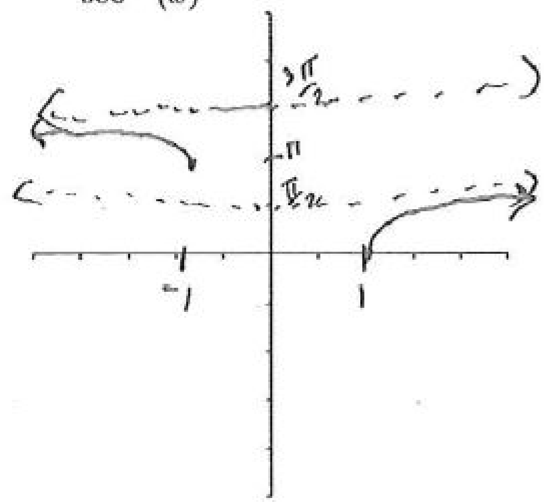
sec(x)



D =

R =

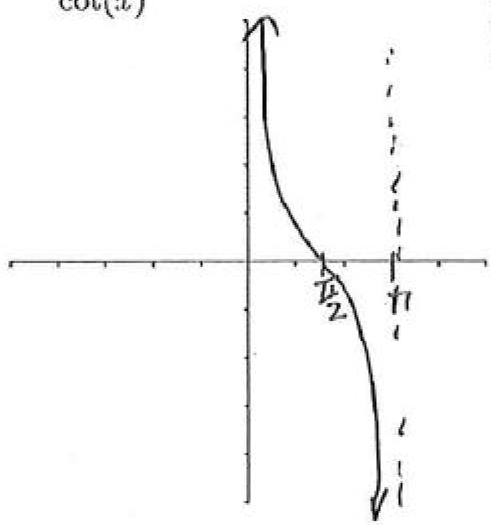
$\sec^{-1}(x)$



D =

R =

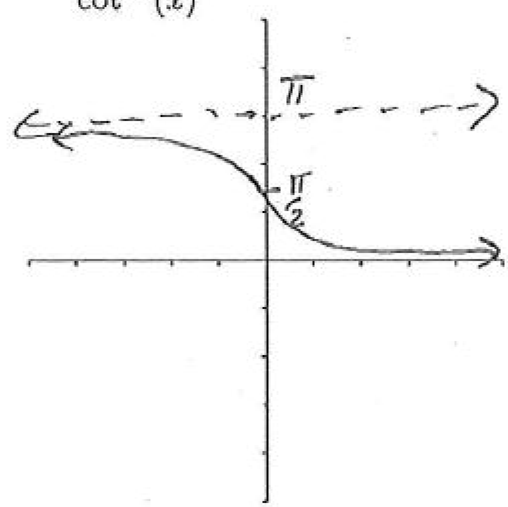
cot(x)



D =

R =

$\cot^{-1}(x)$



D =

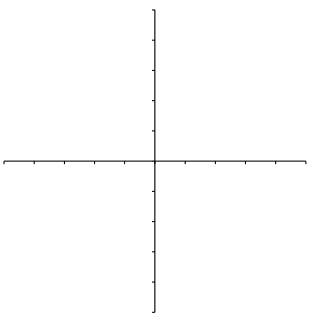
R =

Inverse Trig Functions			
Function	**Inverse**	**Domain of Inverse**	**Range of Inverse**
$\sin x$	$\sin^{-1} x$ or $\arcsin\ x$	$[-1, 1]$	$[\frac{-\pi}{2}, \frac{\pi}{2}]$
$\cos x$	$\cos^{-1} x$ or $\arccos\ x$	$[-1, 1]$	$[0, \pi]$,
$\tan x$	$\tan^{-1} x$ or $\arctan\ x$	$(-\infty, \infty)$	$(\frac{-\pi}{2}, \frac{\pi}{2})$
$\csc x$	$\csc^{-1} x$ or $\text{arccsc}\ x$	$(-\infty, -1) \cup (1, \infty)$	$(0, \frac{\pi}{2}] \cup (\pi, \frac{3\pi}{2}]$
$\sec x$	$\sec^{-1} x$ or $\text{arcsec}\ x$	$(-\infty, -1) \cup (1, \infty)$	$[0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$
$\cot x$	$\cot^{-1} x$ or $\text{arccot}\ x$	$(-\infty, \infty)$	$(0, \pi)$

Note: The range of the inverses are very important! These make answers unique (& correct!)

Important: Where you put your $^{-1}$ matters:
$\arcsin x = \sin^{-1} x \neq \dfrac{1}{\sin x}$. Instead $(\sin x)^{-1} = \dfrac{1}{\sin x} = \csc x$ and $\sin(x^{-1}) = \sin(\frac{1}{x})$.

Example 8.1. *Solve for θ:* $\arccos(\dfrac{\sqrt{3}}{2}) = \theta$.



Example 8.2. *Evaluate* $\tan(\arcsin(\dfrac{4}{5}))$.

Example 8.3. *Solve for x:* $y = 1 + 3\sin(2x)$.

8.2 Derivatives of Inverse Functions

If $y = \arcsin(x)$, what is y'?
To solve this, we can use implicit differentiation.
If $y = \arcsin(x)$, then this means $x =$

Since we have a range restriction for $\arcsin(x)$, which is $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$.
Recall, $\sin^2(y) + \cos^2(y) = 1$, so $\cos^2(y) =$

This implies $\cos(y) =$

But for $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$, $\cos(y)$ __________. Thus we can take the positive root.

And since $x = \sin(y) \Rightarrow x^2 =$

Thus we can substitute back and get $y' =$

The Derivative of $\arcsin(x)$ **(or** $\sin^{-1}(x)$**):** $\frac{d}{dx}[\arcsin x] =$

Since $\sin(y) = \cos(\frac{\pi}{2} - y)$, $\arcsin(x)$ and $\arccos(x)$ are just shifted by $\frac{\pi}{2}$.
That is $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$. Thus we can find $(\arccos(x))'$:

The Derivative of $\arccos(x)$ **(or** $\cos^{-1}(x)$**):** $\frac{d}{dx}[\arccos x] =$

Similarly, we can find...
The Derivative of arctan(x) **(or** $\tan^{-1}(x)$**):** $\frac{d}{dx}[\arctan x] =$

The Derivative of arccsc(x) **(or** $\csc^{-1}(x)$**):** $\dfrac{d}{dx}[\text{arccsc } x]=$

The Derivative of arcsec(x) **(or** $\sec^{-1}(x)$**):** $\dfrac{d}{dx}[\text{arcsec } x]=$

The Derivative of arccot(x) **(or** $\cot^{-1}(x)$**):** $\dfrac{d}{dx}[\text{arccot } x]=$

Example 8.4. *Find the derivative of* $y = \arccos(3x)$.

Example 8.5. *Find the derivative of* $y = \sqrt[3]{\arctan x}$.

Example 8.6. *Find the derivative of* $y = x^2 \arcsin(5x)$.

8.3 Antiderivatives of Inverse Trigonometric Functions

We can use our derivatives to find corresponding antiderivatives.

1. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx =$

2. $\displaystyle\int \frac{-1}{\sqrt{1-x^2}}dx =$

3. $\displaystyle\int \frac{1}{x^2+1}dx =$

4. $\displaystyle\int \frac{-1}{x\sqrt{x^2-1}}dx =$arccsc$x$ +C

5. $\displaystyle\int \frac{1}{x\sqrt{x^2-1}}dx =$ arcsec$x$ +C

6. $\displaystyle\int \frac{-1}{x^2+1}dx =$ arccot$x$ +C

Example 8.7. *Evaluate* $\displaystyle\int \frac{\arctan(x)}{x^2+1}dx.$

Example 8.8. *Evaluate* $\displaystyle\int \frac{x^2}{\sqrt{1-x^6}}dx.$

Example 8.9. *Evaluate* $\displaystyle\int \frac{1}{x^2+9}dx$.

Thus, to generalize, $\displaystyle\int \frac{1}{x^2+a^2}dx =$

Example 8.10. *Evaluate* $\displaystyle\int \frac{1}{2x^2+9}dx$.

8.4 Techniques for Integration

Strategies for Integration, So Far...

Step 1: Check to see if you recognize the function as a derivative of a function you already know.
Ex: $\int \sec^2 x dx = \tan x + C$

Step 2: Check to see if you recognize the function as a derivative of an inverse trigonometric function you already know.
Ex: $\int \frac{1}{1+x^2} = \arctan x + C$
Ex:

Step 3: Check for simple substitution. (Look for derivatives hanging out or inside functions or a way to make $w =$"complex part")
Ex: $\int \frac{\arctan x}{1+x^2} dx$

Step 4: Try To simplify or rewrite the integrand by using using algebra or a trigonometric identity/property.
Ex: $\int \sin(x)\sin(2x)dx = \int \sin(x)(2\sin(x)\cos(x))dx = \int 2\sin^2(x)\cos(x)dx$
Ex: $\int \frac{x^2+x^5}{x^4} dx = \int (x^{-2}+x)dx$

Step 5: Try the "Wouldn't it be nice..." problem solving method: Try to create a substitution or algebra or trigonometic manipulation to make the integrand look more like a derivative of a known function.
Ex. $\int \frac{1}{2x^2+9} dx$ (Examples 8.9 and 8.10)

8.5 ICE – Inverse Trig Functions

2 sides

1. Evaluate the integrals.

   a) $\displaystyle\int \frac{1}{\sqrt{25-8x^2}}dx$

   b) $\displaystyle\int \frac{e^{2x}}{\sqrt{1-e^{4x}}}dx$

c) $\int \frac{1}{\sqrt{x}(1+x)}dx$

d) $\int \frac{x}{3x^2-2}dx$

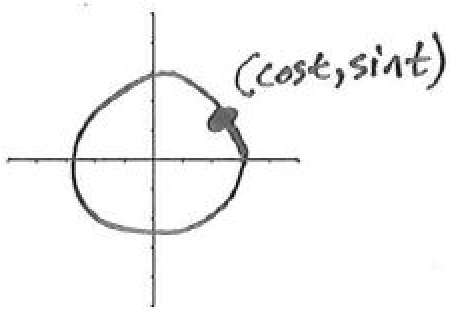
2. A balloon is rising from the ground at the rate of 6m/s from a point 100m from an observer, also on the ground. Use inverse trigonometric functions to determine how fast the angle of inclination of the observer's line of sight is increasing when the balloon is at an altitude of 150m.

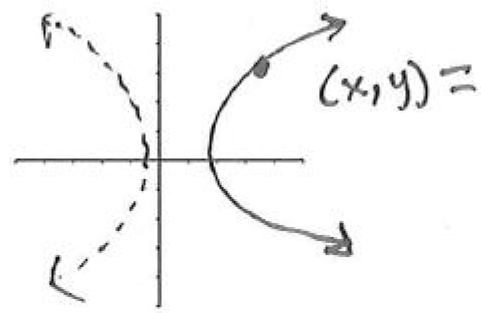
9 Hyperbolic Functions

9.1 Introduction of Hyperbolic Functions

You may remember from other math classes that we can represent the points of the unit circle using parametric equations. That is, each point (x, y) on the circle can be represented by ____________________. [We will cover this more in Calc 3.]. We can do the same with the unit hyperbola.

Recall a hyperbola is a curve of the form $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

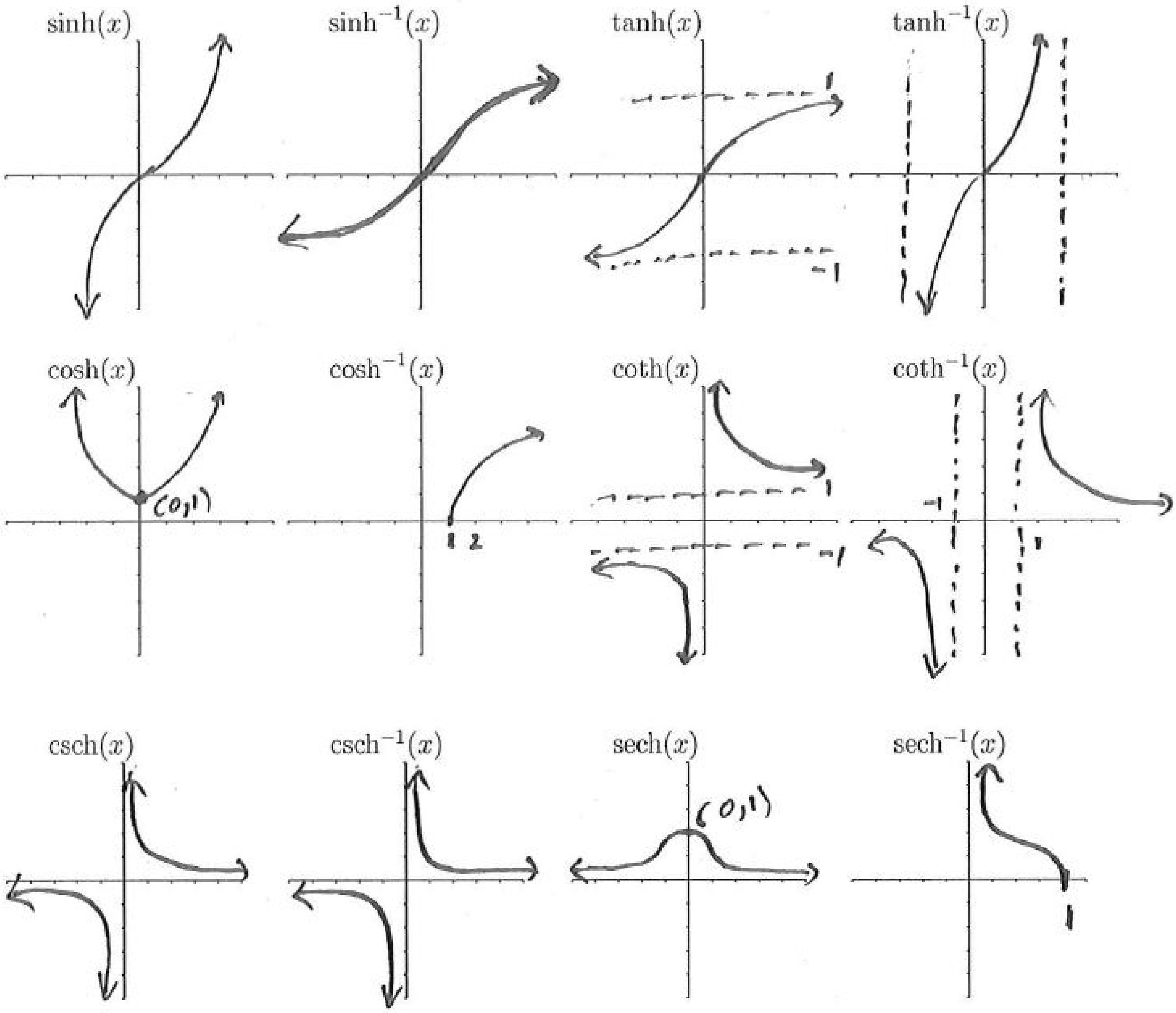




We can represent each point on the right half of the unit hyperbola $x^2 - y^2 = 1$ as a point:
What are these weird functions? They arise from combinting the exponential functions e^x and e^{-x} and are analogous to trig functions.

Hyperbolic Functions				
Function	**Notation**	**Definition**	**Domain**	**Range**
Hyperbolic Sine . .	$\sinh(x)$		$(-\infty, \infty)$	$(-\infty, \infty)$
Hyperbolic Cosine . .	$\cosh(x)$		$(-\infty, \infty)$	$[1, \infty)$
Hyperbolic Tangent . .	$\tanh(x)$		$(-\infty, \infty)$	$(-1, 1)$
Hyperbolic Cosecant .	$\text{csch}(x)$		$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
Hyperbolic Secant .	$\text{sech}(x)$		$(-\infty, \infty)$	$(0, 1)$
Hyperbolic Cotangent .	$\coth(x)$		$(-\infty, 0) \cup (0, \infty)$	$(-\infty, -1) \cup (1, \infty)$

Graphs of Hyperbolic Functions and Their Inverse Functions



Identities of Hyerbolic Functions

1) $\sinh(-x) =$

2) $\cosh(-x) =$

3) $\cosh^2 x - \sinh^2 x =$

4) $\sinh(a \pm b) = \sinh a \cosh b \pm \cosh a \sinh b$

5) $\cosh(a \pm b) = \cosh a \cosh b \pm \sinh a \sinh b$

6) $\coth^2 x - 1 = \text{csch}^2 x$

7) $1 - \tanh^2 x = \text{sech}^2 x$

Inverse Hyperbolic Functions				
Function	**Notation**	**Definition**	**Domain**	**Range**
Inverse Hyperbolic Sine	$\sinh^{-1}(x)$ or arcsinh x	$\ln(x+\sqrt{x^2+1})$	$(-\infty,\infty)$	$(-\infty,\infty)$
Inverse Hyperbolic Cosine	$\cosh^{-1}(x)$ or arccosh x	$\ln(x+\sqrt{x^2-1})$	$[1,\infty)$	$[0,\infty)$
Inverse Hyperbolic Tangent	$\tanh^{-1}(x)$ or arctanh x	$\frac{1}{2}\ln(\frac{1+x}{1-x})$	$(-1,1)$	$(-\infty,\infty)$
Inverse Hyperbolic Cosecant	$\text{csch}^{-1}(x)$ or arccsch x	$\ln(\frac{1}{x}+\sqrt{\frac{1}{x}+1})$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
Inverse Hyperbolic Secant	$\text{sech}^{-1}(x)$ or arcsech x	$\ln(\frac{1}{x}+\sqrt{\frac{1}{x}-1})$	$(0,1)$	$[0,\infty)$
Inverse Hyperbolic Cotangent	$\coth^{-1}(x)$ or arccoth x	$\frac{1}{2}\ln(\frac{1+x}{x-1})$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$

9.2 Derivatives of Hyperbolic Functions and their Inverses

Example 9.1. *Determine* $\dfrac{d}{dx}\sinh x$.

Example 9.2. *Determine* $\dfrac{d}{dx}\cosh x$.

Derivatives of Hyperbolic Functions

1) $\dfrac{d}{dx}(\sinh x) =$

2) $\dfrac{d}{dx}(\cosh x) =$

3) $\dfrac{d}{dx}(\tanh x) =$

4) $\dfrac{d}{dx}(\text{csch } x) =$

5) $\dfrac{d}{dx}(\text{sech } x) =$

6) $\dfrac{d}{dx}(\coth x) =$

Neat! We get similar derivative relationships like we have for trig functions!

Example 9.3. *Differentiate* $y = \sinh(\cosh x)$.

We can derive derivatives of inverse hyperbolic functions in ways that are similar to finding derivatives of inverse trig functions. That is we let $y = \cosh^{-1} x \Rightarrow x = \cosh y$ and use implicit differentiation and hyperbolic function identities to determine $\frac{d}{dx} \cosh^{-1} x$.

Derivatives of Inverse Hyperbolic Functions

1) $\dfrac{d}{dx}(\sinh^{-1} x) =$

2) $\dfrac{d}{dx}(\cosh^{-1} x) =$

3) $\dfrac{d}{dx}(\tanh^{-1} x) =$

4) $\dfrac{d}{dx}(\text{csch}^{-1} x) =$

5) $\dfrac{d}{dx}(\text{sech}^{-1} x) =$

6) $\dfrac{d}{dx}(\coth^{-1} x) =$

Example 9.4. *Differentiate* $y = \sinh^{-1}(2x)$.

9.3 Antiderivatives of Hyperbolic Functions and their Inverses

We can use our derivatives to find corresponding antiderivatives. Here are the ones you should memorize.

1. $\displaystyle\int \cosh x dx =$
2. $\displaystyle\int \sinh x dx =$
3. $\displaystyle\int \text{sech}^2 x dx =$
4. $\displaystyle\int \frac{1}{\sqrt{1+x^2}} dx =$
5. $\displaystyle\int \frac{1}{\sqrt{x^2-1}} dx =$
6. $\displaystyle\int \frac{1}{1-x^2} dx =$

Example 9.5. *Evaluate* $\displaystyle\int \frac{\cosh x}{\cosh^2 x - 1} dx$.

9.4 ICE – Hyperbolic Functions

1. Differentiate $y = x \sinh(\sin x)$.

2. Differentiate $y = \cosh^{-1}(\sqrt{3x})$.

3. Evaluate $\displaystyle\int \frac{e^x}{1 - e^{2x}} dx$.

4. Evaluate $\displaystyle\int \frac{e^x}{1 + e^{2x}} dx$.

5. Evaluate $\int_0^1 \frac{1}{\sqrt{16x^2+1}}dx$

6. Show that $\frac{d}{dx}\arctan(\tanh x) = \operatorname{sech}(2x)$

10 L'Hopital's Rule

We have explored many different methods for finding limits.

a) $\lim_{x \to 3} \dfrac{x^2 - x - 6}{x - 3}$

b) $\lim_{x \to 0} \dfrac{\sin x}{x}$

What if we wanted to find $\lim_{x \to 2} \dfrac{x - 2}{\ln(x - 1)}$

Today we will learn another method for finding limits with indeterminate forms.

10.1 Indeterminate Quotients

Defintion: Indeterminate Forms are expressions that

L'Hospital's (or L'Hopital's) Rule: Suppose f and g are both __________ and $g'(x) \neq$ __________ on an interval I that contains a. If we have an indeterminate form of type __________ or __________, then $\lim_{x \to a} \dfrac{f(x)}{g(x)} =$__________ (If the limit on the right side exists.) This also work for taking one sided limits or infinite limits.

Pf. Omitted; see text.

