Northwestern University

Name: SOLUTIONS
NetID:

Math 224: Integral Calculus

Winter 2016 Final Exam

Roy-Fortin	8:00 am	
Roy-Fortin	9:00 am	
Sweet	10:00 am	
Jin	11:00 pm	
Jin	12:00 pm	
Cuzzocreo	1:00 pm	
Maltenfort	2:00 pm	

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

Instructions:

- You must show your work to receive full credit except when noted otherwise or on True/False questions.
- Do not write above the horizontal line on each page.
- Put a check mark next to your section in the table to the left.
- Read each problem carefully.
- Write legibly and in complete sentences.
- Cross off anything you do not wish graded.
- This exam has 16 pages with 12 questions, for a total of 200 points. Make sure that all pages are included.
- You may not use books, notes, or electronic devices.
- You may ask proctors questions to clarify problems on the exam.
- Graded work should only be written in the provided spaces (not on scratch papers) unless otherwise approved by a proctor.
- You have 120 minutes to complete this exam.

Good luck!

1. Are the following statements are true or false? Circle your answer. No explanation is necessary.

(a) (3 points) If two positive sequences $\{a_n\}$ and $\{b_n\}$ converge, then the sequence $\left\{\frac{a_n}{b_n}\right\}$ always converges.

TRUE or **FALSE**

$a_n = \frac{1}{n}$ $b_n = \frac{1}{n^2}$

$a_n / b_n = n$

(b) (3 points) Suppose the Taylor series of $f(x)$ centered at $a = 5$ has radius of convergence $R = \infty$. Then for all values of x, the following equality must hold

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n.$$

TRUE or **FALSE**

Taylor series does not always converge to f itself

(c) (3 points) If a power series centered at $a = 4$ converges at $x = 2$, then it must converge at $x = 5$.

TRUE or **FALSE**

2 is 2 away from 4, so radius is at least 2 and 5 falls within this radius

(d) (3 points) The finite sum $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{99}$ gives an estimate of the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ with an error that is at most 0.01.

error $\leq b_{100} = \frac{1}{100} = 0.01$

$b_n = \frac{1}{n}$

TRUE or **FALSE**

(e) (3 points) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

TRUE or **FALSE**

Test for Divergence

(f) (3 points) The trapezoid approximation of the integral $\int_a^b f(x)\,dx$ using n equal subintervals is equal to

$$\frac{1}{2}(L_n + R_n),$$

where L_n and R_n are the left endpoint and right endpoint approximations, respectively.

TRUE or **FALSE**

$L_n = \Delta x \left(f(x_0) + \cdots + f(x_{n-1})\right)$

$R_n = \Delta x \left(f(x_1) + \cdots + f(x_n)\right)$

$\frac{1}{2}(L_n + R_n)$

$= \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$

2. (16 points) Find the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x-4)^n}{n \cdot 2^n}.$$

State which tests you are using and why they apply.

$$a_n = \frac{(-1)^n (x-4)^n}{n \cdot 2^n}$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x-4|^{n+1}}{(n+1)2^{n+1}} \frac{n2^n}{|x-4|^n} = \lim_{n\to\infty} \frac{|x-4|}{2} \frac{n}{n+1}$$

$$= \lim_{n\to\infty} \frac{|x-4|}{2} \frac{1}{1+\frac{1}{n}} = \frac{|x-4|}{2}$$

converges when $\frac{|x-4|}{2} < 1$ by ratio test.