Example 10.1. *Evaluate* $\lim_{x \to 2} \dfrac{x - 2}{\ln(x - 1)}$

Example 10.2. *Evaluate* $\lim_{x \to 0} \dfrac{e^{2x} - 1}{\sin x}$

Example 10.3. *Evaluate* $\lim_{x \to \infty} \dfrac{(\ln x)^2}{x}$

Example 10.4. *Evaluate* $\lim_{x\to 0} \frac{x^2}{1+\cos x}$

10.2 Indeterminate Products

Now let's talk about indeterminate products that are indeterminate forms of type :

Fix: write $f \cdot g =$ __________ or __________

Example 10.5. *Evaluate* $\lim_{x\to\infty} \sqrt{x}e^{\frac{-x}{3}}$

Example 10.6. *Evaluate* $\lim_{x\to 1^-} (1-x)\tan(\frac{\pi x}{2})$

10.3 Indeterminate Differences

If we have an indeterminate form of a difference, that is of the type:

Fix: Try factoring out a common factor, rationalize the denominator, write as a common denominator, or try to write it in the form $\frac{\infty}{\infty}, \frac{-\infty}{-\infty}$ or $\frac{0}{0}$.

Example 10.7. *Evaluate* $\lim_{x\to 0^+} (\frac{1}{x} - \frac{1}{e^x - 1})$

Example 10.8. *Evaluate* $\lim_{x\to\infty}(x - \sqrt{x^2+1})$

10.4 Indeterminate Powers

If we have an indeterminate forms that involve powers are of the type:

Fix: Use a logarithmic substitution for the limit or write the function as an exponential.

Example 10.9. *Evaluate* $\lim_{x\to\infty}(x + e^x)^{\frac{1}{x}}$

Example 10.10. *Evaluate* $\lim_{\theta \to 0^+} (\sin\theta)^{\tan\theta}$

10.5 ICE – L'Hopital's Rule

1. Evaluate the limits.

   a) $\lim_{x \to 0} \dfrac{x - \sin x}{x^3}$

   b) $\lim_{x \to 0} \dfrac{1}{x} \arcsin(4x)$

   c) $\lim_{x \to \infty} (1 + \dfrac{1}{x})^x$

   d) $\lim_{x \to 0} \dfrac{x^2 + 5x}{x + \cos x}$

e) $\lim_{x\to\infty} \frac{x^2+5}{x\ln x}$

2. Show that $\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$. This shows that the exponential function approaches infinity faster than any power of x.

3. Use the idea from the previous question to show that the logarithmic function approaches infinity more slowly than any positive power of x.

4. Consider $\lim_{x\to\infty} \frac{\sqrt{ax+b}}{\sqrt{cx+d}}$ What happens if we try to use L'Hospital's Rule? Can we find this limit another way?

11 Integration By Parts

Un-doing the Product Rule

Integration by substitution corresponds with the differentiation rule:
Today we will learn the "product rule" for integration. First what is our product rule?

Integration by Parts:

$\displaystyle\int f(x)g'(x)dx =$

$\displaystyle\int u\ dv =$

$\displaystyle\int_a^b u\ dv =$

Note: We usually wait until the end to add the constant of integration.

Example 11.1. *Evaluate* $\displaystyle\int_1^4 xe^x dx.$

Example 11.2. *Evaluate* $\displaystyle\int x^2\cos(x)dx.$

Example 11.3. *Evaluate* $\int \ln x dx$.

Integral of Natural Log: $\int \ln x dx =$

Integral of Log: $\int \log_b x dx = x \log_b(x) - \frac{1}{\ln(b)} x + C$

11.1 Strategy for Integration By Parts

What type of functions do we want to pick for our u and dv? How do you choose?

For our u:

For our dv:

Helpful Acronym: LIATE

Logarithmic **I**nverse Trigonometric **A**lgebraic **T**rigonometric **E**xponential

Example 11.4. *Determine which functions make sense for u and for dv:*

a) $\int x^2 3^x dx$

b) $\int \arcsin x dx$

c) $\int (x^2 + 5x - 2)e^x dx$

d) $\int x \sec^2 x dx$ e) $\int x \ln(x+2) dx$ f) $\int x^4 \sinh x dx$

11.2 Cycling Integrals

Sometimes integration by parts seems to not be working... but it is...
Cycling integrals often occurs when both u and dv don't change much when you differentiate or integrate. Or when it seems like you keep getting *almost* the same thing OVER AND OVER Again.

Example 11.5. *Evaluate* $\int e^x \cos(2x) dx$.

Example 11.6. *Evaluate* $\int (\tan x)^2 dx$.

Example 11.7. *Evaluate* $\int \sin(\ln x) dx$.

11.3 ICE – Integration By Parts

2 sides!

1. Evaluate $\displaystyle\int x \sin(3x) dx$

2. Evaluate $\displaystyle\int \arctan x dx$

3. Use integration by parts to prove $\int x^n e^x dx = x^n e^x - n \int x^{n-1} e^x dx$.
   Then use it to evaluate $\int x^3 e^x dx$.

4. Evaluate $\displaystyle\int \frac{x}{x^2+9}dx$

5. Can I use any of my methods for integration to evaluate $\displaystyle\int \frac{4x+10}{x^2+2x-8}dx$? Why?

6. Can I use any of my methods for integration to evaluate $\displaystyle\int \tan^2 x dx$? Why?

7. Evaluate $\displaystyle\int_0^2 \ln(5^x)5^x dx$ [Hint: Careful, do we need integration by parts?].

8. Evaluate $\displaystyle\int 2^{\sqrt{x}}dx$ [Hint: Substitute first].

12 Integration Techniques for Higher Trig Functions

Suppose we want to evaluate $\int \tan^2 x dx$. What can we do?
$\int \tan^2 x dx$

Often when we have integrals with **higher** powers of trigonometric functions, we need to use trigonometric identities to solve them.

12.1 Trig Identities

Here are some some of the main identities we will use.

Basic Identities
$\sin^2\theta + \cos^2\theta = 1$
$1 + \tan^2\theta = \sec^2\theta$
$1 + \cot^2\theta = \csc^2\theta$

Power reduction formulas
$\sin^2\theta = \dfrac{1}{2}(1 - \cos(2\theta))$
$\cos^2\theta = \dfrac{1}{2}(1 + \cos(2\theta))$

Double-angle formulas
$\sin(2\theta) = 2\sin\theta\cos\theta$
$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$
$= 2\cos^2\theta - 1$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \cos\theta\sin\phi$
$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$
$\sin(A)\cos(B) = \frac{\sin(A-B)+\sin(A+B)}{2}$
$\sin(A)\sin(B) = \frac{\cos(A-B)-\cos(A+B)}{2}$
$\cos(A)\cos(B) = \frac{\cos(A-B)+\cos(A+B)}{2}$

Addition Formulas

12.2 Strategy for Integrating Powers of Sine and Cosine

Strategy for Evaluating $\displaystyle\int \sin^n x \cos^m x dx$:

Case 1: If the power of cosine is odd, save one cosine factor and use
Then use substitution of $w = \sin x$. See Example 1.

Case 2: If the power of sine is odd, save one sine factor and use
Then use substitution of $w = \cos x$. See Example 3.

Case 3: If the **both** powers of sine and cosine are even,
use our Power Reduction Formulas: $\sin^2 x = \frac{1}{2}(1 - \cos(2x))$ or $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$

***Notice except for Case 3, we are preparing our integral for substitution by pulling out our substitution for dw.*

Example 12.1. *Evaluate* $\displaystyle\int \cos^3 x \sin^{-2} x dx.$

Example 12.2. *Evaluate* $\int \cos^4 x dx$.

Example 12.3. *Evaluate* $\int \sin^5 x dx$.

12.3 Strategy for Integrating Powers of Tangent and Secant

Now suppose we have an integral of the form $\int \tan^n x sec^m x dx$, what derivatives could we see in this product?

Strategy for Evaluating $\int \tan^n x \sec^m x dx$:

Case 1: If the power of tangent is odd and you have *at least one secant*,
save one $\sec x \tan x$ factor and use

So that the rest of the integral is only in terms of secant (except for our one saved $\sec x \tan x$ factor). Then use substitution of $w = \sec x$. See Example 5.

Case 2: If the power of secant is even, save one $\sec^2 x$ factor and use
So that the rest of the integral is only in terms of tangent (except for our one saved $\sec^2 x$ factor). Then use substitution of $w = \tan x$.

If Case 1 and Case 2 don't apply, you need to be creative (e.g. $\int \tan^2 x dx$) or use substitution (e.g. $\int \tan x dx$).

Example 12.4. *Evaluate* $\int \tan^3 x \sec x dx$.

Example 12.5. *Evaluate* $\int \sec x dx$.

12.4 Strategy for Integrating Powers of Cotangent and Cosecant

Strategy for Evaluating $\int \cot^n x \csc^m x dx$:

*Note these are analogous to tangent and secant.

Case 1: If the power of cotangent is odd and you have *at least one cosecant*, save one $\csc x \cot x$ factor and use
So that the rest of the integral is only in terms of cosecant (except for our one saved $\csc x \cot x$ factor). Then use substitution of $w = \csc x$.

Case 2: If the power of cosecant is even, save one $\csc^2 x$ factor and use
So that the rest of the integral is only in terms of cotangent (except for our one saved $\csc^2 x$ factor). Then use substitution of $w = \cot x$.

If Case 1 and Case 2 don't apply, you need to be creative. (for example $\int \csc x dx$)

Example 12.6. *Evaluate* $\int \cot^4 x \csc^4 x dx$

12.5 Using Addition Trigonometric Properties

When we have integrals of the form $\int \sin(ax)\cos(bx)dx$, $\int \sin(ax)\sin(bx)dx$,or $\int \cos(ax)\cos(bx)dx$, we need to use the addition properties of trig functions.

Example 12.7. $\int \sin(8x)\cos(3x)dx$

12.6 ICE – Higher Trig Techniques for Integration

2 sides!

1. Evaluate $\int 4x^3 \tan(2x^4) \sec^3(2x^4) dx$
   (What do you do first?)

2. Evaluate $\int (1 + \sin(3x))^2 dx$

3. Evaluate $\int \sin^3 x\sqrt{\cos x}dx$

4. Evaluate $\int \cos x \cot^2 x dx$

13 Inverse Trigonometric Substitution Techniques for Integration

How would we evaluate $\int \frac{x}{\sqrt{x^2+4}} dx$?
How about $\int \dfrac{1}{\sqrt{x^2+4}} dx$?
How about $\int \dfrac{1}{\sqrt{3-x^2}} dx$?

What problem do we have when we try to evaluate $\displaystyle\int \frac{1}{x^2\sqrt{x^2+4}} dx$? or $\displaystyle\int \frac{1}{(4-x^2)^{\frac{3}{2}}} dx$

In this section we are going to again use the main trig identities used to simplify trigonometric integrals from the previous section.
These are:

Which form does $4 - x^2$ "somewhat" look like?

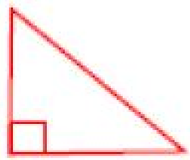
Let's consider the following substitution:

Example 13.1. *Use this substitution to evaluate* $\displaystyle\int \frac{1}{(4-x^2)^{\frac{3}{2}}} dx.$

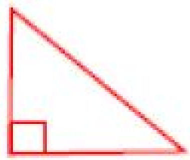
This technique is called __________ __________.

13.1 Strategy for Inverse Trigonometric Substitution

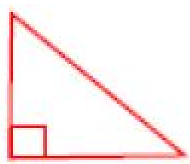
Case 1: We see the expression $\sqrt{a^2+x^2}$, we use the substitution __________
(with the restricted domain $[\frac{-\pi}{2}, \frac{\pi}{2}]$) Then we will use the identity: $1+\tan^2\theta = \sec^2\theta$
picture:



Case 2: We see the expression $\sqrt{a^2-x^2}$, we use the substitution __________
(with the restricted domain $[\frac{-\pi}{2}, \frac{\pi}{2}]$) Then we will use the identity: $1-\sin^2\theta = \cos^2\theta$
picture:



Case 3: We see the expression $\sqrt{x^2-a^2}$, we use the substitution __________
(with the restricted domain $[0,\pi]$ or $[0,\frac{\pi}{2})\cup[\pi,\frac{3\pi}{2})$)
Then we will use the identity: $\sec^2\theta - 1 = \tan^2\theta$
picture:



Note: This method is usually used for integrals with $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, but it can be also used for integrals with a^2+x^2, a^2-x^2, x^2-a^2 or other variations.)

Steps for Inverse Trig Substitution:

1. Look for the easy substitution or an inverse trig function you recognize.
2. Substitute for x picking either Case 1, 2, or 3 (see above).
3. Rewrite integral in terms of θ and solve integral.
4. Substitute answer back in terms of x. Note sometimes you need to use trig identities and/or the triangle for this.

In this section, assume all domains are nice if not otherwise noted.

Example 13.2. *Evaluate* $\displaystyle\int_1^2 \frac{1}{x^2\sqrt{x^2+4}}dx$

Example 13.3. *Evaluate* $\displaystyle\int \frac{1}{\sqrt{3-x^2}}dx$ *For* $x \in (-\sqrt{3}, \sqrt{3})$.

Example 13.4. *Evaluate* $\int \sqrt{9-x^2}dx$ *for* $x \in (-3,3)$.

Example 13.5. *Evaluate* $\int \frac{1}{\sqrt{x^2+2x-3}}dx \qquad$ *For* $x \in (1, \infty)$.

13.2 ICE – Inverse Trig Substitution

2 sides!

1. Evaluate $\displaystyle\int \frac{1}{(x^2+1)^{\frac{3}{2}}}dx$.

2. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{5-x^2}}dx$ for $x \in (1,2)$.

3. Can we approach the integrals $I_1 = \int \frac{x+1}{x^2+2x+3}dx$ $I_2 = \int \frac{x+1}{x^2+4x+3}dx$ and $I_3 = \int \frac{x+1}{x^2+4x+5}dx$ using any of our methods of integration thus far? Explain.

4. Evaluate $\displaystyle\int \frac{x-2}{\sqrt{x^2+1}}dx$. Hint: Split the integral up into a sum. You should not need to use inverse trig substitution unless you want to.

14 Integration Using Partial Fractions

14.1 Partial Fractions

Today we will discuss a method of integration for rational equations in which substitution may not help us.

To start, how would we add $\dfrac{3}{x-2}+\dfrac{1}{x+4}=$

Now suppose we were asked to evaluate $\displaystyle\int \frac{4x+10}{x^2+2x-8}dx$

Suppose we were given $\dfrac{4x+10}{x^2+2x-8}$. How do we find two simple fractions (called __________ __________) whose sum is $\dfrac{4x+10}{x^2+2x-8}$

14.2 Partial Fractions Procedure:

Given $\displaystyle\int \frac{R(x)}{Q(x)}dx$ (and using simple substitution or algebra techniques won't work).

Step 1: Make sure the (degree of $R(x)$) < (degree of $Q(x)$).
If it isn't, use polynomial division so that we can write the rational expression/partial fraction decomposition as $\dfrac{R(x)}{Q(x)} = S(x) + \dfrac{R'(x)}{Q(x)}$ with $\deg R'(x) < \deg Q(x)$. (This is like writing $\frac{10}{7} = 1 + \frac{3}{7}$)

Step 2: Factor the denominator $Q(x)$ into linear factors and non-factorable quadratic factors (complex roots with determinant $b^2 - 4ac < 0$). See note at bottom of page.

Step 3: Form the partial fraction depending on the following cases:

Case 1: $Q(x)$ can be factored into a product of non-repeated linear factors
e.g $Q(x) = (x-2)(x-3)(x)$.

The partial fraction decomposition is $\dfrac{R(x)}{Q(x)} =$

Case 2: $Q(x)$ can be factored into a product of linear factors, but at least one factor is repeated
e.g: $Q(x) = (x-3)(x-2)^3$
The partial fraction decomposition is $\dfrac{R(x)}{Q(x)} =$

KEY: We have a partial fraction for *each* power of $(x-r)$ up & including the mth power!

Case 3:The factored form of $Q(x)$ has a non-repeated, non-factorable quadratic factor
e.g $Q(x) = (x-3)(x^2+3)(x^2-x+1)$
The partial fraction decomposition is $\dfrac{R(x)}{Q(x)} =$

KEY: We have an $A_i x + B$ instead of an A_i for the numerator of the quadratic factor.

Case 4: The factored form of $Q(x)$ has a repeated, non-factorable quadratic factor
e.g $Q(x) = (x-3)(x^2+3)^2$
The partial fraction decomposition is $\dfrac{R(x)}{Q(x)} =$

KEY: This is a combination of Case 2 and Case 3.

Step 4: Clear denominators. (Multiply both sides by $Q(x)$)

Step 5: Solve for coefficients (the A's, B's, C's, etc).

Note: This procedure is theoretically possible because every polynomial with real coefficients can be factored into linear factors and irreducible quadratic factors. Example: $x^5 + 3x^4 + 4x^3 + 12x^2 + 4x + 12 = (x+3)(x^2+2)^2$. This answers the question, "Why do we stop at Case 4 and not have irreducible cubics, quartics, etc?" Even though this factoring is *theoretically* possible, it isn't always easy to factor a polynomial. (It was shown in the 1800's that polynomials of degree 5 or higher cannot be solved, in general, by using only arithmetic and square roots).

Example 14.1. *Evaluate* $\int \frac{1}{x(x^2+1)}dx$

Example 14.2. *Evaluate* $\int \frac{2x^2 - 7x - 24}{x^2 - 4x - 12} dx$

Example 14.3. *Evaluate* $\displaystyle\int \frac{dx}{e^x + e^{2x}}$

14.3 ICE – Partial Fractions

1. Explain why the coefficients A and B cannot be found if we set $\frac{x^2}{(x-4)(x+5)} = \frac{A}{x-4} + \frac{B}{x+5}$

2. True or false: The "easiest way" to evaluate $\displaystyle\int \frac{6x+1}{3x^2+x}dx$ is with partial fractions.

3. Evaluate $\displaystyle\int \frac{x+1}{x(x^2+4)}dx$.

4. Evaluate $\displaystyle\int \frac{1}{x - \sqrt[3]{x}} dx$ Hint: Use the substitution $x = u^3$.

15 Approximating Integrals Using Numerical Methods

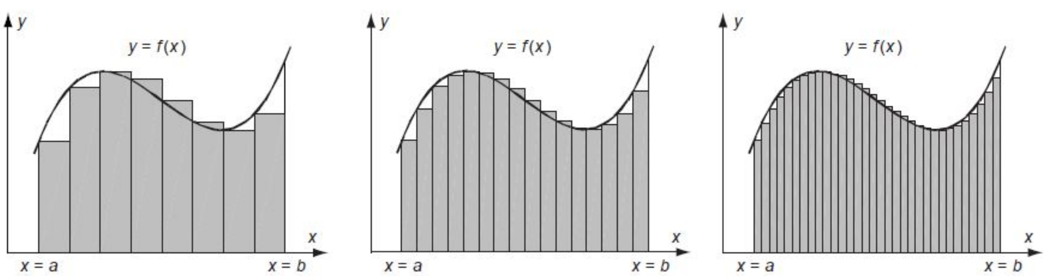
15.1 Approximating Integrals Review

We have been discussing sophisticated integration techniques. Unfortunately sometimes we have functions that aren't derivatives of a nice function. For example, the Gaussian function $f(x) = e^{-x^2}$ is not the derivative of any elementary function. That is, $\int e^{-x^2}dx$ does not exist! Other examples include the elliptical integral: $\int \sqrt{1-x^4}dx$, $\int \ln(\ln(x))dx$, $\int \frac{e^x}{x}dx$, $\int e^{e^x}dx$, $\int \frac{1}{\ln x}dx$, $\int \frac{\sin x}{x}dx$, $\int \sin(x^2)dx$, and $\int \sqrt{x}\cos(x)dx$.
But we can approximate $\int_a^b e^{-x^2}dx$ for finite limits and also approximate the function using something called Taylor Polynomials (coming soon to a Calculus class you are in!)

Furthermore, the improper integral $\int_{-\infty}^{\infty} e^{-x^2}dx$ can be evaluated using multivariate calculus and polar coordinates and shown to equal $\sqrt{\pi}$. (You can look forward to this in Calculus III!!) We will talk about crazy improper integrals later on in Calculus II!

Recall, in Calculus I, we discussed how we could approximate the area under a curve using __________.

$$\int_a^b f(x)\ dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x_i.$$



We usually use end points of the subintervals to find the height of our rectangles.[3] We also discussed how we could use just any point in the subintervals or the midpoints of subintervals.
When we use the **left** endpoint of our intervals to find the height of our rectangles, this is called a **Left** Riemann Sum.
When we use the **right** endpoint of our intervals to find the height of our rectangles, this is called a **Right** Riemann Sum.
When we use the **midpoint** of our intervals to find the height of our rectangles, this is called a **Midpoint** Approximation or the Midpoint Rule.

We are going to review the Midpoint Rule and introduce two other techniques.

[3] Image taken from suggested textbook Stewart's *Calculus* 8th edition, ©2016

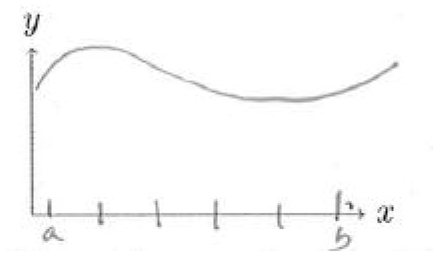
15.2 Midpoint Rule

Procedure for Using Midpoint Rule:

1. Divide the interval $[a, b]$ into n subintervals of **equal** width: $\Delta x = \dfrac{b-a}{n}$
   Let $x_0 = a < x_1 < ... < x_n = b$ where $x_i = a + i * \Delta x$

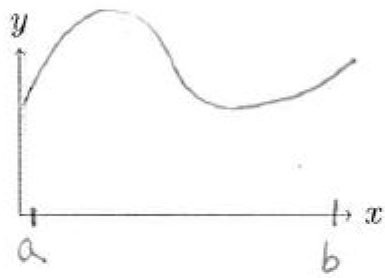
2. Find corresponding y values. Let $y_i =$ __________ This gives the height of our rectangles.

3. Sum up the areas of the rectangles.



Example 15.1. *Use the Midpoint Rules to approximate* $\int_1^5 x^2 dx$ *using 2 rectangles.*

15.3 The Trapezoid Rule



The Area of a Trapezoid =

Procedure for Using Trapezoid Rule:

1. Divide the interval $[a, b]$ into n subintervals of **equal** length: $h = \Delta x = \dfrac{b-a}{n}$
   Let $x_0 = a < x_1 < ... < x_n = b$ where $x_i = a + i * \Delta x$
2. Find corresponding y values. Let $y_i = f(x_i)$ This gives the bases of our trapezoids.
3. Sum up the areas of the trapezoids.

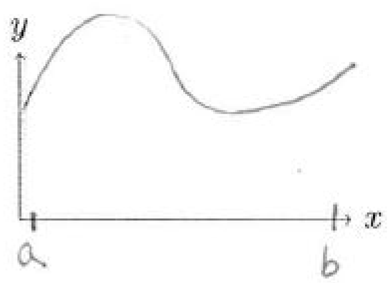
Trapezoid Rule:
$\displaystyle\int_a^b f(x)dx \approx \Delta x[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + ...f(x_{n-1}) + \frac{1}{2}f(x_n)]$
or $\approx \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + ...2f(x_{n-1}) + f(x_n)]$

where $h = \Delta x = \dfrac{b-a}{n}$

Example 15.2. *Use the Trapezoid rule to approximate* $\displaystyle\int_0^8 xe^x dx$ *using 4 trapezoids.*

15.4 Simpson's Rule

Now we will use parabolas instead of trapezoids or rectangles. (Woo!)



The area under the parabola determined by (x_0, y_0), (x_1, y_1), and (x_2, y_2) is

Procedure for Using Simpson's Rule:

1. Divide the interval $[a, b]$ into n subintervals of **equal** length. n MUST be **EVEN!**: $\Delta x = \dfrac{b-a}{n}$ Let $x_0 = a < x_1 < ... < x_n = b$ where $x_i = a + i * \Delta x$
2. Find corresponding y values. Let $y_i = f(x_i)$ This gives points of the parabolas.
3. Sum up the areas of the parabolas.

Simpson's Rule:

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4)...4f(x_{n-1}) + f(x_n)]$$

where $\Delta x = \dfrac{b-a}{n}$ and n is **even**.

Simpson's rule is related to the Trapezoid and Midpoint Rule:

Example 15.3. *Use Simpson's rule to approximate* $\int_0^8 xe^x dx$ *using* $n = 4$.

15.5 Approximating Error in Numerical Methods

The error in an approximation is the amount that needs to be added to the approximation to make it exact.

The error in the Midpoint Rule is $E_M = |\int_a^b f(x)dx - M_n|$

The error in the Trapezoid Rule is $E_T = |\int_a^b f(x)dx - T_n|$

The error in Simpson's Rule is $E_S = |\int_a^b f(x)dx - S_n|$

Error Bounds: Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$.
Then $|E_M| < \frac{K(b-a)^3}{24n^2}$ and $|E_T| < \frac{K(b-a)^3}{12n^2}$

Suppose $|f^{(4)}(x)| \leq K$ for $a \leq x \leq b$. Then $|E_S| < \frac{K(b-a)^5}{180n^4}$

Example 15.4. *Find the maximum error in using S_6 to calculate $\displaystyle\int_1^3 \cos(2x)dx$.*

Example 15.5. *a) Find the maximum error in our trapezoid approximation, T_4 for $\displaystyle\int_0^8 xe^x dx$*

b) How large should we take n to be in order to guarantee that our trapezoid approximation is within 0.1?

15.6 ICE – Numerical Approximation

Recall there is no elementary indefinite integral for the Gaussian function $f(x) = e^{-x^2}$. This function shows up in many applications including statistics and quantum mechanics. Let's use numerical methods to approximate the Gaussian Function over the interval $[1, 13]$

Given $\displaystyle\int_1^{13} e^{-x^2}\,dx$

a) Set up the sums to find T_4 and S_4

Turn over!

b) How large do we have to choose n so that T_n is accurate to within 0.001?

16 Strategy for Integration

So many integration techniques! How do we choose?!?!
Actually, it seems like we have a lot of choices, but most are just variations of ________.
My best suggestion for gaining intuition about integration strategy is

Gameplan for integration:

1. Do I recognize the integrand as a function I know the antiderivative for? Check for derivatives of $\arctan x$, $\arcsin x$, $\text{arctanh} x$ etc.

2. Can I use a basic substitution? Often you can set w as an "inside" function or the whole denominator (so our integral looks like $\frac{1}{w}$). Be on the lookout for "inside" functions that have derivatives hanging out in the integrand or if you have a linear expression (constant derivative -yay!) inside another function(eg if you have $\sqrt[n]{ax+b}$). And remember you can set w as a sum of terms.

3. Can I simplify the integrand to make my life easier? (Expand a binomial, find a common denominator, use polynomial division, rewrite in terms of $\sin x$, $\cos x$?)

4. Are there only higher powers of trig functions? If so, use higher trig substitution and/or trig identities.

5. Do I see a version of this expression $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$, $\sqrt{a^2-x^2}$? Use Inverse Trig substitution *You may need to Complete the Square!*

6. Do I have integrand that is a rational expression? Then I may be able to use partial fractions.

7. Can I use integration by parts? Often you will see a product of functions or a "loner" like $\ln x$, $\arctan x$, etc. If you have a product, one function will be something you cannot integrate right away but can differentiate, and the other is a function you can integrate.

8. Remember often you must use more than one method. (Usually one of these methods is substitution)

Still stuck? Then try again using a different method. Most of the time you need a creative substitution as a primary step before seeing a suitable method. Or you need to try integration by parts. Go back and see if completing a square will help you. Also beaware of "cycling integrals!"

Last ditch effort: Think back and try to relate the integral to another integral you may have done in class, homework, or ICE. This is where practice really really helps!

Example 16.1. *For each integral, suggest one approach that you think is appropriate to evaluate the integral.*

a) $\displaystyle\int \frac{x^2 - x}{2x^3 - 3x^2 + 2}dx$

b) $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$

c) $\displaystyle\int \frac{2x^2}{x^2 - 2x + 8}dx$

d) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}dx$

e) $\displaystyle\int \sin^3 x \cos^5 x dx$

f) $\displaystyle\int \frac{1}{4x^2 - 9}dx$

g) $\displaystyle\int x \csc x \cot x dx$

h) $\displaystyle\int 3^x \cos(2x) dx$

i) $\displaystyle\int \arcsin x dx$

j) $\displaystyle\int \frac{1}{\sqrt{1-x^2}\arcsin x}dx$

k) $\displaystyle\int \tan^3 x \sec^5 x dx$

l) $\displaystyle\int \frac{3x-1}{x+2}dx$

Interesting Substitutions:

m) $\displaystyle\int \frac{\ln x}{x\sqrt{1+(\ln x)^2}}dx$

n) $\displaystyle\int \frac{1}{1+e^x}dx$

16.1 Interesting Examples

Example 16.2. *Evaluate* $\displaystyle\int e^{x+e^x}\,dx$

Example 16.3. *Evaluate* $\displaystyle\int (1+\sqrt[3]{x})^4 dx$ *(Neat Substitution Technique!)*

Example 16.4. *Evaluate* $\int \frac{\arctan x}{x^2} dx$

Example 16.5. *Evaluate* $\int \frac{\sec(x)\tan(x)}{\sec^2(x) - \sec(x)} dx$

16.2 ICE – Review of Integral Techniques

1. I have $\sin^n x$, or $\sin^n x \cos^m x$, and n is odd like $\int \sin^3 x \cos^4 x dx$. What do I do?

2. I have $\cos^m x$, or $\sin^n x \cos^m x$ like $\int \sin^4 x \cos^3 x dx$, and m is odd. What do I do?

3. I have $\sin^n x$, $\cos^m x$, or $\sin^n x \cos^m x$, but only even powers like $\int \sin^2 x \cos^2 x dx$. What do I do?

4. I have $\tan^n x \sec^m x$, with at least 1 value of each and n is odd like $\int \tan^3 x \sec^3 x dx$. What do I do?

5. I have $\tan^n x \sec^m x$, with at least 1 value of each and m is even like $\int \sec^4 x \tan^2 x dx$. What do I do?

6. I have $\tan^2 x$ without any even powers of $\sec x$ or $\cot^2 x$ without any even powers of $\csc x$ like $\int \tan^2 x dx$. What do I do?

7. I have a bunch of e^x, a^x, or e^{nx} (usually where normally I would have x and x^n's) like $\int \frac{1}{e^x + e^{2x}} dx$. What can I do?

8. I have a product of two or more functions like like $\int x \sin x dx$ or $\int x \ln x dx$. What can I do?

9. I have a product, one factor is a value of x and the other is a function I can't integrate right away like $\int x \arcsin x dx$. What can I do?

10. I have a product, I can integrate one part of product (usually powers of trig functions), but the problem is there is an x term which makes my life a bit difficult like $\int x \tan^2 \sec^2 x dx$ What can I do?

11. I have a version of square root $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$, $\sqrt{a^2-x^2}$. First I look for a version of Or what can I do?

12. I have a version of square root $\sqrt{x^2 \pm bx + a^2}$, $\sqrt{x^2 \pm bx - a^2}$, $\sqrt{a^2 \pm bx - x^2}$. What can I do?

13. I have integrand that is a rational expression like $\int \dfrac{x+2}{x2-5x+6}dx$. What can I do? (sometimes there are 2 approaches)

14. I have a linear "inside" function like $\sqrt[n]{ax+b})$ like $\int \sqrt{3x+5}dx$. What can I do?

15. I have a term like $\int(1+\cos(2x))^2 dx$ or $\int(1+w^3+6w)(w^3)dx$. What can I do?

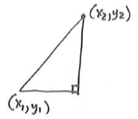
16. I have a fraction with trig functions like $\int \dfrac{\tan x}{\sec x}dx$ or $\int \dfrac{\sin x}{\cos^2 x}dx$. What can I do?

17. I have a function with a sum in the numerator like $\int \dfrac{x+5}{\sqrt{1-x^2}}dx$ or $\int \dfrac{x^2+6}{\sqrt{x^2+a^2}}dx$ What could I do?

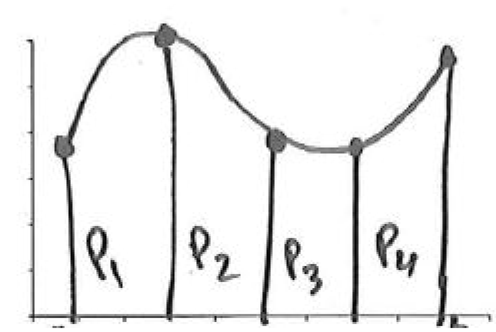
18. I have only 1 function and I don't know how to integrate it like $\int \arctan x$. What can I do?

17 Arc Length

It is easy to calculate distance when someone is traveling in a straight line:



But often this is not the case. So it is helpful to know how to calculate the length of a curve. We will just give an idea of the derivation for the Arc Length Formula. See your text for details.

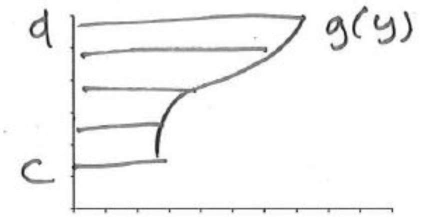


Mean Value Theorem: If f is continuous on $[a, b]$,
then there is a c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Arc Length Formula:
If f' is __________ on [a,b], then the length of the curve $y = f(x)$ over [a,b] is L=

Example 17.1. *Find the arc length of* $y = 3 \ln x - \frac{x^2}{24}$ *over* $1 \leq x \leq 6$.

Remember sometimes when we were finding the area under a curve, it was helpful not to integrate with respect to x, but instead with respect to ________. We have a similar situation here.



Arc Length for the curve $x = g(y)$ (g' continuous) over $x \in (a, b)$ and $y \in (c, d)$ is

Arc Length Function: By the Fundamental Theorem of Calculus, we can generalize the length of a curve so that we a function that represents the length of a curve.
For a curve with initial point $(a, f(a))$, The arc length function is $s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} dt$

Example 17.2. *Set up an integral to find the length of the curve of* $x = \sqrt{1 - y^2}$ *for* $\frac{-1}{2} \leq y \leq \frac{1}{2}$.

Example 17.3. *Find the length of the curve* $y = x^{\frac{2}{3}}$ *between* $x = 0$ *and* $x = 8$.

17.1 ICE – Arc Length

Suppose we want to find the length of the curve $y = f(x) = \ln(x + \sqrt{x^2 - 1})$ on the interval $[1, \sqrt{2}]$ [Notice this function is $f(x) = $arccosh x.] **Hint: USE THIS FACT for parts a and c**.

a) Calculate $f'(x)$. What problem do we have?

b) How can we deal with this problem?

c) Find $f^{-1}(x)$ and use it to write our function in the form $x = g(y)$. Hint: There is a quick way to do this and a long way.

d) Use $x = g(y)$ to calculate the arc length.

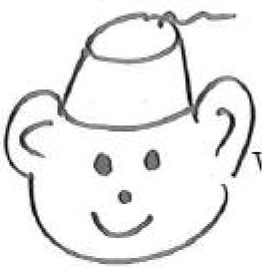
18 Surface Area

18.1 Deriving Surface Area of a Revolution

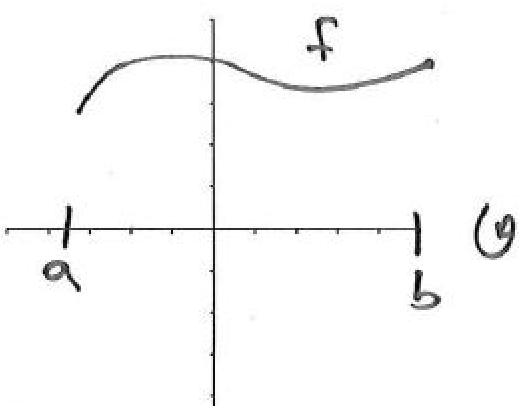
Earlier, we learned how to calculate the volume of solids we make by rotating these areas around lines. Now we are going to learn how to calculate the __________ __________ of these solids. First let's recall the how to find the area of a circular sector

Thus the lateral surface area of our cone is

What about a more complicated example? Say perhaps a section of the circular cone. Which makes us have more of an Abu the Monkey's hat instead of a dunce's hat.



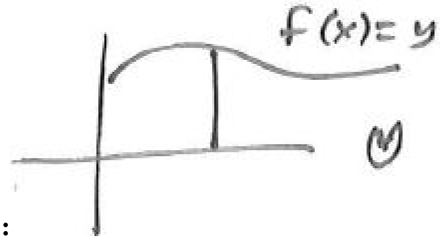
18.2 Surface Area of a Revolution about x-axis



The "typical element" is a section of a __________ with slant length =
and the radius is

Thus the surface area of our typical element is

Important: When rotating around the **x-axis** we will always have: 2π__________ or 2π__________
BUT, when finding our length (arc length) we can do this either with respect to x or y.



Surface Area when rotating about x-axis:

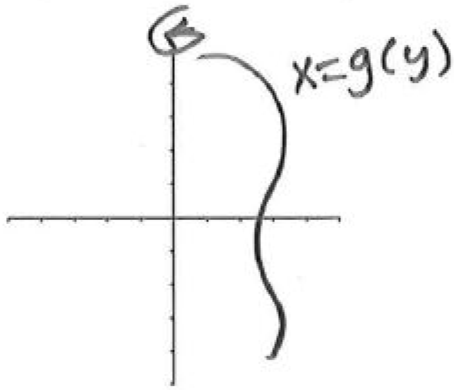
Integrating with respect to x: The surface area obtained by rotating the curve $y = f(x)$ over $a \leq x \leq b$ about the x-axis is S=

Integrating with respect to y: The surface area obtained by rotating the curve $x = g(y)$ over $c \leq y \leq d$ about the x-axis is S=

Example 18.1. *Setup an integral representing the surface area obtained by rotating* $x = \dfrac{y^2}{9} + 2$ *about the x-axis for* $3 \leq x \leq 6$.

Example 18.2. *Set up an integral representing the surface area obtained by rotating* $x = \dfrac{y^2}{9} + 2$ *about the x-axis for* $3 \leq x \leq 6$ *but integrate with respect to x.*

18.3 Surface Area of a Revolution about y-axis



The "typical element" is a section of a __________ with slant length =
and the radius is

Thus the surface area of our typical element is

Important: When rotating around the **y-axis** we will always have: 2π__________ or 2π__________
BUT as before, when finding our length (arc length) we can do this either with respect to x or y.

Surface Area when rotating about y-axis:
Integrating with respect to x: The surface area obtained by rotating the curve $y = f(x)$ over $a \le x \le b$ about the y-axis is S=

Integrating with respect to y: The surface area obtained by rotating the curve $x = g(y)$ over $c \le y \le d$ about the y-axis is S=

Example 18.3. *Set up an integral that finds the surface area obtained by rotating* $x = \sqrt{y-1}$ *about the y-axis for* $1 \le y \le 2$

Example 18.4. *Find the surface area obtained by rotating* $y = 1+x^2$ *about the y-axis for* $2 \le x \le 3$*, and integrate with respect to y.*

Guidelines for finding Surface Area from a curve rotated about either axis:

1. Sketch the general idea of the curve.
2. Determine if we have $2\pi x$ (or $2\pi g(y)$) or $2\pi y$ (or $2\pi f(x)$).
3. Decide which variable you want to integrate with respect to and take the corresponding version of the arc length formula $\sqrt{1+(f'(x))^2}$ or $\sqrt{1+(g'(y))^2}$
4. Rewrite $2\pi x$ in terms of y if you are integrating with respect to y. And rewrite $2\pi y$ in terms of x if you are integrating with respect to x.
5. Make sure you have to correct limits of integrating.
6. Calculate the integral.

18.4 ICE – Surface Area

1. Set up an integral representing the surface area obtained by rotating $y = 1 + e^x$ about the *x-axis* for $0 \leq x \leq 1$

2. Set up an integral representing the surface area obtained by rotating $y = 1 + e^x$ about the *x-axis* for $0 \leq x \leq 1$ but integrate with respect to y.

3. Set up an integral representing the surface area obtained by rotating $y = 1 + e^x$ about the *y-axis* for $0 \leq x \leq 1$

4. Set up an integral representing the surface area obtained by rotating $y = 1 + e^x$ about the *y-axis* for $0 \leq x \leq 1$ but integrate with respect to y.

5. Set up an integral representing the surface area obtained by rotating the region bounded by $y = 1 + e^x$ and $y = \sqrt{x}$ for $0 \leq x \leq 1$ about the line $y = -1$

6. Set up an integral representing the surface area obtained by rotating the region bounded by $y = 1 + e^x$ and $y = \sqrt{x}$ for $0 \leq x \leq 1$ about the line $y = -1$ integrating with respect to y.

19 Using Integrals to Calculate Work and Hydrostatic Force

19.1 Work with constant Force

When we say something is work, usually we mean the total amount of effort we need to put forth to complete a task. In physics it has a more technical meaning that depends on the idea of ________.
What is a force? We often think of it as a stress on an object, a ________ or ________ on an object.
Example: Newton's Second Law of Motion:

Note: We will assume the acceleration due to gravity is $g = 9.8$ m/s^2
If we have constant mass and acceleration, the force F is also a constant and the work done is defined to be :

Force F= and Density $\rho =$ so Mass $m =$

Units:

Metric:	**US:**
Mass is measured in kilograms (kg)	Mass is measured in pounds (lbs)
Displacement is measured in meters (m)	Displacement is measured in feet (ft)
Time is measured in seconds (s)	Time is measure in seconds (s)
Force is measured in Newtons (N)	Force is measured in pounds (lbs)
Work is measured in Joules (J)	Work is measured in foot-pounds (ft-lbs)

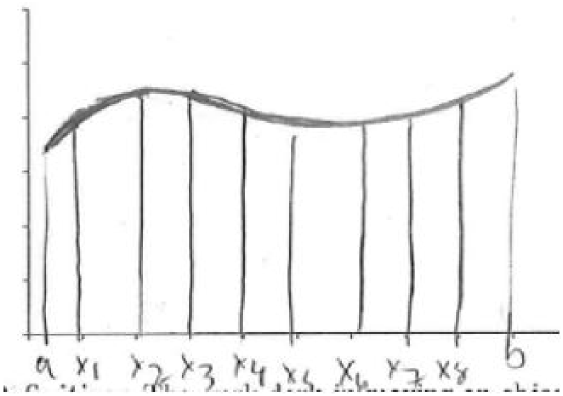
Example 19.1. *How much work does Harriet do when she lifts a 20 kg dumbbell a height of 3 meters?*

Example 19.2. *How much work does Barry do when he lifts a 30 lb weight 7ft off the ground?*

19.2 Work with varying Force

So we can determine the work done when we have a constant force (see examples 1-2). But what do we do when the force is varying?