$\hookrightarrow |x-4| < 2 \longrightarrow x$ in $(4-2, 4+2) = (2,6)$

endpoints

$x=2$: $\sum_{n=1}^{\infty} \frac{(-1)^n(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges by p-series test

$x=6$: $\sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by alternating series test

since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n}$ decreases (denominator increases)

$\boxed{\text{interval of convergence is } (2,6]}$

3. Consider the following sequence:

$$\frac{1}{2}, -\frac{4}{4}, \frac{9}{8}, -\frac{16}{16}, \frac{25}{32}, -\frac{36}{64}, \dots$$

(a) (6 points) Find a formula for the n^{th} term a_n, with the first term corresponding to $n = 1$.

numerators are $1^2, -2^2, 3^2, -4^2, 5^2, -6^2, \dots$

denominators are $2^1, 2^2, 2^3, 2^4, 2^5, 2^6, \dots$

So $\boxed{a_n = \frac{(-1)^{n+1} n^2}{2^n}}$

(b) (6 points) Determine if the sequence converges or diverges. If it converges, find its limit. Justify your answer.

$$\lim_{n\to\infty} \frac{n^2}{2^n} = \lim_{n\to\infty} \frac{2n}{2^n \ln 2} = \lim_{n\to\infty} \frac{2}{2^n (\ln 2)^2} = 0$$

L'Hopital's Rule

since $-\frac{n^2}{2^n} \leq a_n \leq \frac{n^2}{2^n}$

$-\frac{n^2}{2^n} \to 0$, $\frac{n^2}{2^n} \to 0$

$\lim_{n\to\infty} a_n = 0$ by squeeze theorem.

4. Let $f(x) = x^{7/2}$.

(a) (10 points) Find the third-degree Taylor polynomial, $T_3(x)$, for $f(x)$ centered at $a = 1$.

$f(x) = x^{7/2}$ $\quad f'(x) = \frac{7}{2} x^{5/2}$ $\quad f''(x) = \frac{35}{4} x^{3/2}$

$f^{(3)}(x) = \frac{105}{8} x^{1/2}$

$$f(1) + f'(1)(x-1) + \frac{f''(1)}{2}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$$

$$= 1 + \frac{7}{2}(x-1) + \frac{35}{8}(x-1)^2 + \frac{105}{48}(x-1)^3$$

(b) (10 points) Find an upper bound for $|R_3(x)|$ on the interval $\frac{3}{4} \le x \le \frac{5}{4}$. You do not need to simplify your final answer. Justify your reasoning.

$\frac{3}{4} \le x \le \frac{5}{4}$ means $|x-1| \le \frac{1}{4}$ $\qquad f^{(4)}(x) = \frac{105}{16\sqrt{x}}$

$$|R_3(x)| = \left| \frac{f^{(4)}(c)}{4!}(x-1)^4 \right|$$

Taylor Remainder, i.e. "error"

$$= \left| \frac{105}{4! \cdot 16\sqrt{c}} \right| |x-1|^4$$

$$\le \frac{105}{4! \cdot 16 \cdot \sqrt{\frac{3}{4}}} \left(\frac{1}{4}\right)^4$$

Since $\frac{3}{4} \le c$ so $\sqrt{\frac{3}{4}} \le \sqrt{c}$

5. In each part of this problem, write your answer using sigma notation or write the first four nonzero terms of the power series.

   (a) (8 points) Find a power series representation of the function $\dfrac{1}{1-x^3}$ centered at 0.

$$\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n \quad , \quad \text{Set } y = x^3$$

$$\text{So } \frac{1}{1-x^3} = \sum_{n=0}^{\infty} x^{3n}$$

   (b) (8 points) Use your answer from part (a) to evaluate $\displaystyle\int \frac{x}{1-x^3}\,dx$ as a power series centered at 0.

$$\frac{x}{1-x^3} = x \sum_{n=0}^{\infty} x^{3n} = \sum_{n=0}^{\infty} x^{3n+1}$$

$$\text{So } \int \frac{x}{1-x^3}\,dx = \sum_{n=0}^{\infty} \frac{x^{3n+2}}{3n+2} + C$$

6. (a) (6 points) Write the Maclaurin series for $f(x) = e^x$. No explanation is needed.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

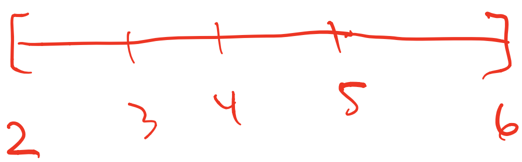
(b) (8 points) Find the exact value of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(\ln 7)^n}{n!}$. Do not prove this series converges.