Suppose I have an object moving along the x-axis and I have a varying force. So I can write my force as a function of x, say, __________.



Common Idea: Notice regardless if we are calculating area, volume, surface area, or work, we slice small enough so that we can convert the problem to its simplest form and calculate the area/volume/surface area/work of the typical slice. If we slice small enough in this example, we can treat the force as a constant and find the work on a typical slice and then add up all the parts.

Definition: The work done in moving an object from a to b of a force $f(x)$ is $W =$__________.
Sometimes we are given the force function, but often we have to create the force function.

Example 19.3. *Suppose an object is located a distance x feet from the origin has a force of* $x^3 + 1$ *acting on it. How much work is done in moving it from* $x = 1$ *to* $x = 2$ *ft.*

Hooke's Law: The force required to maintain a spring stretched x units beyond its natural length is proportional to x:

Note: k is a __________ constant which includes g called the __________ constant & we assume x is not

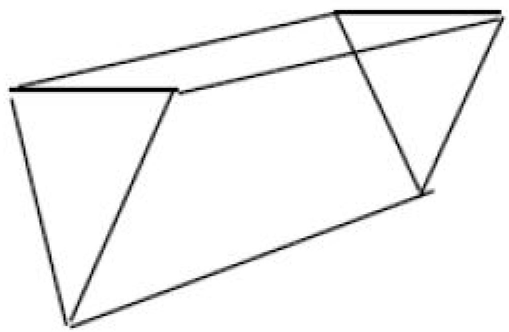
Example 19.4. *A spring has a natural length of 20 cm. If a 25-N force is required to keep it stretched to a length of 30 cm. How much work is required to stretch it from 20 cm to 24 cm? [Careful of units!]*

Note: Hooke's Law is nice because it gives us the general form for the varying force. In the next problems, we have to find the force function.

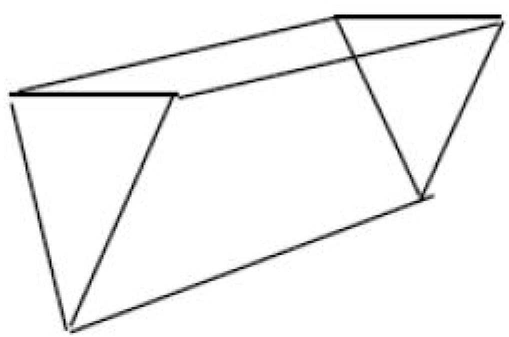
Example 19.5. *a) A 800 lb box of baby chinchillas is at the bottom of a 500 ft pit. Luckily you were able to attach a rope that weighs 2 lb/ft to the top of the box. How much work will you have to do to save the baby chinchillas (bring them to the top)?*

b) Notice that our first step was to draw a picture and set up a coordinate system (set $x = 0$ somewhere). Sometimes other professors/Webassign will use a different convention than Dr. Harsy. So let's use a different coordinate system for this.

Example 19.6. *a) Suppose a tank has a shape of triangular prism with a height of 4 ft and length of 6 ft and a width of 2 ft with a 1 foot spout. Suppose it is completely filled with water. Set up an integral to represent the work required to empty the tank by pumping all the water out of the spout. Use the fact that the weight of water is 62.5 lb/ft*^{3}.



b) Answer the same question as part a. Only now suppose the tank is only filled up to a height of 3 ft.



19.3 Hydrostatic Force

We can also use methods similar to calculating work to calculate the force we feel due to water pressure.

Remember Force F= and Density $\rho =$ so Mass $m =$ & V=

When we swim under water, the water above us has a force on us. We feel "pressure" we define pressure $P =$

That is Pressure = __________ $\times$ __________ $\times$ __________.
Our units for pressure are N/m^2 or a Pascal. Sometimes we use kPa (kilopascal).

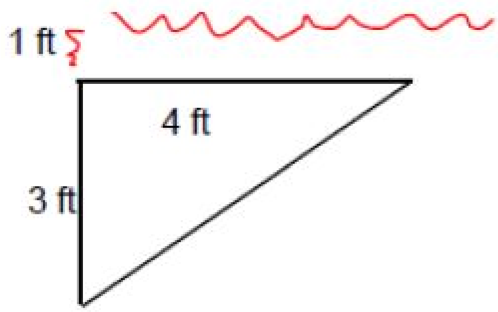
Fact 1: The density of water is $\rho = 1,000$ kg/m^3 or 62.5 lb/ft^3
Fact 2: The pressure at any point in a liquid is the same in all directions.
Fact 3: The pressure in any direction at a depth d in a fluid with density ρ is $P =$ __________
Fact 4: If we have a constant pressure P acting on a surface with area A. Then the hydrostatic force that acts on the area is $F =$

Careful, if the pressure varies with the depth, then we have to set up an integral resulting from taking the limit of a Riemann Sum of the Force at each depth.

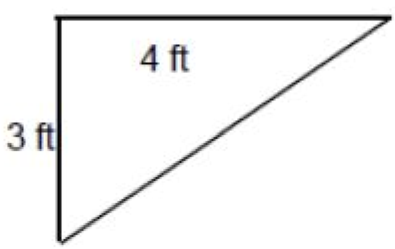
Example 19.7. *A tank is 4 meters wide, 5 meters long, 3 meters high and contains a liquid with density 800 kg/m^3 to a depth of 2 meters.*
a) Find the hydrostatic force on the bottom of the tank.

b) Set up an integral to find the hydrostatic force on one end of the tank.

Example 19.8. *a) Set up an integral that expresses the hydrostatic force against one side of the vertical plate that is submerged in water with the shape indicated by the picture below.*

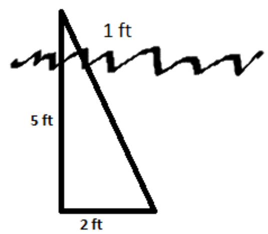


b) Now suppose the same plate is floating 1 ft above the water. Set up an integral that calculates the hydrostatic force for this situation.

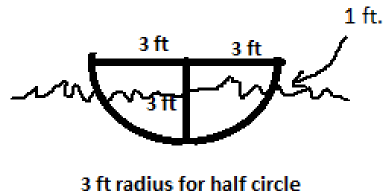


19.4 ICE – Work and Hydrostatic Force

1. Set up an integral that expresses the hydrostatic force against one side of the vertical plate that is submerged in water with the shapes indicated by the pictures below.

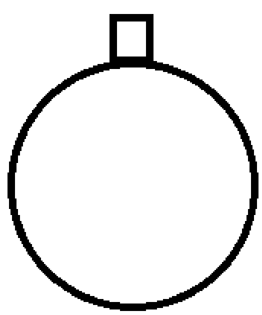


a)



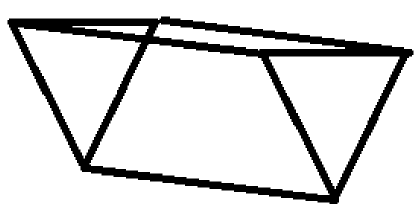
b)

2. Suppose a tank has a shape of a sphere with a radius of 3 ft with a 1 foot spout. Suppose it is completely filled with water. **Set up the integral** used to find the **work** required to empty the tank by pumping all the water out of the spout. Use the fact that the weight of water is 62.5 lb/ft^3. **Hint: Set $x = 0$ at the diameter.**



3. A chain lying on the ground is 10m long and its mass is 80 kg. Set up the integral to determine how much work is required to raise one end of the chain to a height of 6 meters? Use $g = 9.8$ m/s^2.

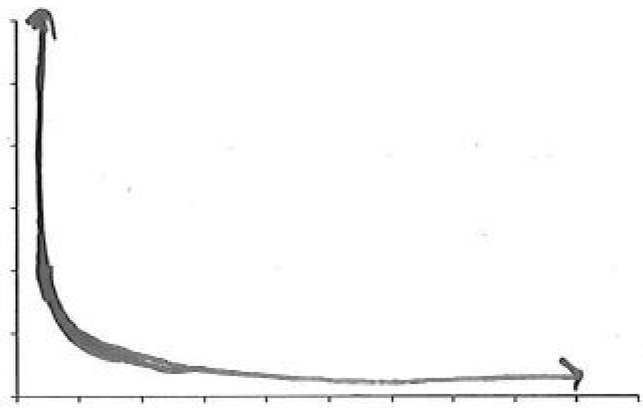
4. A trough is filled with a liquid of density 800 kh/m^3. The ends of the trough are equilateral triangles with sides 8 m long and vertex at the bottom. **Set up** an integral to determine the **hydrostatic force** on one end of the trough. [Hint the height of an equilateral triangle with side length 8 is $4\sqrt{3}$].



20 Improper Integrals

20.1 Calculating Improper Integrals

When we consider definite integrals $\int_a^b f(x)dx$, we assume that a and b are finite AND $f(x)$ is bounded. Consider the region bounded by $x = 1$, $y = \frac{1}{x^2}$ and $y = 0$.



We can write the area (formally) as

This is only symbolic since we don't have a definition of the integral with infinite limits of integration.
Solution: $\int_a^\infty f(x)dx :=$

Now we can solve the question above:

Note: If we have $\int_{-\infty}^\infty f(x)dx$

Example 20.1. *Determine* $\int_2^\infty \frac{1}{x}dx$

Example 20.2. *Find the area of the region bounded by* $y = \dfrac{1}{\sqrt[3]{x}}$, $y = 0$, $x = 0$, $x = 1$.

Example 20.3. *Determine* $\displaystyle\int_0^2 \frac{2}{(x-1)^3}\,dx$

Example 20.4. *Is* $\displaystyle\int_2^3 \frac{2}{(x-1)^3}dx$ *an improper integral?*

20.2 Comparison Tests for Improper Integrals

Sometimes even if we cannot find the exact value for an improper integral, we can determine whether the integral diverges or converges by comparing it to a known integral. This idea will come in handy later when we learn about sequences and series!

Direct Comparison Theorem:
Suppose f and g are __________ functions with $0 \leq g(x) \leq f(x)$ for $x \geq a$, then
1) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$

2) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$

Note: If $\int_a^\infty f(x)dx$ is divergent or $\int_a^\infty g(x)dx$ is convergent, then we know __________

How to choose comparison functions:

$f(x) = \dfrac{x^3+x+1}{x^4-x^2-1}$ compare with $g(x) =$

Example 20.5. *Is* $\displaystyle\int_2^\infty \frac{x^3+x+1}{x^4-x^2-1}dx$ *convergent or divergent?*

Example 20.6. *Show that* $\displaystyle\int_1^\infty \frac{(\sin^2 x)}{x^2} dx$ *is convergent.*

Example 20.7. *Show that* $\displaystyle\int_1^\infty \frac{x}{x^3 - x - 1} dx$ *is convergent. What is the problem with using a direct comparison?*

The Limit Comparison Theorem: Suppose f and g are positive functions on $[a, \infty)$ and we know whether or not $\int_a^\infty g(x)dx$ converges or diverges. Let $\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$, then

1) If L is finite ($0 < L < \infty$) then $\int_a^\infty g(x)dx$ and $\int_a^\infty f(x)dx$ either
BOTH ____________________ or BOTH ____________________.

2) If L=0 AND $\int_a^\infty g(x)dx$ __________ then $\int_a^\infty f(x)dx$ __________

3) If $L = \infty$ AND $\int_a^\infty g(x)dx$ __________ then $\int_a^\infty f(x)dx$ __________

Now use the LCT to finish Example 20.7.

20.3 ICE – Improper Integrals

1. Use the Comparison Theorem to determine whether the following integrals converge or diverge. That is find a comparison function, calculate its improper integral over the same integral, and then apply either the Direct Comparison Theorem or the Limit Comparison Theorem.

   a) $\displaystyle\int_0^{\pi} \frac{\sin^2 x}{\sqrt{x}}\,dx$

   b) $\displaystyle\int_4^{\infty} \frac{1}{\sqrt{x}-1}\,dx$

c) $\displaystyle\int_4^\infty \frac{x^3+x^2+1}{x^4+2}\,dx$

d) $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x+1}\,dx$

2. Evaluate the improper integral $\displaystyle\int_2^5 \frac{1}{x-2}\,dx$ or show that it diverges.

21 Sequences

21.1 Introduction to Sequences

A sequence is a list (usually of numbers) that has a particular order assigned to each term (member) of the list $\{2, 4, 6, 8, 10, ...\}$. Often we represent sequences as $< a_n >$, a_n, $a_n{}_{n=k}^{\infty}$ or $\{a_1, a_2, a_3, ...\}$. We call n the __________ and it just tells us where the term is ordered in the list. Usually, but not always, n starts at 0 or 1.

We can represent our sequences in several different ways. Given $\{2, 4, 6, 8, 10, ...\}$, we can write this as more of a formula: $a_n =$

We can also write this sequences in an implicit formula (or recursive relation): $a_1 = 2$,$a_n =$

Often writing a sequence in an explicit formula is the most useful way: $a_n =$

Example 21.1. *Find an explicit formula for the following sequence,* $\{\frac{2}{3}, \frac{4}{4}, \frac{6}{5}, \frac{8}{6}, \frac{10}{7}, ...\}$

Example 21.2. *Find an explicit formula for the following sequence,* $\{\frac{1}{2}, \frac{-1}{4}, \frac{1}{6}, \frac{-1}{8}, \frac{1}{10}, ...\}$

Some sequences cannot be written in an explicit way or are very difficult to do so.

Example 21.3. *Find an implicit formula for the following sequence,* $\{1, 1, 2, 3, 5, 8, 13, ...\}$.

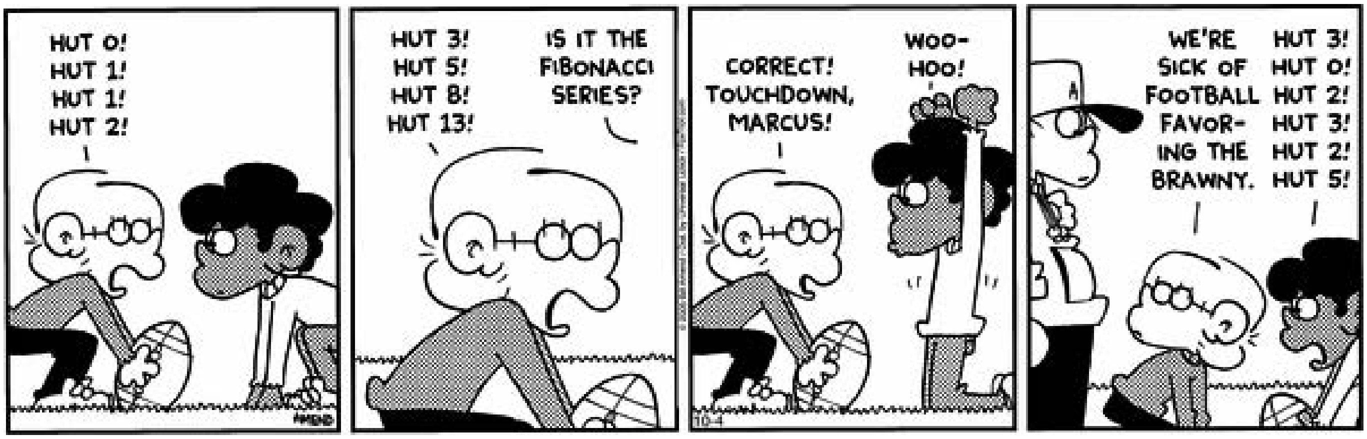


Image from: Amend, B. "FoxTrot.com." Cartoon from Oct. 11, 2005. http://www.foxtrot.com

Note: There actually is a closed form for the Fibonacci Sequence: $F_n = \dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$

Example 21.4. *Find an explicit or implicit formula for* $a_n = \{-1, 1, -1, 1, -1, ...\}$.

Example 21.5. *Find an explicit or implicit formula for* $a_n = \{2, 3, 5, 7, 11, 13, 17, ...\}$.

21.2 Limits of Sequences

We define limits of sequences in the same spirit as limits of functions.

Definition of a limit:
The limit of a sequence $< a_n >$ is L if as n increases, a_n gets closer to ________.
That is, $\lim_{n\to\infty} a_n = L$. Note L needs to be finite.
And we say that the limit of a_n as n approaches ∞ is L.
If a sequence has a limit we say it is a ________ sequence. Otherwise we say the sequence is ________ or that the limit does not exist, or the sequence diverges to $\pm\infty$.

Example 21.6. *Which of our sequence examples so far have limits?*

Definition of a Sequence with an Infinite limit:
Give a sequence $< a_n >$, $\lim_{n\to\infty} a_n = \infty$ means for any $M > 0$, we can find an integer N such that if $n > N$, then __________.

Example 21.7. *Give an example of a sequence that diverges to ∞.*

Definition for the Nonexistence of a Limit: We say $\lim_{n\to\infty} a_n \neq L$ if for some positive ϵ, there is NO integer $N > 0$ satisfying $|a_n - L| < \epsilon$ whenever $n > N$.

Example 21.8. *Given an example of a sequence that has no limit.*

21.3 Properties and Theorems for Sequences

Let (a_n), (b_n) be sequences, and let c be a constant. Then if $a_n \to L$ and $b_n \to K$ where L and K are finite (so a_n and b_n are convergent, then

a) $\lim_{n\to\infty} (a_n \pm b_n) =$

b) $\lim_{n\to\infty} (ca_n) =$

c) $\lim_{n\to\infty} (a_n \cdot b_n) =$

d) $\lim_{n\to\infty} (\frac{a_n}{b_n}) =$

e) $\lim_{n\to\infty} a_n^P =$ __________ if __________ and __________.

Sequences as Functions If $\lim_{n\to\infty} f(n) = L$ and $f(n) = a_n$ where n is an integer, then $\lim_{n\to\infty} a_n =$ __________

Example 21.9. *Determine the limit of* $a_n = \{\frac{n^2}{\ln(n^2)}\}_{n=2}^{\infty}$ *if it exists.*

Continuous Functions of Sequences If $\lim_{n\to\infty} a_n = L$ and f is a continuous function at L, then $\lim_{n\to\infty} f(a_n) =$ __________

Example 21.10. *Determine the limit of* $a_n = \{e^{\frac{1}{2n}}\}_{n=-5}^{\infty}$ *if it exists.*

Example 21.11. *Determine the limit of* $a_n = \{\cos(n\pi)\}_{n=-5}^{\infty}$ *if it exists.*

Example 21.12. *Intuitively what does the sequence* $a_n = \{\frac{(-1)^{n+1}}{2n}\}$ *seem to approach? Can we really treat it like a function of n?*

Squeeze Theorem for Sequences: If $b_n \leq a_n \leq c_n$ for $n \geq n_0$ and $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = L$, then

Example 21.13. *Determine the limit of* $a_n = \{\dfrac{\cos n}{n}\}_{n=1}^{\infty}$ *if it exists.*

The most romantic theorem in mathematics yields to a corollary since $\qquad \leq a_n \leq$

Absolute Convergence for Sequences: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n =$ __________

Example 21.14. *What is the limit of* $a_n = \dfrac{(-1)^n}{n}$*?*

Note: Sometimes determining the limit of sequences is difficult, but we can tell whether or not a limit does exist.

Definition: A sequence is **monotonic** if it is either increasing or decreasing.

$< a_n >$ is increasing if __________ $\geq$ __________ for all $n \geq 1$

$< a_n >$ is decreasing if __________ $\geq$ __________ for all $n \geq 1$

$< a_n >$ is strictly increasing (decreasing) if we change the inequalities to $>$ or $<$.

Definition: $< a_n >$ is **bounded above** if there exists a number $M > 0$, such that __________ for all $n \geq 1$

$< a_n >$ is **bounded below** if there exists a number $m > 0$, such that __________ for all $n \geq 1$

A sequence is **bounded** if it is bounded above AND bounded below.

Note: We can have eventually bounded above/below sequences and eventually monotone sequences.

Example 21.15. *Determine whether the sequence* $a_n = \{1 + \frac{1}{n}\}_{n=1}^{\infty}$ *is bounded. Is it monotonic?*

Example 21.16. *Give an example of a sequence that has the following properties:*

a) Bounded but not monotonic;

b) Monotone but not bounded below;

c) Monotone but not bounded above;

d) Monotone but not bounded ;

e) Monotone and bounded;

Monotonic Sequence Theorem: Every bounded, monotonic sequence is ___________.
proof omitted. See text.

Example 21.17. *Determine whether the sequence* $\{a_n\}_{n=1}^{\infty}$ *has a limit where* $a_1 = 1$ *and* $a_{n+1} = 1 + \dfrac{1}{a_n}$.

21.4 ICE – Sequences

1. Construct 3 different sequences that all converge to 4 -one increasing, one decreasing, and one neither increasing nor decreasing.

2. Determine whether the following statements are True or false. Include reasons or counterexamples:

a) Every bounded sequence is convergent.

b) Every sequence that is bounded above is convergent.

c) Every monotonic sequence is convergent.

d) If $a_n = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\}$ and $b_n = \{1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots\}$, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

e) An alternating sequence can be convergent.

f) The only way a sequence can diverge is if the terms tend to infinity as n tends to infinity.

3. Determine whether the following sequences converge or diverge. If they converge find the limit.

a) $a_n = \{\frac{\sqrt{n}}{1+\sqrt{n}}\}_{n=1}^{\infty}$

b)$b_n = \{\frac{-1}{4}, \frac{2}{9}, \frac{-3}{16}, \frac{4}{25}, \ldots\}$

c) $c_n = \{2, 7, 9, 2, 7, 9, ...\}_{n=1}^{\infty}$

d) $d_n = \{\frac{n}{1+\sqrt{n}}\}_{n=1}^{\infty}$

4. Determine for which x, the sequence $\{x^n\}_{n=1}^{\infty}$ is convergent.

22 Series

22.1 Sequence of Partial Sums

If we add up an infinite sequence $\{a_n\}_{n=1}^{\infty}$ we get an infinite sum of the form:
We call this a series and we denote the series as

Each series has a sequence of ________ sums $\{S_n\}$.
The nth term of this sequence is the sum of the first ________ terms of $\{a_n\}$.
So $S_1 =$
$S_2 =$
$S_3 =$
$S_4 =$
$S_n =$

Key Idea: If our sequence of PARTIAL SUMS $\{S_n\}$ has a limit L. Then the SERIES $\sum\limits_{n=1}^{\infty} a_n$ converges to L. That is, $\sum\limits_{n=1}^{\infty} a_n =$

If our sequence of PARTIAL SUMS $\{S_n\}$ diverges, then the series $\sum\limits_{n=1}^{\infty} a_n$ ________.

Example 22.1. *For the following series, find the first 4 terms of the sequence of partial sums and use it to find an expression for the nth partial sum S_n. Determine if the series converge or diverge?*
a) $\sum\limits_{n=1}^{\infty} 2n$

b) $\sum\limits_{n=1}^{\infty} \dfrac{1}{n(n+1)}$.

c) $\sum_{n=1}^{\infty}(-1)^n$

22.2 Geometric Series

Consider the series of the form $\sum_{n=0}^{\infty} ar^n$

What are some of the partial sums?

a is the __________ term of the series and r is called the ratio.

Sum of A Geometric Series: If a and r are real numbers and $|r| < 1$ then $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$. If $|r| \geq 1$, the series diverges.

Pf. Omitted (see text).

Sometimes a geometric series show up in different forms:

$\sum_{n=1}^{\infty} ar^n$ $\qquad$ $\sum_{n=0}^{\infty} ar^{n-3}$

$\sum_{n=1}^{\infty} ar^{n+1}$ $\qquad$ $\sum_{n=0}^{\infty} a\frac{r^n}{q^m}$

For each of these, you want to make sure the series is actually a geometric series, which sometimes requires doing some algebra to simplify it enough to identify the ratio (that is get it in the form of:____________________),

you actually don't need to do all the algebra to get it in the standard form: $\sum_{n=0}^{\infty} ar^n$ because...

Once you have identified the ratio, and determined it is a convergent geometric series, Dr. Harsy recommends the following way to determine the sum:

Better Way to Remember Sum of a Geo Series: $\sum$geo series=

Example 22.2. *Evaluate the following geometric series.*

a) $\sum_{n=2}^{\infty} \frac{1}{5^n}$

b) $\sum_{n=2}^{\infty} (\frac{7}{5})^n$

c) $\sum_{n=0}^{\infty} e^{-n}$

d) $\sum_{n=1}^{\infty} \frac{2^n}{5^{n+1}}$

e) $\sum_{n=1}^{\infty} (\frac{1}{2^n} - \frac{1}{2^{n+2}})$

22.3 Telescoping Series

We can calculate a geometric series using a formula, but not many infinite series can be solved using a formula. But we can do this when we have another class of series called __________ series.

Example 22.3. *Evaluate* $\sum_{n=1}^{\infty}(\frac{1}{2^n} - \frac{1}{2^{n+2}})$ *(not using geometric series.)*

Example 22.4. *Evaluate* $\sum_{n=1}^{\infty}(\frac{1}{(n+1)(n+2)})$

22.4 Properties of Series

The Harmonic Series: The Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Pf. See Text. You can show the for each term of the partial sums $S_{2^n} > 1 + \frac{n}{2}$ so then $S_n \to \infty$. So the series diverges.

Theorem If $\sum_{n=1}^{\infty} a_n$ is convergent then $\lim_{n\to\infty} a_n =$

CAREFUL: The converse of this statement is not true.
Just because $\lim_{n\to\infty} a_n = 0$, does **NOT** mean $\sum_{n=1}^{\infty} a_n$ is convergent!!!
Remember the Harmonic Series: $\sum_{n=1}^{\infty} \frac{1}{n}$ **diverges** even though $\lim_{n\to\infty} \frac{1}{n} =$__________!

We can use the theorem just to show that a sequence does not converge.
Hence the next theorem:

Test for Divergence If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then $\sum_{n=1}^{\infty} a_n$ __________.

Example 22.5. *Show that* $\displaystyle\sum_{n=1}^{\infty} \frac{-n^2 - n + 5}{5n^2 + 1}$ *is divergent.*

Example 22.6. *Can we use the Test for Divergence for* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$*?*

Properties of Series
Let $\sum_{n=1}^{\infty} a_n = L$, $\sum_{n=1}^{\infty} b_n = M$ (so we have CONVERGENT series), and let c be a constant. Then

a) $\displaystyle\sum_{n=1}^{\infty} (a_n \pm b_n) =$

b) $\displaystyle\sum_{n=1}^{\infty} c a_n =$

.

22.5 ICE – Series

Two sides!

1. Determine if the following series converge or diverge. Evaluate those that converge.

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{(-5)^n}$.

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^{n-1}}{(-5)^n}$. [Hint: factor out a $\frac{-1}{5}$ and write this as a geometric series.]

   c) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$. [Hint: Using partial fraction, we see that $\frac{2}{n^2+4n+3} = \frac{1}{n+1} - \frac{1}{n+3}$.]

d) $\sum_{n=1}^{\infty}(\frac{5}{4^n} + \frac{4}{n})$

2. Find the values for x for which $\sum_{n=1}^{\infty} \frac{(x+3)^n}{2^n}$ converges. Then find the sum of the series for those values of x.

3. The n^{th} partial sum of a series of a_n is $s_n = 3 - \frac{n}{2^n}$. Find a_n and $\sum_{n=1}^{\infty} a_n$.

23 Tests for Convergence for Series

Up until now, we have had the following ways to determine whether or not a series, $\sum a_n$, converges:

1. Find a formula for the sequence of partial sums S_n. If S_n converges to a finite limit then $\sum a_n$ converges. If S_n diverges to ∞, $-\infty$, or the limit doesn't exist, $\sum a_n$ diverges. **Note: When we recognize a telescoping series, we find S_n**

2. Write $\sum a_n$ in the form of a geometric series, $\sum ar^n$ (**if you can, remember this doesn't often work**). If $|r| < 1$, our geometric series converges. If $|r| \geq 1$, the series diverges.

3. Use the Test for Divergence. **But remember this only tells us if a series diverges and not if it converges!**

The next week we will be talking about more tests for convergence. Remember this is important because eventually we will want to use series to model functions so making sure these models converge is important.

23.1 The Test for Divergence

The Test for Divergence: If $\lim_{n\to\infty} a_n \neq 0$ or does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 23.1. *Determine whether the series* $\displaystyle\sum_{1}^{\infty} \frac{2n+1}{n-3}$ *converges or diverges.*

Example 23.2. *Determine whether the series* $\displaystyle\sum_{1}^{\infty} \frac{1}{\sqrt{n}}$ *converges or diverges.*

23.2 Integral Test

Recall in our notes on Improper Integrals, we determined that $\displaystyle\int_1^{\infty} \frac{1}{x} dx$ __________ and $\displaystyle\int_1^{\infty} \frac{1}{x^2} dx$ __________

In fact we can confirm that $\displaystyle\int_1^{\infty} \frac{1}{x^k} dx$ diverges for __________ and converges for __________.

Note: This is not true for $\displaystyle\int_0^{1} \frac{1}{x^k} dx$.

We can use improper integrals to help us determine whether or not our series converge:

Integral Test for Series: Suppose f is a 1) continuous, 2) positive, and 3) decreasing (or eventually decreasing) function on $[b, \infty)$. Let $a_n = f(n)$. Then

1) If $\displaystyle\int_b^{\infty} f(x)dx$ is convergent then $\displaystyle\sum_{n=b}^{\infty} f(n) = \sum_{n=b}^{\infty} a_n$ is ____________________

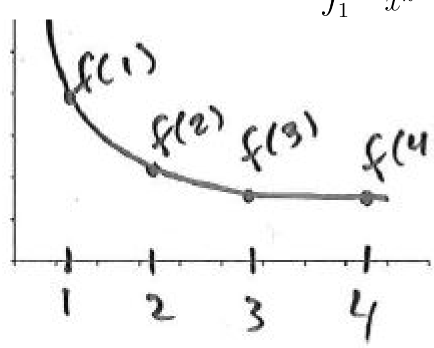
2) If $\displaystyle\int_b^{\infty} f(x)dx$ is divergent then $\displaystyle\sum_{n=b}^{\infty} f(n) = \sum_{n=b}^{\infty} a_n$ is ____________________

Note: You must always justify the use of this test by checking that our function satisfies the conditions of the theorem.

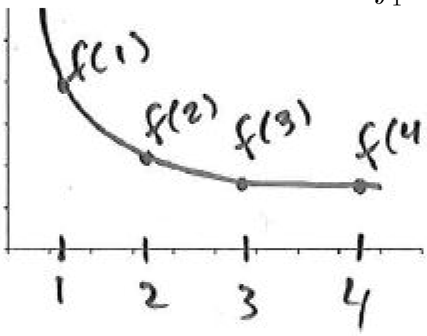
Sometimes you can modify this, if your function is eventually decreasing after n=k and use $\displaystyle\int_k^{\infty} f(x)dx$ for the test.

We will omit the proof and instead explore the reasoning for this using a few examples.

Let's pick a k such that $\displaystyle\int_1^{\infty} \frac{1}{x^k} dx$ converges.



Now let's pick a k such that $\int_1^\infty \frac{1}{x^k} dx$ diverges.



Example 23.3. *Use the integral test to determine whether* $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ *converges or diverges.*

Remember you always MUST justify that you can use the integral test.

Example 23.4. *Use the integral test to determine whether* $\sum_{n=2}^{\infty} \frac{1}{n \ln(n)^2}$ *converges or diverges.*

23.3 The P-Test

Using our results from improper integrals, we know $\int_1^\infty \frac{1}{x^p} dx$ diverges for __________ and converges for __________.
This gives us a very useful test when we apply the integral test. We use it so often, we often think of it as its own thing:

P-test for Series: $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent if __________ and convergent if __________.

Definition: A series in the form of $\sum \frac{1}{n^P}$ is called a *P-Series.*

Example 23.5. *a) Give an example of a P-Series that converges.*

b) Now give an example of a P-Series that diverges.

Example 23.6. *Determine if the sum of the sequence* $\{1, \frac{1}{4\sqrt{2}}, \frac{1}{9\sqrt{3}}, \frac{1}{16\sqrt{4}}, \frac{1}{25\sqrt{5}}, \ldots\}$ *converges or diverges.*

23.3.1 Estimating Sums of Series using Improper Integrals

Important: If we use our integral test for $a_n = f(n)$ and $\int_1^\infty f(x)dx = B < \infty$, this DOES NOT mean that $\sum_{n=1}^{\infty} a_n = B$. [This makes sense if we go back to our examples from the first page.]

For example: $\int_1^\infty \frac{1}{x^2}dx = 1$. But $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

So far we have used improper integrals to determine whether or not a series converges. We can also use improper integrals to approximate the sum of the series.

Remember we look at the sequence of partial sums, S_n to determine the sum of a series $\sum_{n=1}^{\infty} a_n$. If $\lim_{n\to\infty} S_n = S$, then $\sum_{n=1}^{\infty} a_n = S$. This means that the error made for the nth partial sum, denoted R_n, should be getting relatively small as n increases.

So the error in the approximating our sum using Sn is $R_n = S - S_n =$

So $R_n =$

Remainder Estimate for the Integral Test: Suppose $f(n) = a_n$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum_{n=1}^{\infty} a_n$ is convergent. If $R_n = S - S_n$,

then __________ $\leq R_n \leq$ __________

Since $R_n = S - S_n \Rightarrow$ __________+__________ $\leq S \leq$ __________+__________

Example 23.7. *How many terms of the series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *must be summed to obtain an approximation that is within* 10^{-2} *of the exact value of the series?*

23.3.2 ICE – Integral and P-Test

Two sides!

1. Explain why the integral test can't be used to determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{\sqrt{n}}$ converges or diverges.

2. For what values of p does the sum $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^p}$ converge.

3. Euler found that the exact sum of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

   Use this to find the sum of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ AND $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(n+1)^2}$.

23.4 Direct Comparison Test for Series

Recall, we could determine whether an improper integral converged or diverged by comparing it with another improper integral.
So if f and g are continuous functions with $0 \leq g(x) \leq f(x)$ for $x \geq a$, then
1) If $\int_a^\infty f(x)dx$ is convergent, then $\int_a^\infty g(x)dx$__________
2) If $\int_a^\infty g(x)dx$ is divergent, then $\int_a^\infty f(x)dx$__________

We can do the same thing with series!

Direct Comparison Test for Series: Suppose $\sum a_n$ and $\sum b_n$ are series with **POSITIVE** terms. Then
1) If $a_n \leq b_n$ for all n, and $\sum b_n$ __________, then $\sum a_n$__________
2) If $a_n \geq b_n$ for all n, and $\sum b_n$ __________, then $\sum a_n$__________
Note: Just like with integral comparison. If $a_n \leq b_n$ and $\sum b_n$ diverges OR if $a_n \geq b_n$ and $\sum b_n$ converges, then we know __________ !!!

Helpful Series to compare with:

Example 23.8. *Determine whether* $\displaystyle\sum_{n=3}^{\infty} \frac{5}{\sqrt{n-2}}$ *converges or diverges.*

Example 23.9. *Determine whether* $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(n)}{n^4}$ *converges or diverges.*

23.5 Limit Comparison Test for Series

We use the comparison test if there is an obvious series we can compare our series to. But sometimes life is HARD!

What about if we had $\displaystyle\sum_{n=3}^{\infty} \frac{5}{\sqrt{n+2}}$ for example 1 instead of $\displaystyle\sum_{n=3}^{\infty} \frac{5}{\sqrt{n-2}}$

Rather than trying to find a series to compare to we can use another test.

Limit Comparison Test: Suppose $\sum a_n$ and $\sum b_n$ are series with **POSITIVE** terms AND $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$. Then

1) If L is finite ($0 < L < \infty$) then $\sum a_n$ and $\sum b_n$ either BOTH __________ or __________

2) If L=0 AND $\sum b_n$ __________ then $\sum a_n$ __________

3) If $L = \infty$ and $\sum b_n$ __________ then $\sum a_n$ __________

Example 23.10. *Use the limit comparison test to determine if* $\displaystyle\sum_{n=3}^{\infty} \frac{5}{\sqrt{n+2}}$ *converges or diverges.*

Example 23.11. *Determine if* $\displaystyle\sum_{n=1}^{\infty} \frac{5n^5 - 2n^2 + 3}{3n^7 - n + 7}$ *converges or diverges.*

Example 23.12. *What happens if we try to use the limit comparison test to determine if* $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2 + 5}$ *converges or diverges by comparing it to* $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n}$*?*

23.5.1 Estimating Sums of Series Using Comparison Series

Recall the error made for the nth partial sum,R_n of $\sum a_n$ is $R_n = S - S_n =$
Suppose we compare $\sum a_n$ with $\sum b_n$. Let T_n represent the error of the nth partial sum of $\sum b_n$.
Then if $a_n \leq b_n$ for all n, then __________

If b_n is a p-series, then we can use the Remainder Estimate for the Integral Test and $R_n \leq T_n \leq \displaystyle\int_n^\infty f(x)dx$ where $f(n) = a_n$.

If b_n is a geometric series we also have that $R_n \leq T_n \leq \displaystyle\int_n^\infty f(x)dx$ where $f(n) = a_n$.

Example 23.13. *Estimate the error in approximating* $\sum\limits_{n=3}^{\infty} \frac{n}{3^n(n+1)}$ *if we use the sum of the first 5 terms,* S_5*, to approximate the series.*

23.5.2 ICE – Comparison Tests

1. Determine if the following series converge or diverge. Two sides!

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + n^5 - 1}{3n^9 + 2}$.

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{1 + \cos(n)}{\sqrt{2}^n}$.

   c) $\displaystyle\sum_{n=5}^{\infty} \frac{1}{n - 2\sqrt{n}}$.

d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 + 2\sqrt{n}}$.

e) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n) + n + 9}{(n+2)^3}$.

23.6 The Alternating Series Test

So far we can test series using the Integral Test, P-Test, the Direct Comparison Test, and The Limit Comparison Test. But all of these require us to have series with only __________ terms. What if we have some negative terms along with our nonnegative terms?

We have a test for series that alternating between positive and negative terms. An alternating series can be written in the form: $\sum_{n=k}^{\infty}(-1)^{n+1}a_n$

Example: $\sum_{n=1}^{\infty}(-1)^{n+1}$ is alternating

Alternating Series Test:
The alternating series $\sum(-1)^{n+1}a_n$ converges if...
1) The terms of a_n are nonincreasing. That is $a_{n+1} \leq a_n$. [Note: eventually nonincreasing is ok.]
2) $\lim_{n\to\infty} a_n = 0$

and diverges if either of the two above conditions fails.

Warning: If we do NOT have an alternating series and have a series with all positive terms, then $\lim_{n\to\infty} a_n = 0$ __________ __________ imply convergence. This is a special case for an alternating series.

Example 23.14. *Determine whether* $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ *converges or diverges.*

The Alternating Harmonic Series: $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ converges to $\ln(2)$

Example 23.15. *Determine whether* $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}-1}$ *converges or diverges.*

Example 23.16. *Determine if* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^4}{3n^4 - 5}$ *converges or diverges.*

Example 23.17. *Determine if* $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(\frac{\pi n}{2})}{n}$ *converges or diverges.*

23.6.1 Estimating Remainders in Alternating Series

The error made for the nth partial sum,R_n (the remainder) of a convergent alternating series $\sum(-1)^{n+1}a_n$ is bounded by the $(n+1)^{th}$ term of the sequence.
That is $|R_n| \leq$

Example 23.18. *How many terms of* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^8}$ *do we need to take to get the error bounded by* $\frac{1}{10^8}$*?*

23.7 Absolute Convergence Test and Conditional Convergence

Any sequence $a_n \leq$ __________ and so we can always compare $\sum a_n \leq$ with __________. By the comparison test, if $\sum |a_n|$ converges then so does __________

Definition: A series $\sum a_n$ is __________ convergent if $\sum |a_n|$

Absolute Convergence Test (Special case of Comparision Test):
If a series is absolutely convergent, it __________

Can we have a series that converges, but is not absolutely convergent?

Example:

Definition: A series that is convergent but not absolutely convergent is called __________ convergent.

Example 23.19. *Is* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$ *absolutely convergent, conditionally convergent, or divergent?*

Example 23.20. *Determine whether* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^5}}$ *is absolutely convergent, conditionally convergent, or divergent?*

23.7.1 ICE – Alternating Series Test

1. Determine which of the following are true or false. Explain or give a counter example.
   1) A series that converges must converge absolutely.
   2) A series that converges absolutely must converge.
   3) A series that converges conditionally must converge.
   4) If $\sum a_n$ diverges then $\sum |a_n|$ diverges.
   5) If $\sum a_n$ converges conditionally then $\sum |a_n|$ diverges.
   6) If $\sum a_n^2$ converges then $\sum a_n$ converges.

2. Is $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ convergent? absolutely convergent? [Hint: use alt. series test and comparision]

3. Is $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{n^2+1}}$ convergent? absolutely convergent?

4. Is $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{3^n - 1}$ convergent? absolutely convergent?

5. For which values of p is $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n^p}$ convergent? absolutely convergent?

23.8 The Ratio Test

Recall, $n! =$
So $5! =$
Consider the series $\sum_{n=1}^{\infty} \frac{1}{n!}$. Can we determine whether it converges or diverges?

Ratio Test: Given a series $\sum a_n$. Let $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$.

1) If $0 \leq L < 1$, then $\sum a_n$ is __________ __________.
2) If $L > 1$, then $\sum a_n$ is __________.
3) If $L = 1$, then the Ratio Test is inconclusive. We don't know if $\sum a_n$ diverges or converges.

Example 23.21. *Determine if* $\sum_{n=1}^{\infty} \frac{1}{n!}$ *converges or diverges.*

Example 23.22. *Determine whether* $\sum_{n=1}^{\infty} \frac{n!}{100!}$ *converges or diverges.*

Example 23.23. *Use the Ratio Test to determine whether* $\sum_{n=1}^{\infty} \frac{1}{n}$ *converges or diverges.*

23.9 The Root Test

Consider the series $\sum_{n=1}^{\infty} (\frac{5n^2 - 3}{8n^2 + 7})^n$. Can we determine whether it converges or diverges?

Root Test: Given a series $\sum a_n$. Let $\lim_{n \to \infty} \sqrt[n]{|a_n|} = K$.

1) If $0 \leq K < 1$, then $\sum a_n$ is __________ __________.
2) If $K > 1$, then $\sum a_n$ is __________.
3) If $K = 1$, then the Root Test is inconclusive.

Example 23.24. *Determine if* $\sum_{n=1}^{\infty} (\frac{5n^2 - 3}{8n^2 + 7})^n$ *converges or diverges.*

Example 23.25. *Determine if* $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2 3^n}{(n+2)^2}$ *converges or diverges.*

Example 23.26. *Determine if* $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(4n)}{4^n}$ *converges or diverges.*

23.9.1 ICE – Ratio and Root Tests

1. Determine if the following series are convergent or divergent.

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{4^n}$

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{e^n}$

   c) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3^n}$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{(10)^n}{\arctan(n)^n}$

e) Using the Root test what values of a, will $\displaystyle\sum_{n=1}^{\infty} a^n$ converge for? diverge? Does this agree with our geometric series test?

f) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{4^{1+4n}}$ [Hint: Factor out a $\frac{1}{4}$.]