$$\sum_{n=0}^{\infty} \frac{(\ln 7)^n}{n!} = e^{\ln 7} = 7$$

Set $x = \ln 7$ in part (a)

7. This problem concerns the integral $\displaystyle\int_2^6 \frac{1}{x^2-1}\,dx$.

(a) (8 points) Approximate the integral by using the Trapezoid Rule with $n = 4$ subintervals. You do not need to simplify your final numerical answer.



$$\Delta x = \frac{6-2}{4} = 1$$

$$\int_2^6 \frac{1}{x^2-1}\,dx \approx \frac{\Delta x}{2}\left(f(2)+2f(3)+2f(4)+2f(5)+f(6)\right)$$

$$= \frac{1}{2}\left(\frac{1}{3}+\frac{2}{8}+\frac{2}{15}+\frac{2}{24}+\frac{1}{35}\right)$$

(b) (8 points) Approximate the integral by using Simpson's Rule with $n = 4$ subintervals. You do not need to simplify your final numerical answer.

$$\int_2^6 \frac{1}{x^2-1}\,dx \approx \frac{\Delta x}{3}\left(f(2)+4f(3)+2f(4)+4f(5)+f(6)\right)$$

$$= \frac{1}{3}\left(\frac{1}{3}+\frac{4}{8}+\frac{2}{15}+\frac{4}{24}+\frac{1}{35}\right)$$

8. (a) (8 points) Let $g(x) = \int_0^{x^8} (\sin t)\, e^{-t}\, dt$. Find $g'(x)$.

Let $F(x)$ be an antiderivative of $(\sin x)e^{-x}$.

$g(x) = F(x^8) - F(0)$

$g'(x) = F'(x^8)\, 8x^7$

$= (\sin x^8)\, e^{-x^8}\, 8x^7$

(b) (8 points) Let $h(x) = \int_0^{5} (\sin t)\, e^{-t}\, dt$. Find $h'(x)$.

h is constant, so $h'(x) = 0$

9. (a) (10 points) Evaluate $\int_0^1 \frac{e^{2\theta}}{1+e^{2\theta}}\,d\theta$.

$u = 1 + e^{2\theta} \qquad du = 2e^{2\theta}\,d\theta$

$\frac{1}{2}\,du = e^{2\theta}\,d\theta$

$$\int_0^1 \frac{e^{2\theta}}{1+e^{2\theta}}\,d\theta = \frac{1}{2}\int_2^{1+e^2} \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u|\Big|_2^{1+e^2} = \frac{1}{2}\ln\left(1+e^2\right) - \frac{1}{2}\ln 2$$

(b) (10 points) Evaluate $\int y \ln y\,dy$.

$u = \ln y \qquad v = \frac{1}{2}y^2$

$du = \frac{1}{y}\,dy \qquad dv = y\,dy$

$$\int y \ln y\,dy = \frac{1}{2}y^2\ln y - \frac{1}{2}\int \frac{y^2}{y}\,dy$$

$$= \frac{1}{2}y^2\ln y - \frac{1}{2}\int y\,dy$$

$$= \frac{1}{2}y^2\ln y - \frac{1}{4}y^2 + C$$

10. Determine if the following infinite series are conditionally convergent, absolutely convergent, or divergent. Clearly state which test(s) you are applying and justify your answer completely.

(a) (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$.

Ratio test

$$\lim_{n\to\infty} \frac{[(n+1)!]^2}{(2(n+1))!} \frac{(2n)!}{(n!)^2}$$

$$= \lim_{n\to\infty} \frac{(n+1)!\,(n+1)!}{(2n+2)!} \frac{(2n)!}{n!\,n!}$$

$$= \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})(1+\frac{1}{n})}{(2+\frac{2}{n})(2+\frac{2}{n})} = \frac{1}{4} < 1$$

Converges absolutely

(b) (10 points) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

Consider $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$

$$0 \le \frac{|\sin n|}{n^2} \le \frac{1}{n^2}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by p-series test,

So $\displaystyle\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ converges by comparison test.