24 Series Review

Objective 1: *Strategy for Series*
So many tests for series! How do we choose?!?!
It helps to recognize the general form of the series and use the test for that form.
My best suggestion for gaining intuition about series strategy is exactly what I suggested when learning integration techniques...
I have also created a flowchart that can help you (see next page).
Game plan for Series:

1. Check for divergence using the **Test for Divergence** to determine if the sequence we are summing does not have a limit of __________

   Form of Series:

   Condition for convergence: Condition for divergence:

   Comment: Cannot be used to prove convergence!

2. **Alternating Series Test:** I have an alternating series or can rewrite my series to be alternating. Form of Series:

   Condition for convergence: Condition for divergence:

   Comment: The remainder/error $R_n \leq a_{n+1}$

3. **Geometric Series:** I see a Geometric Series!
   Form(s) of Series:

   Condition for convergence: Condition for divergence:

   Comment: Limit of the series is

4. **P-Series:** I see a P-Series!
   Form of Series:

   Condition for convergence: Condition for divergence:

   Comment: Use for comparison tests!

5. **Direct Comparison Test:** I see a series that looks similar to a geometric or p-series, but is a little more complicated.
   Form of Series:

   Condition for convergence: Condition for divergence:

   Comment: You have to come up with the series, $\sum b_n$, as a comparison. Remember to justify that $a_n \leq b_n$ or $a_n \geq b_n$. Also if $\sum a_n$ has some negative values, you can compare with $\sum |a_n|$.

6. **Limit Comparison Test:** It looks like I want to use a Comparison Test, but justifying the comparison seems tough.

   Form of Series:

   Condition for convergence: Condition for divergence:

   Comment: You have to come up with the series, $\sum b_n$ to compare to and find $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$

7. **Ratio Test:** I see a factorial ($n!$) or products and too complicated to compare to a p-series or geometric series.
   Form of Series:

   Condition for convergence: Condition for divergence:

   Comment: If $\lim\limits_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = 1$, we know __________ !

8. **Root Test:** I see an n^{th} power.
Form of Series:

Condition for convergence: Condition for divergence:

Comment: If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$, we know ___________ !

9. **Integral Test:** I see a function in which $f(n) = a_n$ and I can easily calculate $\int_k^\infty f(x)dx$
Form of Series:

Condition for convergence: Condition for divergence:

Comment: You must justify the use of the test! And remember the value of the integral is NOT the value of the series.

10. **Telescoping Series:** I see a difference of two fractions (like $\frac{1}{n} - \frac{1}{n+2}$) or I see that the sequence I am summing looks like I could use partial fractions. Then I can find an expression for the sequence of partial fractions and take $\lim_{n\to\infty} S_n$ which will be the sum of my series.
Form of Series:

Condition for convergence: Condition for divergence:

Comment: Sometimes you can expand the fraction and use a comparison test instead!

11. **Finding Sequence of Partial Sums:** I am given the sequence of partial sums S_n or I can easily find it. Then $\sum a_n = \lim_{n\to\infty} S_n$
Form of Series:

Condition for convergence: Condition for divergence:

Example 24.1. *For each Series, suggest one approach that you think is appropriate to determine whether the series converges or diverges.*

a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{e^n}$

b) $\displaystyle\sum_{n=2}^{\infty} (\frac{3n^2+1}{n^2-1})^n$

c) $\displaystyle\sum_{n=1}^{\infty} [(\frac{1}{3})^n - (\frac{2}{3})^{n-1}]$

d) $\displaystyle\sum_{n=1}^{\infty} \sin(n)$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+1}{\sqrt{n^3+4}}$

f) $\displaystyle\sum_{n=1}^{\infty} ne^{-n}$

g) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n^2}{n^5}$

h) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{10^n}{n!}$

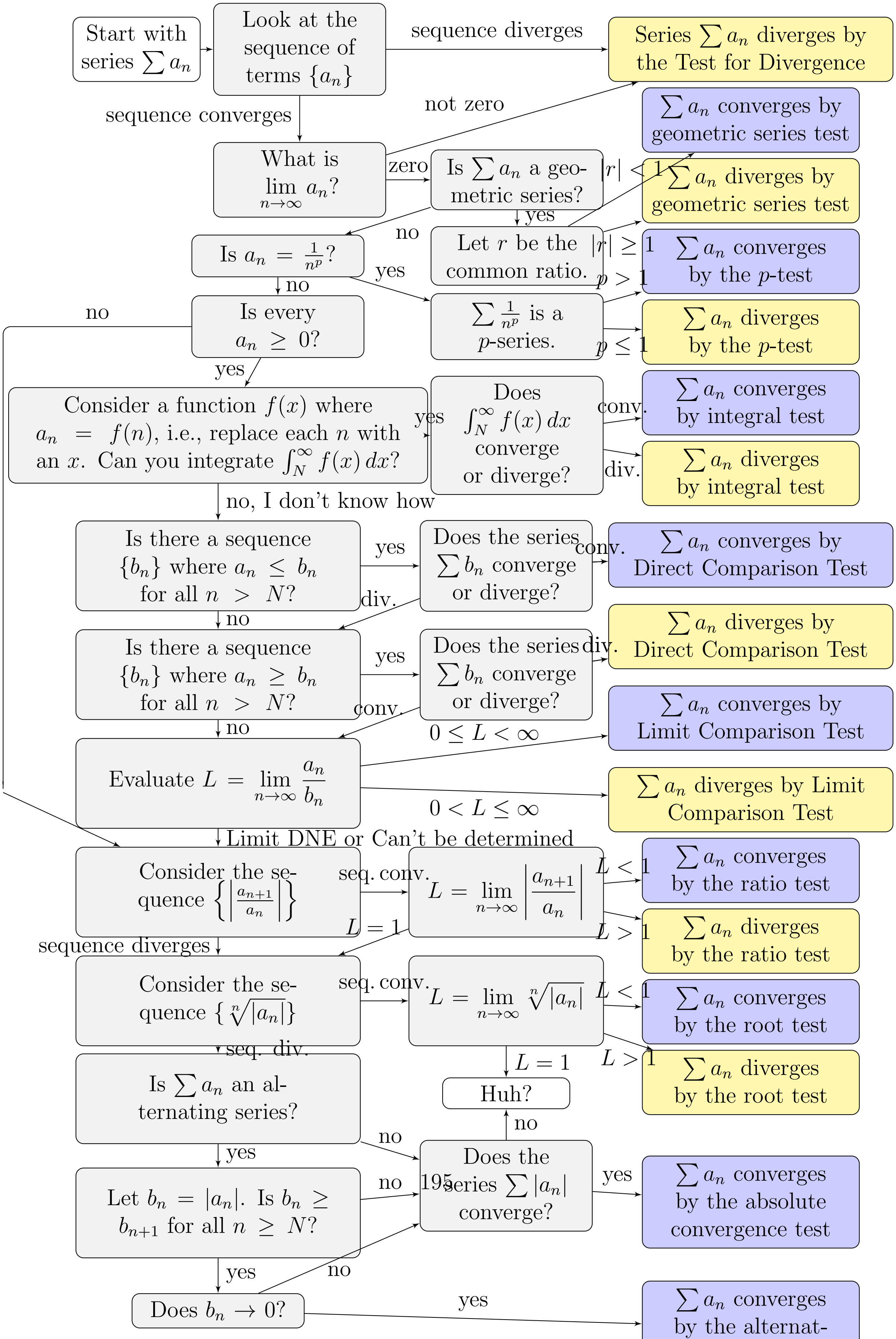
i) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n^2}{n^2}$

j) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

k) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+0.4^n}$

l) $\displaystyle\sum_{n=1}^{\infty} \frac{e^{\frac{1}{n}}}{n^2}$

24.1 Flowchart Summarizing Most Series Convergence Tests



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24.2 Infinite Series Convergence Testing Practice

Here are some good practice series for you to work on during your own study time. For the following series, your task is to make the strongest statement about convergence that is possible.

1. $\displaystyle\sum_{n=3}^{\infty} \frac{(\ln(n))^2}{n}$
2. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \cdot 7^n}$
3. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{e^n}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(5n)^3}$
5. $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{n!}$
6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^{2n+3}}$
7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3+4}$
8. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+4}{n}$
9. $\displaystyle\sum_{n=3}^{\infty} \frac{4^n(n+2)(2n)!}{(2n+2)!}$
10. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}}$
11. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{(n^3+4)^2}$
12. $\displaystyle\sum_{n=0}^{\infty} \frac{6^n+7^n}{8^n}$
13. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n}{e^n+5}$
14. $\displaystyle\sum_{n=3}^{\infty} \frac{4^n}{(n!)^2}$
15. $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{(n^4+1)^2}$
16. $\displaystyle\sum_{n=0}^{\infty} \frac{1+2^n}{3^n}$
17. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n)}{n^3}$
18. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{\left(\frac{5}{4}\right)^n+\left(\frac{1}{4}\right)^n}$
19. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+4}{3-2n^2}$
20. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{e^n+n}$
21. $\displaystyle\sum_{n=1}^{\infty} n^{-7/4}$
22. $\displaystyle\sum_{n=1}^{\infty} \frac{(-7)^n}{n+10000}$
23. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n e^n}{e^n-3}$
24. $\displaystyle\sum_{n=0}^{\infty} e^{3-n}$
25. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n(n+1)}{5^n}$

Flowchart modified by Dr. Harsy from Dr. Edward Kim's original LaTeXdocument.

25 Power Series

We got the Power -Power Series!
So far we have primarily discussed series of numbers. Could we have series with variables?

What if we replaced the ratio (r) in our geometric series $\sum r^n$ with a variable x?
$\sum_{0}^{\infty} x^n$, when will this series converge?

What if we replaced the ratio with $(x-a)$? $\sum_{0}^{\infty}(x-a)^n$ When will it converge now?

We call the series above a _________ series. It is a series in the form:

$$\sum_{n=0}^{\infty} c_n(x-a)^n =$$

Note: c_n is a sequence of __________________ depending on n and should not be functions of x.

Sometimes, we call this a power series in $(x-a)$ or a power series centered at/about _________.

The simplest power series is when we take $a =$ _________. Then we get this power series:

The "Powerful Idea": When you have $\sum_{n=0}^{\infty} c_n(x-a)^n$, you have a <u>battle</u> between adding up infinite terms and $(x-a)^n$ becoming smaller and smaller. If $(x-a)^n$ becomes small enough, our infinite sum may converge!

Power Series Centered at a: $\sum_{n=0}^{\infty} c_n(x-a)^n$ has 3 possibilities:

1) The series converges for 1 value of x. That is, when x=_________
2) The series converges for _________ values of x.
3) The series converges for _________ values of x. That is, there exists a number R such that the series converges for $|x-a| <$ _________ and diverges for $|x-a| >$ _________

Definition: We call R, the radius of __________________.

Note: For case 1) $R =$ _________
and for case 2) $R =$ _________

Definition: The __________ of convergence (or *domain*) of a power series is the set (usually in interval notation) of all values of x for which the series converges.

If $|x - a| = R$, then we don't know what it does. We will have to __________ the series at the endpoints of our interval.

So the interval of convergence could have __________ possibilities

So how do we find this interval of convergence?

Procedure: Find a limit of your series using the Ratio or Root Test. Call this limit, L. L often depends on x, except when $L = 0$ or $L = \infty$.
If $L = 0$, then $R =$ __________ and the interval of convergence is __________
If $L = \infty$, then $R =$ __________ and the interval of convergence is __________

If $0 < L < \infty$, then solve $L <$ __________:
Example: Suppose $L = \frac{1}{5}|2x - 1|$

To find the Radius of Convergence: To find the Interval of Convergence:

Example 25.1. *Find the interval of convergence for* $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}(x-2)^n$

Example 25.2. *Find the interval of convergence for* $\displaystyle\sum_{n=1}^{\infty} n^{2n}(x+1)^n$

Example 25.3. *Find the interval of convergence for* $\displaystyle\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}$

Example 25.4. *Find the interval of convergence for* $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$

25.1 Differentiating and Integrating Power Series

Given a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ with a radius of convergence R, we can consider the sum of the series as a function of __________ and we can integrate and differentiate this function on the interval $(a-R, a+R)$. We can then represent $f'(x)$ and $\int f(x)dx$ as a power series too.

Theorem Differentiating/Integrating Power Series:
Suppose $\sum c_n(x-a)^n$ has a radius of convergence R. Then $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n =$
is continuous and differentiable on $(a-R, a+R)$ and
1) $f'(x) =$

2) $\int f(x)dx =$

Both of these two new power series have the __________ radius of convergence as $f(x)$.
Note: But the interval of convergence may be __________

25.2 ICE – Power Series

1. Find the radius of convergence and the interval of convergence for the following series.
   3 sides!

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^{n+1}}{(n+1)!}$.

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n3^n}$.

c) $\displaystyle\sum_{n=1}^{\infty} n^3(x-5)^n.$

d) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{\ln(n)^n}.$

e) $\displaystyle\sum_{n=1}^{\infty}(n+1)!(x-3)^n$.

f) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{b^n}(x-a)^n$ for $b>0$.

25.3 Using Power Series to Represent Functions

If $|r| < 1$, the geometric series $\displaystyle\sum_{n=0}^{\infty} a(r)^n$ has a sum of __________

The power series $\displaystyle\sum_{n=0}^{\infty} (x)^n$ has a sum of __________ as long as $|x| <$ __________.

So this means we can represent our function using a series: $f(x) = \frac{1}{1-x} =$

In fact we can do this for a lot of functions. It just requires some creativity!

Example 25.5. *Find a power series to represent* $f(x) = \dfrac{1}{1+3x}$.

Example 25.6. *Find a power series to represent* $f(x) = \dfrac{x^5}{1+3x}$.

Example 25.7. *Find a power series to represent* $f(x) = \dfrac{1}{x^3+x^2}$.

Example 25.8. *Use your answer from Example 1 to find a power series to represent* $g(x) = \dfrac{\ln(3x+1)}{3}$.

Example 25.9. *Find a power series to represent* $f(x) = \dfrac{1}{(1+x)^2}$ *What is the radius of convergence?*

25.4 ICE –Introduction to Taylor Series

1. Given the function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$. Find the values of the coefficents $c_0, c_1, c_2, ...$

a) Find c_0 by plugging in $x = a$ into $f(x)$.

b) Find c_1 by plugging in $x = a$ into $f'(x)$

c) Find c_2 by plugging in $x = a$ into $f''(x)$.

d) Find c_3 by plugging in $x = a$ into $f'''(x)$.

e)Find c_4 by plugging in $x = a$ into $f^{(4)}(x)$.

f) Can you see the pattern to find c_n ?

2. a) Evaluate $\int \frac{x}{1+x^3}dx$ as a power series. What is the radius of convergence?

b) What is the maxiumum error if we use the first 5 terms (so up until n=4) of a power series to approximate $\int_0^{0.2} \frac{x}{1+x^3}dx$

3. Find a power series to represent $f(x) = x\arctan(x^2)$ [Hint: We would like to find a function $g(x)$ such that $\int g(x)dx = \arctan(x^2)$. So we may want something of the form $\int \frac{dw}{1+w^2}$ where $w = x^2$.] What is the radius of convergence?

26 Taylor Series

From our ICE sheet we found that if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ and $|x-a| < R$, then the coefficients are given by $c_n =$ __________.

So we can write $f(x) =$

Definition: Taylor Series for f at a:

Definition: Maclaurin Series for f:
(AKA: The Taylor Series for f at 0)

Definition: The m^{th}-degree Taylor Polynomial for f at a:
$T_m =$

Example 26.1. *Find the 5th degree Taylor polynomial, T_5 for $f(x) = \cos x$ centered at 0. Then find the Maclaurin Series for $f(x) = \cos x$. What is the interval of convergence?*

Example 26.2. *Find the Taylor series for $f(x) = \cos x$ centered at $\frac{\pi}{6}$ after first finding T_3.*

26.1 The Binomial Series

We can find the Maclaurin series for $f(x) = (1+x)^k$ where k is any real number. In doing so we must determine $f^{(n)}(0)$ for n=0,1,2,3,.... What results is a series called the __________ series.

The Binomial Series

For any real number k, and $|x| < 1$, then

$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + ...=$

Each coefficient can be written as $\binom{k}{n} = \dfrac{k!}{n!(k-n)!}$ which are called the binomial coefficients.

Note: We must check the convergence at $x = \pm 1$, and the convergence depends on k.

Example 26.3. *Find the first 4 terms of the Maclaurin series for the function $f(x) = \sqrt{1+x}$ and use it to approximate $\sqrt{1.17}$.*

26.2 Using Known Maclaurin Series

Of course you can always derive a Maclaurin Series. But often it is quickier if you can memorize a few common ones. Below is a list of common series that may be useful to memorize.

Maclaurin Series and Radii of Convergence

Function	**Radius**	**Series**	**Expanded**
$\dfrac{1}{1-x}$	1	$\displaystyle\sum_{n=0}^{\infty} x^n =$	$1 + x + x^2 + x^3 + ...$
e^x	∞	$\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} =$	$1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + ...$
$\sin x$	∞	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} =$	$x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + ...$
$\cos x$	∞	$\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} =$	$1 + -\dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + ...$
$\arctan x$	1	$\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} =$	$x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + ...$
$\ln(1+x)$	1	$\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} =$	$x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \dfrac{x^5}{5} - ...$
$(1+x)^k$	1	$\displaystyle\sum_{n=0}^{\infty} \binom{k}{n} x^n =$	$1 + kx + \dfrac{k(k-1)x^2}{2!} + \dfrac{k(k-1)(k-2)x^3}{3!} + ...$

Example 26.4. *a) Find the sum of the series* $1 + \ln 2 + \frac{(\ln 2)^2}{2!} + \frac{(\ln 2)^3}{3!} + ...$

b) Find the sum of the series $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + ...$

Example 26.5. *Find the Maclurin series for* $f(x) = \sin(2x)$ *and* $g(x) = x\sin(2x)$.

Example 26.6. *Find the first four non-zero terms of the Maclaurin series for* $f(x) = (\arctan x)e^x$.

26.3 Using Taylor Polynomials

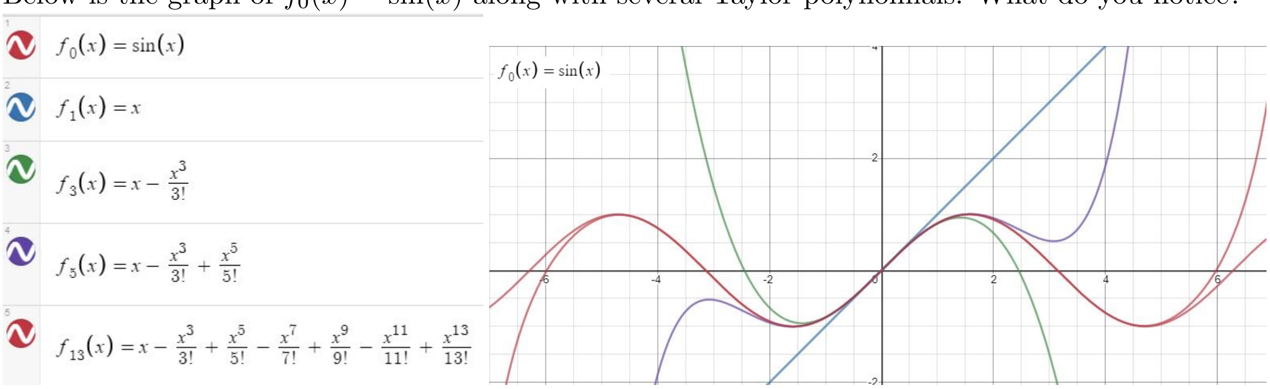
Why is it useful to use Taylor Polynomials to approximate functions?

Recall, in Section 15, sometimes we have functions that aren't derivatives of a nice function. These are called non-elementary functions. Some other examples include: examples include the elliptical integral: $\int \sqrt{1-x^4}dx$, $\int \ln(\ln(x))dx$, $\int \dfrac{e^x}{x}dx$, $\int e^{e^x}dx$, $\int \dfrac{1}{\ln x}dx$, $\int \dfrac{\sin x}{x}dx$, $\int \sin(x^2)dx$, and $\int \sqrt{x}\cos(x)dx$. Some of these functions come up in applications, so not being able to calculate them cane be a problem. Luckily Taylor Polynomials can help!

Example 26.7. *Recall the Gaussian function $f(x) = e^{-x^2}$ is not the derivative of any elementary function. That is, $\int e^{-x^2}dx$ does not exist! Represent this function as a Taylor series.*

Obviously, an infinite series is not practical to use for applications, so instead we use Taylor Polynomials and the appropriate choice of coefficients, you can approximate any function (within a certain radius of convergence) by a power series with the appropriate choice of coefficients. Let's visualize that Radius of convergence.

Below is the graph of $f_0(x) = \sin(x)$ along with several Taylor polynomials. What do you notice?



26.4 Remainder Theorem/Error Analysis for Taylor Series

Naturally if we cut off a Taylor Series at say n=100, we are missing terms and will not get an exact solution. The sum of the missing terms is called the __________ of the Taylor Series and we denote it as $R_n(x)$. Then we can write $f(x) =$__________ +__________

Theorem If $f(x) =$__________+__________ and $\lim_{n\to\infty} R_n(x) =$ __________, for $|x-a| < R$. Then $f(x)$ is equal to its __________ series on the interval $|x-a| < R$..

We can use the following to help show that $\lim_{n\to\infty} R_n(x) =$ __________

Taylor's Inequality if $|f^{(n+1)}(x)| \leq M$ for $|x-a| < d$, then
$|R_n(x)| \leq$

for $|x-a| < d$

Pf. Omitted. See Text.

Example 26.8. *Find the maximum error in approximating* $\cos(0.4)$ *using the 10th degree Maclaurin Series (The 10th degree Taylor Polynomial at 0).*

Example 26.9. *Find the constant M when approximating $(0.1)^6$ using the 3^{rd} degree Maclaurin Series (The 3rd degree Taylor Polynomial at 0).*

Example 26.10. [4] *The resistivity ρ of a conducting wire is the reciprocal of the conductivity and is measured in unites of ohm-meters ($\Omega \cdot m$). The resistivity of a given metal depends on the temperature according to the equation*

$$\rho(t) = \rho_{20} e^{\alpha(t-20)}$$

where t is the temperature in $^\circ C$. α, the temperature coefficient, and ρ_{20}, the resistivity at $20^\circ C$, are constants depending on the metal being used.
Except at very low temperatures, the resistivity varies almost linearly with temperature and so it is common to approximate the expression for $\rho(t)$ by the its first- or second- degree Taylor polynomial at $t = 20$. Find expressions for these linear and quadratic approximations.

[4] From *Stewart's Calculus*

26.5 ICE –Taylor Series

1. Recall $\int \frac{\sin x}{x} dx$ cannot be evaluated. Instead represent this integral as a Taylor Series centered at 0.

2. Recall $\int \sin(x^2) dx$ cannot be evaluated. Instead represent this integral as a Taylor Polynomial of degree 10 centered at 0.

3. Go to Desmos Graphing Calculator. Let's explore how the 2 function above look as we add more terms to our Taylor Polynomial Appoximatings.

4. Find the Taylor series for $f(x) = e^{3x}$ centered at $x = 5$.

5. Find the Maclaurin series for $f(x) = e^x$ by deriving it (don't use a table). What is the Taylor series for $f(x) = e^x$ at $x = -1$?

6. Find the Taylor series for $f(x) = \sin x$ centered at $a = \frac{\pi}{2}$.

7. Find the Taylor series for $f(x) = x^{-2}$ centered at $a = 1$. You can derive this yourself or find the binomial expansion for $(x+1)^{-2}$ Then substitute $x - 1$ in.

8. Find the Taylor series for $f(x) = xe^x$ at 1. Hint: $f^{(n)}(x) = (x+n)e^x$.
   Then find the Maclaurin series for $f(x) = xe^x$ either by using the definition or known series from your table.

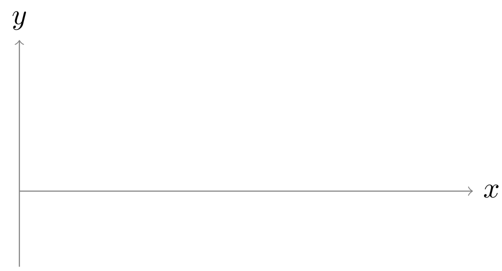
9. Find the first three terms (including 0 terms) of the Maclaurin series for $f(x) = e^x \ln(1+x)$.
   Use your chart of known Maclaurin series from your notes!

27 Review

27.1 ICE -Gabriel's Horn

Consider the object (called Gabriel's Horn) created by rotating the curve $f(x) = \frac{1}{x}$ for $x \geq 1$ about the x-axis.

1. Sketch the graph and think about why this object (made after rotation) is called Gabriel's Horn.



2. Find the volume of Gabriel's Horn using the disk method.

3. Set up an integral to represent the surface area of Gabriel's Horn.

4. Compare this improper integral to another function to show that this integral diverges. Note: You can evaluate the integral. Let $w = \frac{1}{x}$. Then use inverse substitution and then you get an integral with $\sec^2\theta \csc\theta$ so you have to simplify using $\sec^2\theta = 1 + \tan^2\theta$. Then split the integral and integrate (remembering to substitute back twice.)

A Practice Problems and Review for Exams

The following pages are practice problems and main concepts you should master on each exam. Core concepts will have heavier weight than regular concepts.

These problems are meant to help you practice for the exam and are often harder than what you will see on the exam. You should also look over all ICE sheets, Homework problems, Quizzes, and Lecture Notes. Make sure you can do all of the problems in these sets.

Disclaimer: The following lists are topics that you should be familiar with, and these are problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

I will post the solutions to these practice exams on our blackboard site. If you find any mistakes with my solutions, please let me know right away and feel free to email at any hour.

Good Luck,
Dr. H

A.1 Practice Problems for Exam 1

First look over all ICE sheets, Homework problems, Quizzes, and Lecture Notes. Make sure you can do all of those problems. You can also practice with the problems below. Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Here are the 5 Mastery Concepts:
Core Concept 1: Differentiation Techniques (**CORE Concept**)

- Know how the differentiation exponential functions, logarithmic functions, inverse trigonometric functions, hyperbolic functions using the chain rule, product rule, and quotient rule.

Core Concept 2: Integration Techniques (**CORE Concept**)

- Know how the integrate exponential functions, logarithmic functions, and hyperbolic functions.
- Know how to recognize and manipulate derivatives of inverse trig functions and inverse hyperbolic functions

Concept 3: Non-trivial Differentiation Techniques -2 parts

- Know how to use logarithmic differentiation to differentiate a function.
- Know how to find derivatives of inverse functions without solving for the inverse function.
- Know how to show that a function is always increasing and always decreasing and be able to explain why this guarantees that our function is 1-to-1.
- Know why it is important to have a 1-to-1 function.

Concept 4: Differential Equations and Slope Fields

- Be able to identify and create examples of differential equations with different properties (linear, 2nd-order, separable, etc.)
- Be able to sketch a Slope Field of a Differential Equation.
- Use Slope Fields to approximate values of the function and sketch solutions.

Concept 5: Solving Differential Equations

- Be able to solve separable differential equations.
- Be able to solve first ordered linear differential equations.

Derivatives: Know how to differentiate the following functions:

1. e^x
2. $\ln x$
3. $\ln |x|$
4. a^x
5. $\log_a x$ (can use change of basis rule to derive)
6. $\arcsin x$
7. $\arccos x$
8. $\arctan x$
9. $\text{arccot } x$
10. $\sinh x$
11. $\cosh x$
12. $\tanh x$
13. $\text{csch } x$
14. $\text{sech } x$
15. $\text{coth } x$
16. $\sinh^{-1} x$
17. $\cosh^{-1} x$
18. $\tanh^{-1} x$

Antiderivatives: Know how to integrate the following functions:

1. e^x
2. $\frac{1}{x}$ (remember absolute value!)
3. $\tan x$ (can derive by using $\frac{\sin x}{\cos x}$ and substitution)
4. a^x
5. $\dfrac{1}{\sqrt{1-x^2}}$
6. $\dfrac{-1}{\sqrt{1-x^2}}$
7. $\dfrac{1}{1+x^2}$
8. $\dfrac{1}{a^2+x^2}$ (can derive by factoring out the a^2)
9. $\dfrac{1}{\sqrt{a^2-x^2}}$ (can derive by factoring out the a^2)
10. $\dfrac{-1}{\sqrt{a^2-x^2}}$ (can derive by factoring out the a^2)
11. $\sinh x$
12. $\cosh x$
13. $\text{sech}^2 x$
14. $\dfrac{1}{\sqrt{1+x^2}}$
15. $\dfrac{1}{\sqrt{x^2-1}}$
16. $\dfrac{1}{1-x^2}$

1. Differentiate the following functions.

a) $y = \ln[(x^3 + 3x^2)^5]$, for $x > 0$

b) $h(x) = \ln(\frac{1}{x}) + \frac{1}{\ln x}$

c) $f(x) = \arcsin(x^2)$

d) $k(x) = \tanh \sqrt{x}$

e) $f(x) = \log_2 \tan(5x^3)$

f) $k(x) = \cosh^{-1}(\sinh x)$

g) $g(x) = \arctan(\arccos \sqrt{x})$

h) $y = x \cdot \pi^{3x}$

2. Find the equation to the line tangent to $y = (x-1)e^x + 3\ln x + 2$ at $x = 1$.

3. Differentiate the following functions using logarithmic differentiation.

a) $y = (5^x + x)^{\sec x}$

b) $y = (x^2 + 1)^{\ln x}$

4. Consider $f(x) = 3x^5 + x - 2$.
a) Show that f is one-to-one (Hint: Show f is increasing everywhere. Why would this tell us that f is one-to-one?]

b) Find $(f^{-1})'(2)$

5. Evaluate $\int (2^{\tan\theta} \sec^2 \theta) d\theta$

6. Evaluate $\int \frac{1}{2x^2 + 9} dx$

7. Evaluate $\int_0^1 (x^2 + 1)e^{x^3+3x} dx$

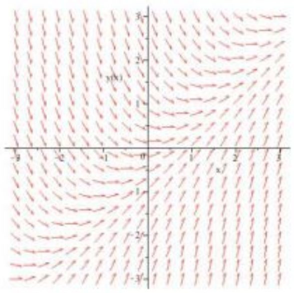
8. Evaluate $\int \frac{x-1}{3x^2 - 6x + 2} dx$

9. Evaluate $\int \cosh x + \frac{1}{\sqrt{1+x^2}} dx$

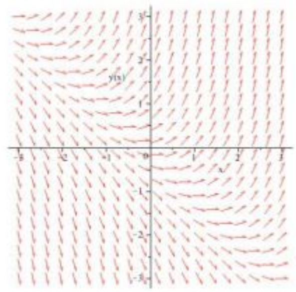
10. The half-life of radium is 1690 years. If 100 grams of radium are present initially, how much will remain after 200 years? Find an exact answer, not just a decimal approximation.

11. Which of the following is the slope field for $\dfrac{dy}{dx} = x + y$?

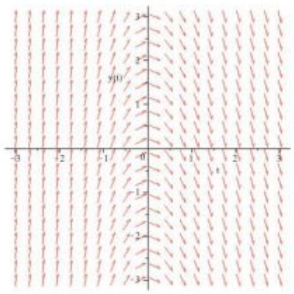
(A)



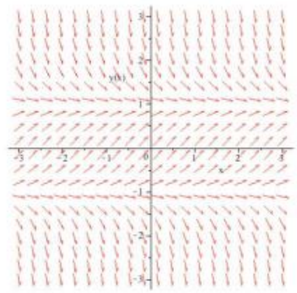
(B)



(C)



(D)



12. What does the differential equation $\dfrac{dy}{dx} = 2y$ tell us about the slope of the solution curves at at any point?
(a) The slope is always 2.
(b) The slope is equal to the x-coordinate.
(c) The slope is equal to the y-coordinate.
(d) The slope is equal to two times the x-coordinate.
(e) The slope is equal to two times the y-coordinate.
(f) None of the above.

13. Be able to recognize and construct differential equations with particular properties like (3-ordered, linear, separable, non-separable)

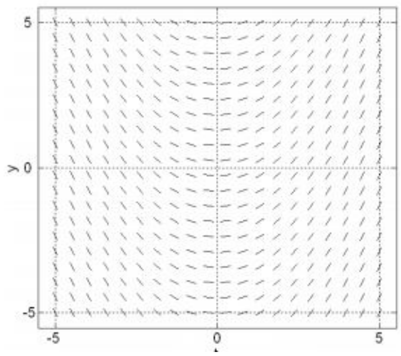
14. Solve the following differential equations

(a) $y'(t) = \dfrac{3t^2}{y}$ (b) $\dfrac{dy}{dx} = e^{\frac{y}{2}} \sin(x)$

15. Solve the following differential equations

(a) $x\dfrac{dy}{dx} = 2x^2 + 4,\ y(1) = 2$ (b) $y'(x) = -2y - 4,\ y(0) = 0$

16. The slope field for a differential equation is given below. Sketch the particular solution for the initial condition $y(0) = -5$ and use it to approximate the value of $y(2.5)$ and $y(5)$.



17. Find the general and particular solutions to the following differential equations

(a) $y' + 4y = e^{-3x}$, $y(0) = 4$

(b) $xy' + 4y = 10x$, $y(1) = 0$, $x > 0$

A.2 Practice Problems for Exam 2

First look over all ICE sheets, Homework problems, Quizzes, and Lecture Notes. Make sure you can do all of those problems. You can also practice with the problems below. Also look over material from previous exams! Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Here are the 4 New Mastery Concepts:

CORE Concept 6: Advanced Integration Techniques: Integration by Parts and Partial Fractions

- Recognize and know how to integrate using Integration by Parts
- Recognize and know how to integrate using Partial Fractions

CORE Concept 7: Advanced Integration Techniques: Higher Trig and Inverse Trig Substitution

- Recognize and know how to integrate using Higher Trigonometric Techniques
- Recognize and know how to integrate using Inverse Trigonometric Substitution

Concept 8: Identifying Advanced Integration Techniques:

- You will be given integrals in which you should identify which technique is best to use when integrating and describe your first step for solving the problem.
- Recognize when to integrate using Substitution, Recognition of a derivative of a known function, Integration by Parts, Higher Trigonometric Techniques, Inverse Trigonometric Substitution, and Partial Fractions

Core Concept 9: L'Hopital's Rule (**CORE Concept**)

- See last page on Practice Problems

Assume all domains of integrals are ok for indefinite integrals.

1. Find the partial fraction for $\dfrac{x}{x^4-1}$. You do not need to solve for the coefficients.
2. Find the partial fraction for $\dfrac{x^2+2x-1}{2x^3+3x^2-2x}$. You do not need to solve for the coefficients.
3. Evaluate $\int \arcsin x dx$
4. Evaluate $\int \dfrac{1}{\sqrt{x^2+5}^3} dx$
5. Evaluate $\int \dfrac{1}{x^2\sqrt{4-x^2}} dx$
6. Evaluate $\int \dfrac{10}{(x-1)(x^2+9)} dx$
7. Evaluate $\int \dfrac{dx}{\sqrt{4+x^2}}$
8. Evaluate $\int \tan^4 x \sec^2 x dx$
9. Evaluate $\int \dfrac{1}{(x-1)(x+5)^2} dx$
10. Evaluate $\int \dfrac{dx}{\sqrt{x^2+2x+5}}$
11. Evaluate $\int \sec x dx$.
12. Evaluate $\int x \tan x \sec^2 x dx$
13. Evaluate $\int \sin^4 x dx$
14. Evaluate $\int \dfrac{dx}{\sqrt{x^2-1}}$
15. Evaluate $\int \dfrac{e^{2x}}{e^{2x}+5e^x+6} dx$

more problems on next page

16. Determine the limits if they exist.

a) $\lim_{x \to 0} \dfrac{e^x - x - 1}{x^2}$

b) $\lim_{x \to 0} (1 - 2x)^{\frac{1}{x}}$

c) $\lim_{x \to 0^+} \dfrac{\sin(ax)}{\sin(bx)}$ for $a, b > 0$

d) $\lim_{x \to 0^+} (x + e^{\frac{x}{3}})^{\frac{3}{x}}$

e) $\lim_{x \to \infty} xe^{-x}$

A.3 Practice Problems for Exam 3

First look over all ICE sheets, Homework problems, Quizzes, and Lecture Notes. Make sure you can do all of those problems. You can also practice with the problems below. Also look over material from previous exams! Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Here are the 4 (maybe 5) new Mastery Concepts:

Concept 10: Numerical Integration

- You will need to know how to use the Trapezoid Rule.
- You will need to know how to use Simpson's Rule.
- You will need to know how to find the error in a trapezoid approximation and a Simpson's approximation, but I will give you the error formulas.

Concept 11: Surface Area

- I will ask you to set up integrals that represent the surface area from 3-dimensional shapes made from rotating functions around lines.
- Understand which radius and arc length formula to use depending on which variable you are integrating with respect to and which axis you are rotating around.
- See questions 3-6.

Concept 12: Work and Hydrostatic Force (**CORE Concept**)

- Know how to set up and sometimes solve integrals that represent the work done depending on the situation
- Be able to set up an integral that calculates hydrostatic force.
- Be able to set up an integral that calculates the work done in pumping out water from a tank.
- See Questions 7-13.

Concept 13: Improper Integrals

- Know how to solve improper integrals using a limit.
- Know how to use comparison tests to check if improper integrals diverge or converge.

Note: I may include Concept 14 (See Practice Exam 4) on Exam 3 depending on the schedule, but I will clarify this in class. So you may want to look at the sequence problems on the Review for Exam 3.

1. Set up an integral representing the surface area obtained by rotating $y = \sqrt{1+e^x}$ about the *x-axis* for $0 \leq x \leq 1$, integrate with respect to x.

2. Set up an integral representing the surface area obtained by rotating $y = \sqrt{1+e^x}$ about the *x-axis* for $0 \leq x \leq 1$, integrate with respect to y.

3. Set up an integral representing the surface area obtained by rotating $y = \sqrt{1+e^x}$ about the *y-axis* for $0 \leq x \leq 1$, integrate with respect to x.

4. Set up an integral representing the surface area obtained by rotating $y = \sqrt{1+e^x}$ about the *y-axis* for $0 \leq x \leq 1$ but integrate with respect to y.

5. A 200 ft cable that weighs 10 lb/ft is used to lift a 1600 lb crate of chinchillas out of a 100 foot hole. How much work is required to save the chinchillas (lift them out of the hole)?

6. Suppose a tank has a shape of a half a sphere (spherical part on the bottom) with a radius of 4 ft and a 1 foot spout. Suppose it is completely filled with water. **Set up the integral** used to find the **work** required to empty the tank by pumping all the water out of the spout. Use the fact that the weight of water is 62.5 lb/ft^3.

7. Suppose a tank has a shape of triangular prism with a height of 4 ft and length of 8 ft and a width of 6 ft with a 2 foot spout. Suppose it is filled to a height of 3 ft with water. Set up an integral to represent the work required to empty the tank by pumping all the water out of the spout. Use the fact that the weight of water is 62.5 lb/ft^3.

8. Suppose a tank has a shape of rectangular prism with a height of 3 ft and length of 5 ft and a width of 2 ft with a 1 foot spout. Suppose it is completely filled with water. Set up an integral to represent the work required to empty the tank by pumping all the water out of the spout. Use the fact that the weight of water is 62.5 lb/ft^3.

9. A right triangular plate with height of 3 ft, a base of 4ft and hypotenuse of 5ft is submerged 2 ft below the surface of the water. Express the hydrostatic force against one side of the plate.

10. A semi-circular plate with diameter 8m is partially submerged in water so that all but the top 2 meters is submerged in the water. Express the hydrostatic force against one side of the plate. The straight part of the semi-circle is out of the water.

11. A square plate with side length of 8m is partially submerged in water so that all but the top 3 meters is submerged in the water. Express the hydrostatic force against one side of the plate.

12. Evaluate the improper integral or show that it diverges. $\displaystyle\int_1^\infty \frac{\ln x}{x^3}dx$

13. Determine whether the integral converges or diverges by using the comparison test: $\displaystyle\int_1^\infty \frac{\cos x}{1+x^2}dx$
.

14. Evaluate the improper integral or show that it diverges. $\displaystyle\int_0^4 \frac{1}{(2-3x)^{\frac{1}{3}}}dx$

15. Use the LCT to determine whether or not $\displaystyle\int_1^\infty \frac{x}{x^3-x-1}dx$ converges or diverges

16. a) Use $n=4$ in Simpson's Rule **and** in the Trapezoid Rule to approximate the area under the curve given by $f(x)=x^3$ between $x=1$ and $x=3$.

    b) Find the maximum error in T_4 and S_4

A.4 Practice Problems for Exam 4

First look over all ICE sheets, Homework problems, Quizzes, and Lecture Notes. Make sure you can do all of those problems. You can also practice with the problems below. Also look over material from previous exams! Disclaimer: The following is a list of topics that you should be familiar with, and a list of problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

Here are the New Mastery Concepts:

Concept 14: Theoretical Understanding of Sequence & Series Properties, Definitions, & Theorems

1. These will most likely be short answer or true or false. See problems 5-10
2. Look back through your sequence and series notes.
3. Know that a monotonic and bounded sequence converges.
4. Know the definition of monotonic, bounded, convergent, and divergent sequences.
5. Know the relationship between the sequence of partial sums and series.

CORE Concept 15: Interval of Convergence for Power Series

- See questions 18-21 on Practice Problems.
- Don't forget to check the endpoints.

Concept 16: Geometric Series

- Determine whether or not a geometric series converges or diverges and find the sum if it is convergent.
- See Questions 2,4, and 11 on Practice Problems

Concept 17: Series Tests

- Know how and when to use the Test for Divergence
- Know how and when to use the Integral test
- Know how and when to use the Comparison test and Limit Comparison test
- Know how and when to use the Alternating Series test
- Know how and when to use the Ratio test
- Know how and when to use the Root test
- See Questions 3, 5-10, 12-17 on Practice Problems

Core Concept 18: Taylor Series

- Know how to find the Taylor or Maclurin Series for a function
- See Problems 24,25, 26
- I will give you the formula for the Taylor Series.

1. Find the sum of the telescoping series: $\sum_{n=1}^{\infty}(\frac{3}{n(n+3)})$

2. Find the sum of the geometric series $\sum_{n=1}^{\infty}\frac{5}{4^n}$ or state that it diverges.

3. Determine whether $\sum_{n=1}^{\infty}\frac{(n+1)^2}{n(n+4)}$ exists. Find the limit if you can. If the limit does not exist, state why.

4. Determine whether $\sum_{n=1}^{\infty}(5(1/2)^n - 3(1/2)^{n+1})$ exists. Find the limit if you can. If the limit does not exist, state why.