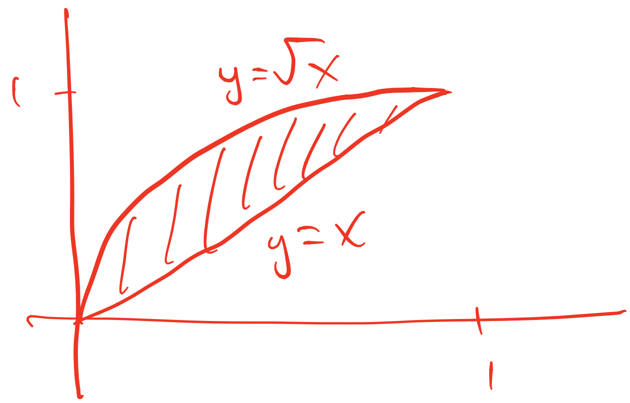
So $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges absolutely.

11. (a) (6 points) Sketch the region bounded by the graphs of $y = x$ and $y = \sqrt{x}$. Shade the region.

$x = \sqrt{x} \rightarrow x^2 = x$

$\rightarrow x(x-1) = 0$

$x = 0$ or $x = 1$



(b) (10 points) Rotate the shaded region from part (a) about the x-axis. Find the volume of the resulting solid of revolution.

$$\text{Volume} = \int_0^1 \left[\pi(\sqrt{x})^2 - \pi(x)^2\right] dx$$

$$= \pi \int_0^1 (x - x^2)\, dx$$

$$= \pi \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$$

$$= \pi \left[\frac{1}{2} - \frac{1}{3}\right]$$

12. (16 points) Evaluate $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

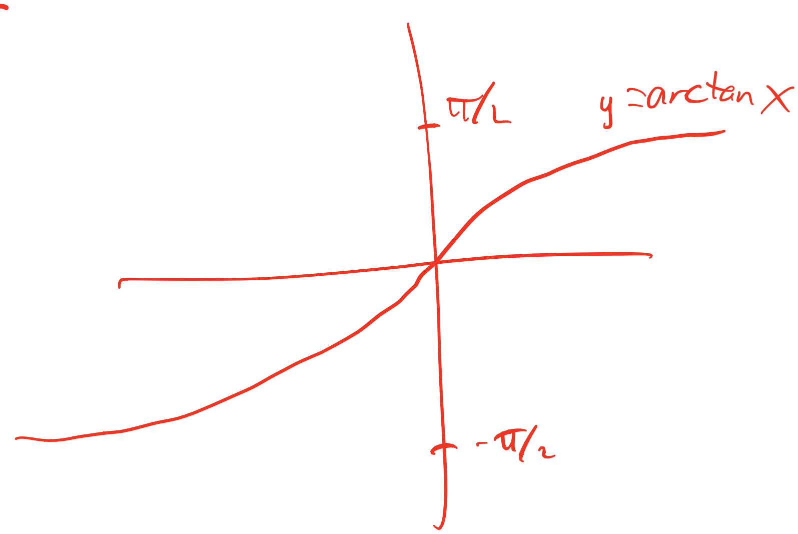
$$= \lim_{b \to -\infty} \int_{b}^{0} \frac{1}{1+x^2}\,dx + \lim_{c \to \infty} \int_{0}^{c} \frac{1}{1+x^2}\,dx$$

$$= \lim_{b \to -\infty} \arctan x \Big|_{b}^{0} + \lim_{c \to \infty} \arctan x \Big|_{0}^{c}$$

$$= \lim_{b \to -\infty} -\arctan b + \lim_{c \to \infty} \arctan c$$

$$= -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2}$$

$$= \boxed{\pi}$$



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