5. Suppose I have a series $\sum_{n=1}^{\infty}a_n$ and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = 1$. What conclusion, if any, can be drawn about the convergence of $\sum_{n=1}^{\infty}a_n$? Refer to any Tests, Theorems, or Definitions you use.

6. Suppose I have a series $\sum_{n=1}^{\infty}a_n$ and $\lim_{n\to\infty}\frac{a_n}{\frac{1}{n}} = \frac{1}{2}$. What conclusion, if any, can be drawn about the convergence of $\sum_{n=1}^{\infty}a_n$? Refer to any Tests, Theorems, or Definitions you use.

7. Suppose I have a series $\sum_{n=1}^{\infty}a_n$, $\sum_{n=1}^{\infty}b_n$ diverges, and $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$. What conclusion, if any, can be drawn about the convergence of $\sum_{n=1}^{\infty}a_n$? Refer to any Tests, Theorems, or Definitions you use.

8. Does the series $\sum_{n=1}^{\infty}\sin(\frac{n\pi}{4})$ converge? If so, find the sum. If not state why. Refer to any Tests, Theorems, or Definitions you use.

9. Does the series $\sum_{n=2}^{\infty}(-1)^n\sqrt{n^2-1}$ converge absolutely, conditionally, or diverge? Refer to any Tests, Theorems, or Definitions you use.

10. Consider the series $\sum_{n=1}^{\infty} \frac{3}{n}$

a) Write the first three terms of the sequence of terms.

b) Write the first three terms of the sequence of partial sums.

c) Does the sequence of terms converge? If so what is its limit? If not, explain why it diverges.

d) Does the series converge? If so what is its limit? If not, explain why it diverges referring to any theorems or tests used.

11. Does the series $\sum_{n=5}^{\infty} \frac{4^{n+1}}{7^{n-1}}$ converge? If so, find its sum. If not, explain why it diverges.

12. Use the limit comparison test to determine whether $\sum_{n=1}^{\infty} \frac{n^2-4}{3n^4+10}$ converges or diverges.

13. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{0.1}}$ is conditionally convergent, absolutely convergent, or divergent.

14. Determine whether $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ is conditionally convergent, absolutely convergent, or divergent.

15. Determine whether $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+1}{3^n}$ is conditionally convergent, absolutely convergent, or divergent.

16. Determine whether $\sum_{n=1}^{\infty} \frac{(-5)^{2n}}{n^2 9^n}$ is conditionally convergent, absolutely convergent, or divergent.

17. Determine whether $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$ is conditionally convergent, absolutely convergent, or divergent.

18. Find the radius of convergence and the interval of convergence of the power series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{\sqrt{n}}$.

19. Find the radius of convergence and the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{x^n}{(\ln n)^n}$.

20. Find the radius of convergence and the interval of convergence of the power series $\sum_{n=1}^{\infty} \frac{(2n)!(x-1)^n}{(n!)^2}$.

21. Find the radius of convergence and the interval of convergence of the power series $\sum_{n=0}^{\infty} \frac{n^n(x+2)^n}{3^n}$.

22. Find a power series representation for $f(x) = \frac{x}{1-x^2}$ for $-1 < x < 1$

23. Find a power series to represent $f(x) = \frac{2}{3+x}$.

24. Find the first four terms of the Maclaurin series expansion of the function $f(x) = \sqrt{1+x}$ at 0 for $-1 < x < 1$. Use the Binomial series or derive it yourself.

25. Find the $T_4(x)$ for $f(x) = \sin x$ at a=1.

26. Find the Taylor series expansion of $f(x) = \sinh x$ at x=0. Recall $\sinh 0 = 0$ and $\cosh 0 = 1$.

B Homework

Please write your solutions on these homework pages and show enough of your work so that I can follow your thought process. This makes it easier for me to grade. Also please **staple** the homework together before you turn it in. Sometimes I have my stapler, but there is also a stapler in my office and at the CaMS Study Tables in AS-107-S.

Follow the instructions for each question. If I can't read your work or answer, you will receive little or no credit!

B.1 Calculus II HW 1: Due Fri 9/4 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers. **There are Two sides!**

1. Differentiate the following functions -You do not need to simplify once you have correctly done the differentiation:

   (a) $y = \dfrac{2}{(3x-1)^2} + \sqrt{7x+2}$

   (c)$h(x) = x^2 \tan(5x)$

   (b) $g(x) = [\sec(\cos x)]^3$

   (d) $f(x) = \dfrac{x-1}{2x-3}$

2. Decide if the following are invertible and explain your reasoning.

   a) $f(d)$ is the total number of gallons of fuel an airplane has used by the end of d minutes of a particular flight.

   b) $f(n)$ is the number of students in your calculus class whose birthday is on the n^{th} day of the year.

3. Evaluate the following integrals.

(a) $\int (\frac{2}{x^3} + \sqrt{x} + 1)dx$

(b) $\int (3x+1)^2 dx$

(c) $\int_0^4 \frac{x}{\sqrt{x^2+5}}dx$

(d) $\int \sin(x)\sec^2(\cos(x))dx$

B.2 Calculus II HW 2: Due Fri 9/11 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers. There are four pages which are two-sided!

1. Consider $f(x) = 3x^3 + x - 2$.
   a) Use the derivative to show that f is increasing everywhere.

   b) Why does our solution to (part a) show that f is one-to-one?

   c) Find $(f^{-1})'(2)$

2. Differentiate $c(x) = e^{e^x}$.

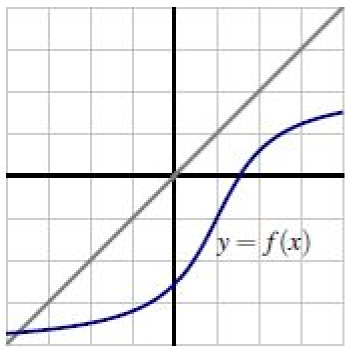
3. The famous Math Kitties, Eva and Archer are having a discussion about calculus. Archer says that to differentiate $f(x) = \ln(\frac{x^2 \cdot e^x}{x+1})$, he needs to first use both the chain rule and the quotient rule to determine this derivative. When he does this, he gets: $f'(x) = \frac{(x+1)(x^2e^x + 2xe^x) - x^2e^x}{(x+1)^2} \cdot \frac{x+1}{x^2e^x}$. Eva scoffs and says that while Archer is indeed correct, he created more work for himself. She claims that one can use logarithmic properties to calculate this derivative without using any product or quotient rules. Is Eva right? If so, calculate $f'(x)$ using her method and (bonus) show that you get the same answer as Archer.

4. The number of cats in a kitty daycare that are infected with stomach flu, t days after the start of an outbreak, is approximately $F(t) = 60te^{-t}$.
The Inept Dr. Van Nostran says: "Because $F'(t) = 0 \cdot 1 \cdot e^{-t} = 0$, no kitties are ever infected." Explain what mistake(s) the Inept Dr. made, correct his work, and write one clear sentence that explains the meaning of $F'(0)$ in real-world terms.

5. Evaluate the following:
a) $\displaystyle\int \frac{3\cos x + 9x^2 + 3}{\sin x + x^3 + x}dx$

b) $\displaystyle\int \frac{x}{5 + 3x^2}dx$

6. Consider the graph of $y = f(x)$ provided in the figure below and use it to answer the following questions.



(a) Use the provided graph to estimate the value of $f'(1)$.

$f'(1) \approx$ __________.

(b) Sketch an approximate graph of $y = f^{-1}(x)$. Label at least three distinct points on the graph that correspond to three points on the graph of f.

(c) Based on your work in (a), what is the value of $(f^{-1})'(-1)$? Why?

7. Consider how Archer used his method of switching the variables and solving for the dependent variable to find the inverse for $f(x) = \dfrac{2x+1}{x-1}$ for $x \neq 0$.. Describe why his work for the function is problematic.

$$y = \frac{2x+1}{x-1}, \quad x \neq 1$$

$$x = \frac{2y+1}{y-1}$$

$$(y-1)x = 2y+1$$

$$yx - x = 2y+1$$

$$yx - 2y = 1 + x$$

$$y(x-2) = x+1$$

$$y = \frac{x+1}{x-2}, \quad x \neq 1$$

B.3 Calculus II HW 3: Due Fri 9/25 Name: ______________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers. There are four pages which are two-sided!

1. Use logarithmic differentiation to find the derivative of $y = x^{\cos(x)}$.

2. We are given that the doubling time for a bacteria culture with an initial number of 60 cells is 10 hours. If we assume the culture grows at a rate proportional to its size, when will we have a population reach 87?

3. Find the general solution to the following differential equations

a) $\frac{dy}{dx} = 3x^2y - 4y$

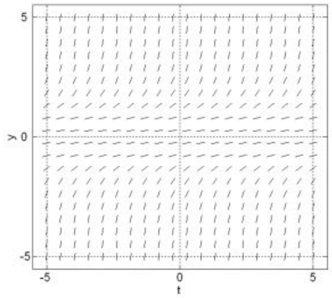
b) $y' = \dfrac{7e^{2x}}{y^2}$

4. Find the general solution to the following differential equations

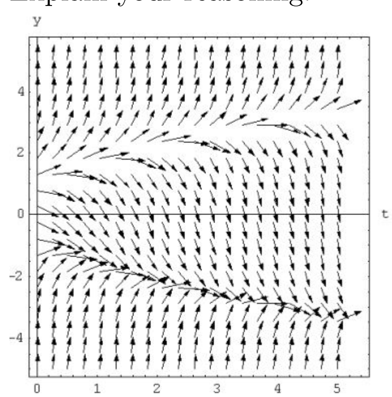
a) $e^{-x}y' + e^{-x}y = 4$

b) $\frac{dy}{dx} + 3x^2y = x^2$, $y(0) = -1$

5. Consider the slope field below. Which of the following forms is this differential equation: $\frac{dy}{dt} = f(y)$, $\frac{dy}{dx} = f(t)$, or $\frac{dy}{dt} = f(y,t)$. Explain your reasoning.



6. The arrows in the slope field below have slopes that match the derivative y' for a range of values of the function $y = f(t)$. Suppose that $y(0) = -4$. Draw the particular solution to the differential equation with this initial value condition. What would you predict for $y(5)$? Explain your reasoning.



7. The incredible math kitties Eva and Archer are looking at the slope field above. Archer says that from the graph he can see that if he follows the solution with the given value $y(0) = 1$, he can see that at $t = 1$, the value of $y' \approx 0$. Is this the correct way to interpret y' from this graph? If not, what is the correct way to interpret y' and y when $t = 1$ with the initial value $y(0) = 1$?

8. Identify which of the following are first ordered linear differential equations and identify $P(x)$ and $Q(x)$ if they are first ordered linear differential equations.

   (a) $\cos x \frac{dy}{dx} - \sin(x)y + x = 1$

   (b) $(\frac{dy}{dx})^2 = x^4y^2 - 1$

   (c) $(\frac{dy}{dx})^3 = 10x^4$

   (d) $\frac{dy}{dx} - x\sin(y) = 2e^x$

   (e) $\frac{dy}{dx} - y\sin(x) = 2e^x$

9. Evaluate the following using hyperbolic functions:
   a) $\int \frac{3\cosh x + 9x^2 + 3}{\sinh x + x^3 + x} dx$

   b) $\int \frac{1}{\sqrt{5+3x^2}} dx$

B.4 Calculus II HW 4: Due Fri 10/16 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers. There are two sides!

1. Evaluate $\int_1^2 x^2 \ln x dx$.

2. Evaluate $\displaystyle\int 2xe^{(x^2+e^{x^2})}dx$. [Hint: Simplify using $[a^x \cdot a^y = a^{x+y}]$ first.

3. Evaluate $\displaystyle\int e^x \cos(2x)dx$.

4. The famous Math Kitties, Eva & Archer are studying for their calculus class. Archer looks at $\int(\sin(x))^3dx$ and sees an inside function $\sin x$. He tells Eva, that he remembers that Dr. Harsy says to look for the simple substitution first, so he is going to try the substitution $w = \sin x$. Explain why this substitution will not work to help evaluate the given integral and suggest a better approach to this integral. Then proceed to evaluate $\int(\sin(x))^3dx$.

5. Use Inverse Trigonometric Substitution to evaluate $\displaystyle\int \frac{x^3}{\sqrt{9+x^2}}dx$.

6. Evaluate $\int (\arcsin x) dx$.

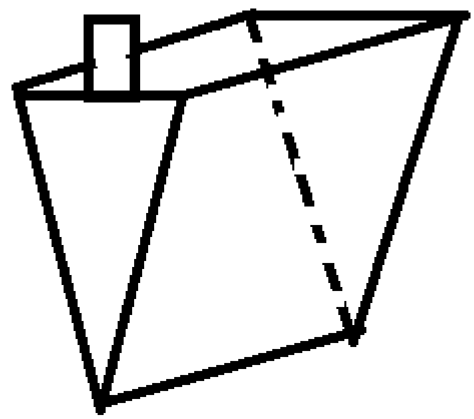
7. Evaluate $\int \frac{x-2}{x^4+x^2} dx$.

8. The famous Math Kitties, Eva and Archer are discussing how to evaluate $\displaystyle\int \frac{1}{4-x^2}dx$. Archer wants to use partial fractions to evaluate the integral, but Eva says she recognizes a version of an inverse hyperbolic function. Who is correct? Are they both correct? If so, evaluate this integral using each of their methods.

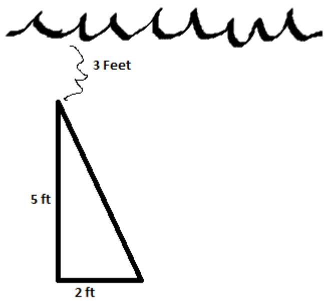
B.5 Calculus II HW 5: Due Fri 11/6 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers. There are two sides!

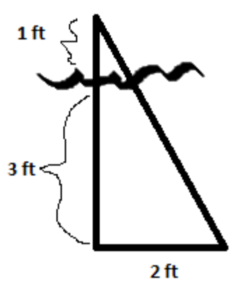
1. Suppose a tank has a shape of triangular prism with a height of 5 ft and length of 8 ft and a width of 4 ft with a 1 foot spout. Suppose it is filled with water up to a height of 3 ft. **Set up an integral** to represent the **work** required to empty the tank by pumping all the water out of the spout. Use the fact that the weight of water is 62.5 lb/ft^3.



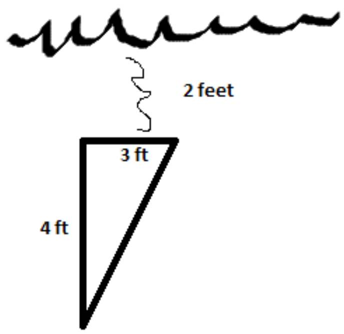
2. Set up an integral that expresses the hydrostatic force against one side of the vertical plate that is submerged in water with the shape indicated by the picture below. The density of water is 62.5 lb/ft^2.



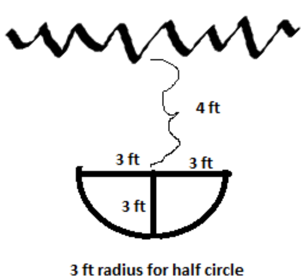
3. Set up an integral that expresses the hydrostatic force against one side of the vertical plate that is submerged in water with the shape indicated by the picture below. The density of water is 62.5 lb/ft^2.



4. Set up an integral that expresses the hydrostatic force against one side of the vertical plate that is submerged in water with the shape indicated by the picture below. The density of water is 62.5 lb/ft^2.



5. Set up an integral that expresses the hydrostatic force against one side of the vertical plate that is submerged in water with the shape indicated by the picture below. The density of water is 62.5 lb/ft^2.



6. **Set up** an integral that represents the area of the surface obtained by rotating the curve $y = e^{2x}$ for $0 \leq x \leq 1$ about the following lines.**DO NOT EVALUATE THE INTEGRAL.**

   (a) about the x-axis with respect to x

   (b) about the x-axis with respect to y

   (c) about the y-axis with respect to x

   (d) about the y-axis with respect to y

7. Evaluate $\int_0^1 \frac{1}{x^{\frac{3}{2}}} dx$ or determine if it diverges.

8. Use a Comparison Theorem or The Limit Comparison Theorem to determine whether $\displaystyle\int_2^{\infty} \frac{1}{1+8x+x^3}\,dx$ converges/diverges.

9. Use a Comparison Theorem or The Limit Comparison Theorem to determine whether $\displaystyle\int_2^{\infty} \frac{x^2+x+2}{1+8x+x^3}\,dx$ converges/diverges.

B.6 Calculus II HW 6: Due Fri 11/20 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers.

1. Determine whether the following sequences are convergent. You must show some form of work or explanation.

   a) $a_n =< \ln(\frac{1+n}{n}) >_{n=1}^{\infty}$

   b) $a_1 = 2$ and $a_{n+1} = \dfrac{1}{3 - a_n}$ [Hint: Show a_n is bounded and decreasing and use a theorem.]

   c) $c_n = \{e^{-n}n^5\}_{n=0}^{\infty}$

   d) $d_n = \{1 + \frac{(-1)^n}{n}\}_{n=0}^{\infty}$

2. Find the sum of $\sum_{n=1}^{\infty} \frac{2}{n^2+2n}$ or show that it diverges. Use the fact that $\frac{2}{n^2+2n} = \frac{1}{n} - \frac{1}{n+2}$.

3. For the following series, find the sum of the series or determine that it diverges by writing the series as a geometric series.

a) $\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{3^n}$

b) $\sum_{n=0}^{\infty} \frac{\pi^n}{3^n}$

c) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^{n-1}}$

d) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}}$

4. We can write $x = 0.999999999...$ as the sum of the geometric series $x = \displaystyle\sum_{n=1}^{\infty} \frac{9}{10^n}$. Show that $0.9999999... = 1$.

5. The incredible Math Kitties, Eva and Archer are trying to determine the convergence of $\sum_{n=3}^{\infty} \frac{n^2-n}{3n^2+n-1}$. Archer says that since the sequence $\{\frac{n^2-n}{3n^2+n-1}\}$ converges, so must the series. Is Archer correct? If not, help Eva correct his logic.

6. The Fabulous Math Kitties, Eva and Archer have found that the nth partial sum for the series $\sum_{n=1}^{\infty} a_n$ is $s_n = \frac{n-1}{n+2}$. Archer says that since the limit of the sequence of partial sums is 1 and not 0, the series diverges. Eva says that the limit of the sequence of partial sums gives us the value that our series approaches, and thus the sum of our series is 1. Which Kitty is correct? Explain why and include what mistakes the kitty with the misconception has.

B.7 Calculus II HW 7: Due Mon 12/7 Name: ____________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit! Please submit this sheet with your answers.

1. Find the radius of convergence and the interval of convergence for the following series.

   a) $\displaystyle\sum_{n=1}^{\infty} 2^n(x-2)^n.$

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{(x)^n}{n^n}.$

c) $\sum_{n=1}^{\infty} n!(x)^{n+1}$.

2. Find a power series to represent $f(x) = \operatorname{arctanh} x$. Use the fact that $\operatorname{arctanh} x = \int \frac{1}{1-x^2} dx$.
Hint: Don't use a Taylor series.

3. Find the third degree Taylor polynomial $T_3(x)$ for $f(x) = e^x \ln(1+x)$ centered at 0. Use your list of Maclurin series from your notes.

4. a) Find $T_5(x)$ (the fifth degree Taylor polynomial) for $f(x) = \cos x$ centered at $\frac{\pi}{2}$.

b) Now find a summand formula for the whole Taylor series.

5. The Esteemed Math Kitties Eva and Archer are determining the convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n+1}$. Archer wants to use the Ratio test. Eva says that the Ratio Test will not work in this case. Who is correct? What test should be conducted?

6. The Studious Math Kitties Eva and Archer are trying to determine the convergence for $\sum(-1)^n a_n$. Archer has found that this series converges by the Alternating Series Test, and says that this must mean $\sum a_n$ also converges. Is he correct?

B.8 Calculus II HW 8: Due Fri 12/11 Name: ____________

This is your last HW for the year!

1. Please complete the Lewis University evaluation for my Calc II class. In the past it has been posted on Bb, but last semester students received an email from noreply@tk20.com or caleighaoconnell@lewisu.edu with a link to do the evaluations. Once you have completed the evaluation, you will get an email. To get credit and in order for me to calculate whether 90% of the class has completed the evaluation, please forward this email to me. Please give thoughtful, constructive feedback which will help me improve the course. For example saying "You suck" or "You are great" doesn't provide much feedback for me. Saying "You suck because..." or "You are great because..." Also, remember for everything you like about the course, there is at least one other person who dislikes it, so please let me know what you would like to be kept the same about the course.

   Check one:
   ☐ I completed the evaluation.
   ☐ I have not completed the evaluation.

2. Create a Meme about this course. It can be something about the topic we covered this semester, but it should relate in some way to this course. On the last day of school, we will share all of the memes and vote for the best one. [Note: You can use a meme that already exists if it relates to this course, but it is more fun to create your own.] I will create an assignment in blackboard in which you can upload your Meme or you can print it and attach to this paper. You can find pictures of Eva and Archer if you would like to use them in your meme here: `http://bit.ly/EvaArcherPics`.

3. Create a "good" Archer answer relating to something from what we learned in class this semester. So this should be an incorrect answer (but not a *trivial* incorrect answer) that demonstrates a subtle misconception about a concept or topic in this course. Then write what Eva should say in order to help explain what Archer's misconception is and help correct his mistake. You may want to use the backside of this paper.